Abstract—In this paper, the bias on the estimated amplitude, which is caused by Gaussian jitter, is studied in case of the IEEE 1057 standard three-parameter sine-fitting method. Because no analytical study exists, this source of uncertainty is usually not considered. Nowadays, it is becoming more and more important due to the ever increasing sampling rates available in analog-to-digital converters (ADCs), which are used in innumerable applications like high-speed digital oscilloscopes. The effect of additive noise is also taken into account.

Index Terms—Analog-to-digital converter (ADC), jitter, least squares fitting, noise, sine wave, sine wave fit, uncertainty.

I. INTRODUCTION

SINE fitting is a technique used in a never-ending list of applications, from analog-to-digital converter (ADC) testing [1]–[5] and impedance measurement [6] to particle size and velocity determination using laser anemometry [7].

Algorithms for sine wave fitting have been standardized in [4] and [5], and the uncertainty of the sine fitting parameters has been studied [8]–[11]. In [8] and [9], an asymptotic Cramér–Rao bound for the variance of three- and four-parameter sine wave fitting parameters (amplitude, offset, initial phase, and frequency) for a large number of samples is derived by taking into account the presence of additive noise. In [10], the same bounds are evaluated when additive noise is present and the data are quantized. In [11], the performance of the frequency estimator used in the IEEE 1057 standard four-parameter sine fitting algorithm is compared with the Cramér–Rao bound.

The presence of jitter in sampling systems has been studied in the past [12], [13]. With the advent of high-frequency digital oscilloscopes, it has become more important. One of the concerns is the measurement of the amount of jitter present in a system [14]–[17]. Another is the minimization of the effect of jitter on measurements [18].

One of those measurements, i.e., the amplitude of a sine wave, which is the focus of this paper, has received little attention in the past [19]–[21]. There are different estimators used to determine the amplitude of a sine wave given a set of data points. In [19], the IEEE 1057 standard method [4], which minimizes the sum of the square of the residuals, was studied. It was pointed out that a bias in the estimated amplitude arises due to jitter in the sampling instant. It is also shown how to compute that bias in the asymptotic case (infinite number of samples). In [20], it was shown that the presence of additive noise leads to a bias in amplitude estimation and presented an approximate analytical expression to compute it. In [21], other estimators, based on least-square and maximum-likelihood approaches, which minimize the difference between the actual and the estimated sample mean and sample variance, are studied.

In this paper, we present such an expression for the bias of the sine wave amplitude, which was estimated using the IEEE 1057 standard method, caused by jitter on the sampling instant when a finite number of samples corrupted by additive noise are acquired. In the limit when the number of samples goes to infinity, and the additive noise standard deviation is zero, the expression presented here is equal to the expression given in [19].

In Section II, we introduce the sine wave amplitude estimator. In Section III, we focus on the exclusive presence of jitter and the bias it introduces. In Section IV, we present an expression for the computation of the estimator bias given the number of samples, signal-to-noise ratio (SNR), and jitter standard deviation. In Section V, we present the results of Monte Carlo simulation, which validate the approximations made. Finally, in Section VI, we draw some conclusions and layout future work.

II. SINE WAVE FITTING

Consider $M$ data points given by

$$y_i = z_i + d_i = C + A \cos(\omega t_i + \theta_i + \varphi) + d_i$$  \hspace{1cm} (1)

where $C$ is the offset, $A$ is the amplitude, $\varphi$ is the initial phase, $\omega$ is the angular frequency, $t_i$’s are the sampling instants, $\theta_i$ is the jitter in radians, and $d_i$ is the additive noise. The variable $z_i$ represents the sample voltage in the absence of additive noise.

The best fitting sine wave, in a least-square error sense, is given by

$$x = (D^T D)^{-1} D^T y$$ \hspace{1cm} (2)

with $x = [\hat{A} \quad \hat{\omega} \quad \hat{\theta}]^T$, $y = [y_1 \quad y_2 \quad \ldots \quad y_M]^T$ and

$$D = \begin{bmatrix}
\cos(\omega t_1) & \sin(\omega t_1) & 1 \\
\cos(\omega t_2) & \sin(\omega t_2) & 1 \\
\ldots & \ldots & \ldots \\
\cos(\omega t_M) & \sin(\omega t_M) & 1
\end{bmatrix}. \hspace{1cm} (3)$$
The sine wave amplitude is obtained from the in-phase \((A_f)\) and in-quadrature \((A_Q)\) components by
\[
\hat{A} = \sqrt{A_f^2 + A_Q^2}.
\] (4)

The variable \(\omega_o\) is the angular frequency of the fitted sinusoid [1]. Here, it is assumed that the frequency of the signal is known \(\omega_o = \omega_x\). For the cases where it is not, a four-parameter algorithm exists, which also estimates the frequency [4]. The error in the frequency estimator is another source of uncertainty that affects the estimation of the sine wave amplitude, offset, and initial phase. These and other effects like harmonic distortion are not considered here.

