CHAPTER 3
SYSTEM IDENTIFICATION PROCEDURES FOR
EVALUATING SOIL-STRUCTURE INTERACTION EFFECTS

3.1 Introduction

This chapter presents the methods of system identification used to evaluate dynamic properties of soil-structure systems from recordings of earthquake shaking at the sites included in this study. A review of relevant structural dynamics theory is initially presented in Section 3.2 to establish the framework within which these analyses were performed. Two system identification techniques, parametric and nonparametric analyses, are discussed in Sections 3.3 and 3.4. In Section 3.5, the specific system identification procedures used in this study are summarized, while the interpretation of results for different input-output pairs is discussed in Section 3.6.

3.1.1 Objectives

As illustrated schematically in Fig. 3.1(a), the fundamental objective of any system identification analysis is to evaluate the properties of an unknown system given a known input into, and output from, that system. For applications in this study, the system is generally associated with structural flexibility alone, or the structural flexibility coupled with foundation flexibility in rocking and/or translation. The inputs and outputs are various combinations of free-field, foundation, and roof-level recordings (Fig. 3.1b). The input-output pairs used to evaluate modal parameters for various cases of base fixity are discussed in Section 3.6.
Fig. 3.1(a): Schematic of the system identification problem

Fig. 3.1(b): Motions used as inputs and outputs for system identification of structures
The desired results from these system identification procedures are the following system properties:

1. Modal frequencies and damping ratios of the structures for the fixed- and flexible-base cases.
2. Transmissibility functions describing the frequency-dependent variations between input and output motions.

3.1.2 Fundamental Assumptions

A significant assumption made in the system identification analyses described in this chapter is that the dynamic response of soil-structure systems can be described by linear dynamic models with proportional damping. The validity of this assumption is suspect when structures are damaged or yield, or when pronounced soil degradation occurs. However, nonlinear systems generally can be modeled by linear systems with time-dependent parameters (Priestly, 1980). Hence, recursive analyses were employed in this study to track time-dependent changes of linear system parameters. These recursive results provide insight into possible structural damage during strong shaking, and serve as a “check” on simpler analyses assuming linear, time-invariant response.

It was also assumed that input and output motions used for system identification analyses were representative of their respective domains. For foundation motions, this implies that lateral and rocking motions at the same elevation are uniform, which strictly holds only for rigid foundation slabs. Roof motions are assumed to be not influenced by torsional deformations in the structure. Perhaps most significantly, recordings at free-field accelerographs are assumed to be representative of the free-field at large, which is
seldom correct due to spatial incoherence effects. Although a given recorded motion (roof, foundation, or free-field) is unlikely to be perfectly representative of its domain, repeated identifications for sites with multiple input or output recordings generally revealed that identification results were relatively insensitive to the specific output or input motion chosen to represent a given domain.

3.2 Derivation of Transfer Functions from Modal Equations

Transfer functions describe the changes which occur to input signals as they pass through a system and emerge as output signals. In particular, as used here, transfer functions describe the modification of motions between single input and output points. Equations describing transfer functions for structures are derived in this section.

The properties of a linear structure with $n$ degrees of freedom include its mass matrix $m$, stiffness matrix $k$, and damping matrix $c$. The damping matrix is intended to model energy dissipation in the structure, and is assumed to be "proportional" (i.e. a linear combination of the mass and stiffness matrices).

The dynamic displacements of the structure relative to its base are described by the $n \times 1$ vector $u$, with corresponding velocity and acceleration vectors $\dot{u}$ and $\ddot{u}$, respectively. The total displacement vector $u^t$ is the sum of the ground displacement $u_g$ and the dynamic displacement of the structure,

$$u^t(t) = u(t) + 1^T u_g(t)$$  \hspace{1cm} (3.1)
where \( \mathbf{1} \) is a 1xn vector of 1's. The equation of motion for the structure is (Clough and Penzien, 1993):

\[
\mathbf{m}\ddot{\mathbf{u}}(t) + \mathbf{c}\dot{\mathbf{u}}(t) + \mathbf{k}\mathbf{u}(t) = -\mathbf{m}\mathbf{1}^T\ddot{\mathbf{u}}_g(t) \tag{3.2}
\]

The undamped eigenvalue problem for the structure, \( \mathbf{k}\Phi_i = \omega_i^2 \mathbf{m}\Phi_i \), gives the vibration frequencies, \( \omega_i \), and vibration mode shapes, \( \Phi_i \), for each mode \( i \). The generalized mass of mode \( i \) is defined as \( m_i^* = \Phi_i^T \mathbf{m}\Phi_i \), and the generalized influence factor as \( L_i^* = \mathbf{1}^T \mathbf{m}\Phi_i \).