It is assumed that the number of acquired samples \((M)\) exactly covers an integer number of periods \((J)\) of the sine wave. This means that the sine wave frequency \((f_x)\), the sampling frequency \((f_s)\), and the number of samples satisfy
\[
\frac{f_x}{f_s} = \frac{J}{M}, \quad J \in \mathbb{N}.
\] (5)

In [20], it is shown that if this is the case, then we have
\[
\hat{A} = \sqrt{\frac{4}{M^2} \sum_{i,j} y_i y_j \cos [\omega_x (t_i - t_j)]}
\] (6)
where \(i\) and \(j\) go from 1 to \(M\). This is valid even in the presence of additive noise or jitter.

Here, it is investigated how to determine the expected value of (6) and quantify the contributions of jitter and additive noise. The expected value of the sine wave amplitude can be expressed as a summation of four terms as
\[
\mu_{\hat{A}} = A + a_{\hat{A}} + p_{\hat{A}} + ap_{\hat{A}}.
\] (7)

The second term only takes into account the presence of additive noise. The third term only takes into account the presence of jitter. The fourth term is due to the simultaneous presence of both effects. To reach a simple expression for the estimation bias, we will consider
\[
ap_{\hat{A}} \approx 0.
\] (8)

Later on, we will present the results of Monte Carlo simulations, which validate the final expression presented and show the range of values of the number of samples and the noise standard deviation where the approximate expression proposed is valid.

### III. INFLUENCE OF JITTER

One way to proceed in determining the expected value of \(\hat{A}\) is to determine the expected value of \(\hat{A}^2\) first and then use an approximation to compute the expected value of the square root of a random variable using Taylor series. According to [22], a first-order approximation is
\[
\mu_{\hat{A}} \approx \sqrt{\mu_{\hat{A}^2}}.
\] (9)

Using (6), we have
\[
E\{\hat{A}^2\} = \frac{4}{M^2} \sum_{i,j} E\{z_i z_j\} \cos [\omega_x (t_i - t_j)].
\] (10)

The variable \(z_i\) is used instead of \(y_i\), because only the contribution of the jitter noise is considered in this paragraph. An expression for \(E\{z_i z_j\}\) has been derived in the Appendix. The result was
\[
E\{z_i z_j\} = E\{z_i z_j\}_a + E\{z_i z_j\}_b
\] (11)
with \(E\{z_i z_j\}_a\) given by (36) and (37) and \(E\{z_i z_j\}_b\) given by (39).

Inserting (11) into (10) leads to
\[
E\{\hat{A}^2\} = \mu_{zz_a} + \mu_{zz_b}
\] (12)
with
\[
\mu_{zz_a} = \frac{4}{M^2} \sum_{i,j} E\{z_i z_j\}_a \cos [\omega_x (t_i - t_j)]
\] (13)
and
\[
\mu_{zz_b} = \frac{4}{M^2} \sum_{i,j} E\{z_i z_j\}_b \cos [\omega_x (t_i - t_j)].
\] (14)

We will now proceed with the determination of \(\mu_{zz_a}\) considering that we have a null mean normally distributed jitter with standard deviation \(\sigma_j\). By splitting the summation of (13) into three parts, as was done in [20], to include all possible combination of \(i\) and \(j\) and using (36) and (37) leads to
\[
\mu_{zz_a} = \frac{2}{M^2} \sum_{i,j} A^2 \cos^2 (\omega_x (t_i - t_j)) e^{-\sigma_j^2}
\] (15)
\[
+ \frac{2}{M} A^2 - \frac{2}{M^2} A^2 e^{-\sigma_j^2}.
\] (16)

Using \(2 \cos^2(\theta) = 1 + \cos(2\theta)\), (15) can be written as
\[
\mu_{zz_a} = A^2 e^{-\sigma_j^2} + \frac{1}{M^2} A^2 e^{-\sigma_j^2} \sum_{i,j} \cos(2\omega_x t_i - 2\omega_x t_j)
\] (17)
\[
+ \frac{2}{M} A^2 \left(1 - e^{-\sigma_j^2}\right).
\] (18)