The solution to Eq. 3.2 is exactly represented by superposition of all \( n \) vibration modes using generalized coordinates \( X_i(t) \), but can also be reliably approximated by \( J < n \) modes:

\[
\mathbf{u}(t) \equiv \sum_{i=1}^{J} \Phi_i X_i(t) \tag{3.3}
\]

Inserting Eq. 3.3 into Eq. 3.2, multiplying both sides by \( \Phi_j^T \), and taking advantage of mode orthogonality, the equation of motion for mode \( j \) can be expressed in terms of generalized coordinates as:

\[
\ddot{X}_j(t) + 2\zeta_j \omega_j \dot{X}_j(t) + \omega_j^2 X_j(t) = -\frac{L_j^*}{m_j^*} \ddot{u}_g(t) \tag{3.4}
\]

where \( \zeta_j \) is the damping ratio for mode \( j \). The solution to Eq. 3.4 is obtained through the use of Laplace transforms. The Laplace transform of a time-dependent function is expressed here as \( f(t) = f(s)e^{st} \), where \( s \) is the operator in the Laplace domain. Using Laplace transforms, the solution of Eq. 3.4 is:

\[
X_j(s) = \frac{-1}{s^2 + 2\zeta_j \omega_j s + \omega_j^2} \cdot \frac{L_j^*}{m_j^*} \ddot{u}_g(s) \tag{3.5}
\]
Since strong motion recordings are of total acceleration, Eq. 3.1 is applied to Eq. 3.3 before taking its Laplace transform. The result, after differentiating twice with respect to time to convert to acceleration, is:

\[
\dot{\mathbf{u}}^t(s) = \sum_{j=1}^{J} \Phi_j \ddot{X}_j(s) + \mathbf{1}^T \ddot{u}_g(s) \tag{3.6a}
\]

where \( \ddot{X}_j(s) = s^2 \cdot X_j(s) \). Making use of a relation modified from Fenves and Desroches (1994) to substitute into Eq. 3.6(a),

\[
\begin{align*}
\mathbf{1}^T & = \sum_{j=1}^{J} \frac{L_j^*}{m_j^*} \cdot \Phi_j \tag{3.6b}
\end{align*}
\]

and substituting Eq. 3.5 into Eq. 3.6(a), the total structure accelerations can be related to the ground acceleration as follows:

\[
\dot{\mathbf{u}}^t(s) = \left[ \sum_{j=1}^{J} \frac{L_j}{m_j} \cdot \Phi_j \left( 1 - \frac{s^2}{s^2 + 2\zeta_j\omega_j s + \omega_j^2} \right) \right] \ddot{u}_g(s) \tag{3.6c}
\]

To simplify this expression, the total acceleration vector can be written as

\[
\dot{\mathbf{u}}^t(s) = \mathbf{H}(s)\ddot{u}_g(s) \tag{3.7a}
\]

where,

\[
\begin{align*}
\mathbf{H}(s) & = \sum_{j=1}^{J} \frac{L_j}{m_j} \cdot \Phi_j \cdot H_j(s) \tag{3.7b} \\
H_j(s) & = \frac{2\zeta_j\omega_j s + \omega_j^2}{s^2 + 2\zeta_j\omega_j s + \omega_j^2}
\end{align*}
\]

Element \( j \) of the vector quantity \( \mathbf{H}(s) \) in Eq. 3.7(a) represents the transfer function between the ground (input) and degree-of-freedom \( j \) (output) in the superstructure.
Different recording locations within a structure exhibit the same poles, so generally it is adequate to consider only the output at the roof for identifying parameters for the lower, most significant modes. Hence, single-output system identification analyses were used in this study, which reduces the vector $\mathbf{H}(s)$ to a scalar function for the roof response. It should be noted that contributions from all $J$ modes are represented in the single-output solution, although only fundamental-mode parameters are significantly affected by SSI (Jennings and Bielak, 1973).

The amplitude of a particular component of $\mathbf{H}(s)$ is a continuous surface with peaks located at poles which can be related to modal frequencies and damping ratios. When a component of $\mathbf{H}(s)$ is evaluated along the imaginary axis, the transmissibility function $\mathbf{H}(i\omega)$ is obtained, which gives the ratio of output-to-input acceleration as a function of frequency $\omega$. The roof component of $\mathbf{H}(s)$ for an example structure is presented in Section 3.5.2(c).

### 3.3 Nonparametric System Identification

Nonparametric system identification is used to examine the dynamic response of structural systems by estimating the transmissibility function $\mathbf{H}(i\omega)$ for a given input-output pair. Transfer functions $\mathbf{H}(s)$ cannot be directly estimated by nonparametric techniques. Calculation of transmissibility functions and smoothing procedures for these functions are the subject of this section.
3.3.1 Transmissibility Functions

Transmissibility functions are useful for identifying vibration frequencies and frequency ranges over which amplification or de-amplification occurs. This section will describe how transmissibility functions are computed from an input \( x(t) \) and output \( y(t) \) accelerogram pair. The formulations are based on a single input, single output model, meaning that \( H(s) \) in Eq. 3.7 is a scalar quantity denoted as \( H(s) \); similarly \( H(i\omega) \) is denoted here as \( H(i\omega) \).

Fundamentally, the transmissibility function \( H(i\omega) \) represents the ratio of the Fourier transform of the output signal to that of the input signal. However, since the input signal is random, its Fourier transform may not exist (i.e. zero amplitude) at some frequencies, causing the \( H(i\omega) \) ratio to be undefined. For this reason, \( H(i\omega) \) is usually computed from power spectral density functions \( (S_x, S_y) \) and cross spectral density functions \( (S_{xy}) \) of the input and output signals, which always exist (Pandit, 1991).

For an ergodic and random process with zero mean, power spectra and cross power spectral density functions are Fourier transforms of the auto-correlation functions \( R_x(\tau) \) and \( R_y(\tau) \) and the cross-correlation function \( R_{xy}(\tau) \) for processes \( x \) and \( y \) (Clough and Penzien, 1993):

\[
S_x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_x(\tau)e^{-i\omega \tau} d\tau \\
S_y(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_y(\tau)e^{-i\omega \tau} d\tau \\
S_{xy}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{xy}(\tau)e^{-i\omega \tau} d\tau
\] (3.8)
The functions $S_x$ and $S_y$ are real-valued, while $S_{xy}$ is a complex-valued Hermitian function $S_{xy} = S_{yx}^*$ (where $^*$ denotes complex conjugation). The auto-correlation and cross-correlation functions are, in turn, defined from the discrete input and output signals as:

$$
R_x(\tau) = \sum_{t=0}^{N-|\tau|} x(t)x^*(t + \tau)
$$

$$
R_y(\tau) = \sum_{t=0}^{N-|\tau|} y(t)y^*(t + \tau)
$$

$$
R_{xy}(\tau) = \sum_{t=0}^{N-|\tau|} x(t)y^*(t + \tau)
$$

where, as before, $^*$ denotes complex conjugation and $N =$ number of data points in the time histories $x(t)$ and $y(t)$.

Using the spectral density functions defined in Eq. 3.8, $H(i\omega)$ can be computed two ways (Ljung, 1987 and Pandit, 1991):

$$
S_{yx}(\omega) = H(i\omega)S_x(\omega)
$$

$$
S_y(\omega) = H(i\omega)S_{xy}(\omega)
$$

Hence, two estimates of the complex-valued $H(i\omega)$ are possible:

$$
H_1(i\omega) = \frac{S_{yx}(\omega)}{S_x(\omega)}
$$

$$
H_2(i\omega) = \frac{S_y(\omega)}{S_{xy}(\omega)}
$$

The two estimates of $H(i\omega)$ should theoretically be equal, but generally are not due to noise in the signals and other errors associated with the discrete Fourier transform. The $H_1(i\omega)$ estimate is less sensitive to output noise, while $H_2(i\omega)$ is less sensitive to input
noise (Fenves and Desroches, 1994). The quality of the transmissibility function in the presence of noise and other errors can be assessed using the coherence function, which is defined as the ratio of the two estimates of \( H(\text{i}\omega) \) (Ljung, 1987 and Pandit, 1991):

\[
\gamma^2(\text{i}\omega) = \frac{H_1(\text{i}\omega)}{H_2(\text{i}\omega)} = \frac{|S_{xy}(\omega)|^2}{S_x(\omega)S_y(\omega)} \tag{3.12}
\]

The coherence varies between zero and one, and provides insight into the noise spectrum which is proportional to \( 1 - \gamma^2(\text{i}\omega) \). The transmissibility function \( H(\text{i}\omega) \) is well estimated when the coherence is near unity because the signal to noise ratio is large. However, estimates of \( H(\text{i}\omega) \) with coherencies less than one are common in practice, indicating that the absolute value of \( H_2(\text{i}\omega) \) exceeds the absolute value of \( H_1(\text{i}\omega) \). Hence, \( H_2(\text{i}\omega) \) gives larger peaks than \( H_1(\text{i}\omega) \), and for this reason \( H_2(\text{i}\omega) \) was used as the estimator of the transmissibility functions in order to illustrate the modal frequencies as clearly as possible.