Again, since we are considering that the sine wave fit to the data covers an integer number of periods, the summation is 0, which leads to
\[
\mu_{zz_a} = A^2 e^{-\sigma_j^2} + \frac{2}{M} A^2 \left(1 - e^{-\sigma_j^2}\right).
\] (19)

Inserting (39) into (14) leads to
\[
\mu_{zz_b} = \frac{2}{M^2} A^2 e^{-\sigma_j^2} \sum_{i,j} \cos(\omega_x t_i + \omega_x t_j + 2\varphi) \cos[\omega_x (t_i - t_j)]
\] (20)
\[
+ \frac{4}{M^2} C_A e^{-\sigma_j^2} \sum_{i,j} \cos(\omega_x t_i + \varphi) \cos[\omega_x (t_i - t_j)]
\] (21)
\[
+ \frac{4}{M^2} C_A e^{-\sigma_j^2} \sum_{i,j} \cos(\omega_x t_j + \varphi) \cos[\omega_x (t_i - t_j)].
\] (22)
Again, considering the acquisition is done over an integer number of periods of the signal, the double summations in (18) are zero, regardless of $\varphi$, which leads to

$$\mu_{zz} = 0.$$  \hfill (19)

In conclusion, from (12), (17), and (19), we have

$$E\{\hat{A}^2\} = A^2 e^{-\sigma_n^2} + \frac{2}{M} A^2 \left(1-e^{-\sigma_n^2}\right).$$  \hfill (20)

Note that this term does not depend on the offset or on the initial phase of the sine wave.

Inserting (20) into (9) leads to

$$\mu_\hat{A} \approx \sqrt{A^2 e^{-\sigma_n^2} + \frac{2}{M} A^2 \left(1-e^{-\sigma_n^2}\right)}.$$  \hfill (21)

This can also be written as

$$\mu_\hat{A} \approx A \sqrt{1 + \left(\frac{2}{M} - 1\right) \left(1-e^{-\sigma_n^2}\right)}.$$  \hfill (22)

Using the approximation

$$\sqrt{1+x} \approx 1 + \frac{x}{2}, \quad \text{for } x = 1$$  \hfill (23)

we can write

$$\mu_\hat{A} \approx A - A \left(\frac{1}{2} - \frac{1}{M}\right) \left(1-e^{-\sigma_n^2}\right).$$  \hfill (24)

This is an approximate expression for the expected value of the sine wave amplitude in the presence of jitter. The bias, defined as $\mu_\hat{A} - A$, is given by

$$p_\hat{A} \approx -A \left(\frac{1}{2} - \frac{1}{M}\right) \left(1-e^{-\sigma_n^2}\right).$$  \hfill (25)

As expected, the bias is zero in the absence of jitter. In addition, as the number of sample increases, the absolute value of this bias term monotonically increases to a constant value, contrary to what is usual for biased estimators.

IV. BIAS OF THE ESTIMATED SINE WAVE AMPITITUDE

Combining the contribution of jitter [see (25)] and that of additive noise [20] while neglecting their combined effect, the expected value of the sine wave amplitude is

$$\mu_\hat{A} \approx A + \frac{1}{M} \frac{\sigma_n^2}{A} - A \left(\frac{1}{2} - \frac{1}{M}\right) \left(1-e^{-\sigma_n^2}\right)$$  \hfill (26)

where $\sigma_n$ is the additive noise standard deviation. The relative error of the estimation is

$$\varepsilon_A = \frac{\mu_\hat{A} - A}{A} \approx \frac{1}{2M \cdot \text{SNR}^2} - \left(\frac{1}{2} - \frac{1}{M}\right) \left(1-e^{-\sigma_n^2}\right)$$  \hfill (27)

Expression (27) gives the relative bias of the sine wave amplitude estimation using the IEEE 1057 sine fitting algorithm. It can be seen that it does not directly depend on the signal amplitude (only indirectly through the SNR).

In the ideal scenario where $\sigma_\theta = 0$ and $\sigma_c = 0$, the relative error is zero, as expected. Note that the additive noise contribution in (27) is always positive, whereas the jitter contribution is always negative ($M > 2$). This means that the presence of additive noise causes the estimate of the sine wave amplitude to be larger than the true amplitude (on average) and that the jitter has an opposite effect.

V. MONTE CARLO VALIDATION

Several approximations [see (9) and (23)] were made to simplify the results. To validate those approximations and to check the correctness of the derivations, we performed a Monte Carlo analysis of the estimator bias using repeated sets of data points, starting from a sine wave with a known true amplitude and that was corrupted by both Gaussian jitter and additive noise. The number of repetitions ($R$) used varied from case to case to have confidence intervals of similar length.

In Fig. 1, the relative error obtained is depicted as a function of jitter standard deviation (markers). The solid line represents the theoretical value given in (27).

![Fig. 1. Relative error of the Monte-Carlo-based sine wave amplitude as a function of jitter standard deviation (markers). The solid line represents the theoretical value given in (27).](image)

where

$$\text{SNR} = \frac{A}{\sqrt{2\sigma_v}}.$$  \hfill (28)
Fig. 2. Difference between the relative error of the Monte-Carlo-based sine wave amplitude and the theoretical value given by (27) as a function of jitter standard deviation in the absence of additive noise. The confidence intervals are for a confidence level of 99.9% assuming a normal distribution.

Fig. 3. Difference between the relative error of the Monte-Carlo-based sine wave amplitude and the theoretical value given by (27) as a function of jitter standard deviation in the presence of additive noise. The confidence intervals are for a confidence level of 99.9% assuming a normal distribution.

It can be seen that the relative error of the expected value of the estimated amplitude is in accordance with the theoretical value given by (27), that is, the confidence intervals (vertical bars), for a 99.9% confidence level, are all around zero for small amounts of jitter.

As the standard deviation of jitter increases, the approximate expression ceases to be valid. It is possible to say that the theoretical expression is correct to within 0.1% for values of jitter standard deviation lower than 0.1 rad and SNR better than 2.8 (9 dB) regardless of the number of samples.

In Fig. 4, the dependence of the relative estimation error on the number of samples can be observed. Note that the relative estimation error does not go to 0 as the number of samples increases. From (27), we can compute the limit when the number of samples goes to infinity, e.g.,

$$\lim_{M \to \infty} \varepsilon_A \approx \frac{1}{2} \left( e^{-\sigma_\theta^2} - 1 \right). \quad (29)$$

This shows that the estimator is asymptotically biased in the presence of jitter.

VI. Conclusion

The derived expression for the bias of the fitted sine wave amplitude obtained with the three-parameter sine fitting algorithm [see (27)] shows that the estimator is biased when the acquired samples are affected by jitter. The existence of this bias was previously mentioned in [19], but only the case of an infinite number of samples was considered. Here, an expression is presented to compute the relative bias on the estimator given the standard deviations of jitter and additive noise and the finite number of acquired samples.
Expression (27) can be used to correct the bias of the estimator if the amount of jitter and additive noise present is known, which can be accomplished using, for instance, the methods recommended in [4].

We limited here our study to the effect of jitter and additive noise on the estimation of the sine wave amplitude. We proceed doing work on the effect of jitter on other estimators related to the sine fitting, namely, sine wave offset, initial phase and frequency, as well as other derived parameters, such as the module and argument of impedances determined with the help of sine fitting, or signal-to-noise and distortion ratio of ADCs.

The influence of other effects, like harmonic distortion and frequency error, on the bias and the variance of the estimators has to also be studied in the future to achieve a full understanding of the performance of sine fitting algorithms in real conditions.

APPENDIX

Here, we derive an expression for \( E\{z_i z_j\} \) using the definition of \( z_i \) given in (1). We thus have

\[
E\{z_i z_j\} = E\left\{ \left[ C + A \cos(\omega_x t_i + \theta_i + \varphi) \right] \times \left[ C + A \cos(\omega_x t_j + \theta_j + \varphi) \right] \right\}
\]

which can be written as

\[
E\{z_i z_j\} = C^2 + A^2 E\left\{ \cos(\omega_x t_i + \varphi + \theta_i) \cos(\omega_x t_j + \varphi + \theta_j) \right\} + C A E\left\{ \cos(\omega_x t_i + \varphi + \theta_i) \right\} + C A E\left\{ \cos(\omega_x t_j + \varphi + \theta_j) \right\}.
\]

Using \( 2 \cos(\alpha) \cos(\beta) = \cos(\alpha + \beta) + \cos(\alpha - \beta) \) leads to

\[
E\{z_i z_j\} = C^2 + \frac{1}{2} A^2 E\left\{ \cos(\omega_x t_i - \omega_x t_j + \theta_i - \theta_j) \right\} + \frac{1}{2} C A E\left\{ \cos(\omega_x t_i + \varphi + \theta_i) \right\} + C A E\left\{ \cos(\omega_x t_j + \varphi + \theta_j) \right\}.
\]

To simplify the derivation, we will now split (32) in two, considering the terms that depend on \( \varphi \) and those that do not, i.e.,

\[
E\{z_i z_j\} = E\{z_i z_j\}_a + E\{z_i z_j\}_b
\]

where

\[
E\{z_i z_j\}_a = C^2 + \frac{1}{2} A^2 E\left\{ \cos(\omega_x t_i - \omega_x t_j + \theta_i - \theta_j) \right\}
\]

\[
E\{z_i z_j\}_b = \frac{1}{2} A^2 E\left\{ \cos(\omega_x t_i + \omega_x t_j + 2\varphi + \theta_i + \theta_j) \right\} + C A E\left\{ \cos(\omega_x t_i + \varphi + \theta_i) \right\} + C A E\left\{ \cos(\omega_x t_j + \varphi + \theta_j) \right\}.
\]

To compute the expected value in (34), we have to consider two cases: equal or different values of indices \( i \) and \( j \). If they are equal, then \( \theta_i \) and \( \theta_j \) cancel each other out, and we cease to have any random variable in the equation. The expected value is thus simply

\[
E\{z_i z_j\}_a = C^2 + \frac{1}{2} A^2.
\]

On the other hand, if the indices have a different value, considering that \( \theta_i \) and \( \theta_j \) are normally distributed random variables with standard deviation \( \sigma_\theta \), then we have

\[
E\{z_i z_j\}_a = C^2 + \frac{1}{2} A^2 \cos(\omega_x t_i - \omega_x t_j) e^{-\sigma_\theta^2}
\]

since

\[
E\{\cos(a + \xi)\} = \int_{-\infty}^{\infty} \cos(a + \xi) \frac{1}{\sqrt{2\pi} \sigma_\xi} e^{-\frac{\xi^2}{2\sigma_\xi^2}} d\xi = \cos(a) e^{-\frac{a^2}{2\sigma_\xi^2}}
\]

where \( a \) is a generic constant, and \( \xi \) is a generic normally distributed random variable with zero mean and standard deviation \( \sigma_\xi \). Defining \( \xi = \theta_i - \theta_j \), we have \( \sigma_\xi^2 = 2\sigma_\theta^2 \), since \( \theta_i \) and \( \theta_j \) are independent and have the same standard deviation \( \sigma_\theta \).
Using a similar approach, we have

\[
E\{z_i z_j\}_b = \frac{1}{2} A^2 \cos(\omega_x t_i + \omega_x t_j + 2\varphi) e^{-\sigma^2}
+ CA \cos(\omega_x t_i + \varphi) e^{-\frac{1}{2}\sigma^2}
+ CA \cos(\omega_x t_j + \varphi) e^{-\frac{1}{2}\sigma^2}.
\]  

(39)

REFERENCES


Francisco Corréa Alegria (M’05) was born in Lisbon, Portugal, on July 2, 1972. He received the Diploma, M.S., and Ph.D. degrees in electrical engineering and computers from the Technical University of Lisbon, Lisbon, in 1995, 1997, and 2002, respectively. Since 1994, he has been a member of the instrumentation and measurement research line with the Instituto de Telecomunicações, Technical University of Lisbon. Since 1997, he has been a member of the teaching and research staff with the Instituto Superior Técnico (IST), Technical University of Lisbon. His current research interests include ADC characterization techniques, automatic measurement systems, sensors, and signal processing.

António Cruz Serra (M’00–SM’03) was born in Coimbra, Portugal, on December 17, 1956. He received the Diploma degree in electrotechnical engineering from the University of Oporto, Oporto, Portugal, in 1978 and the M.S. and Ph.D. degrees in electrical and computer engineering from the Technical University of Lisbon, Lisbon, Portugal, in 1985 and 1992, respectively. He is currently a Full Professor of instrumentation and measurement with Instituto Superior Técnico (IST), Technical University of Lisbon, where he has been a Member of the Teaching and Research Staff since 1978 and President since 2009. Since 1994, he has been a member of the instrumentation and measurement research line with the Instituto de Telecomunicações. His current research interests include electrical measurements, ADC modeling, testing and standardization, and automatic measurement systems.