### 3.3.2 Smoothing of Frequency Response Functions

Unsmoothed power spectral density, cross spectral density, and coherence functions computed using the procedures in Section 3.3.1 have a spiky appearance which can make the interpretation of these frequency response functions difficult. For this reason, some smoothing was performed using periodograms as estimates of spectral density functions (Oppenheim and Schafer, 1989). Periodograms employ an averaging procedure which smoothes the spectra by reducing the randomness associated with the estimation procedures.
The averaging procedure is based on the method by Welch (1967). The records are divided into equal length segments in time which may overlap. The Fast Fourier Transform (FFT) of each segment with a data window is computed. The periodogram is the average of the square of the FFT amplitudes for all the segments. In the case of the coherence functions, the averaging procedure is performed on each spectral density function in Eq. 3.12 before computing the coherence.

There is a tradeoff between the smoothness of a periodogram and error, or bias, relative to the true, unsmoothed spectrum (Pandit, 1991). As the number of segments used to compute the periodogram is increased, the results become smoother, but the frequency resolution is decreased and the peaks are flattened. In this study, periodograms were generally computed using four equal-length segments without overlap.

Tapering windows were used for each segment in a periodogram to reduce the statistical dependence between sections due to overlap and to diminish side lobe interference or “spectral leakage” while increasing the width of spectral peaks (Krauss et al., 1994). A Kaiser window with a factor of 15.7 was used for this purpose.

3.4 Parametric System Identification

3.4.1 Introduction

Problems can arise in identifying vibration properties of a structure solely by spectral analysis. In essence, recovery of the general transfer function surface $H(s)$ from a discrete estimate of $H(i\omega)$ with limited frequency resolution by the FFT can be problematic (Pandit, 1991). For this reason, it is desirable to identify a parameterized
model of the structure in the discrete time domain from which a more robust estimate of
the structure's vibration properties can be computed.

The model of the structure represented in continuous time by Eq. 3.7 must initially be
converted to an equivalent model in the discrete time domain. The parameters describing
this discrete time model are then estimated by least squares procedures to minimize the
error between the model and the recorded output. This section will describe how these
steps were performed for this study. As in Section 3.3.1, these formulations are based on
a single input, single output model, so the transfer function is denoted as H(s).

3.4.2 Representation of Continuous Transfer Functions in Discrete Time

The equation of motion employed in Section 3.2 (Eq. 3.2) was based on a continuous
time representation of a linear structural system. However, earthquake recordings of
ground and structure motion are digital, and hence are data in the discrete time domain.
In this section, the continuous time transfer function in Eq. 3.7 is modified to develop an
equivalent discrete time representation.

A number of methodologies are presented in the literature for conversion from the
continuous to discrete domain; these are summarized by Safak (1988 and 1991), Franklin
and Powell (1980), and Åström and Wittenmark (1984). The methodology employed in
this study is to approximate the transfer function by the hold-equivalence technique
(Franklin and Powell, 1980). The continuous input is approximated by piece-wise
constants (zero-order hold), and is passed through the continuous system to calculate the
corresponding discrete output. The discrete transfer function is then determined by taking
the ratio of the Z-transforms of the discrete output to that of the input. The result of this procedure is (Franklin and Powell, 1980),

\[ H(z) = \left(1 - \frac{1}{z}\right)Z\left[\frac{H(s)}{s}\right] \]  

(3.13)

where \( H(z) \) is the discrete time transfer function, \( z \) is the complex Z-transform operator, and \( Z[f] \) denotes the Z-transform of the function \( f \). Applying Eq. 3.13 to Eq. 3.7(b),

\[ H(z) = \sum_{j=1}^{J} \beta_{ij}z^{-1} + \beta_{2j}z^{-2} \]

(3.14)

where the \( \alpha \) and \( \beta \) parameters are related to the parameters in Eq. 3.7(b) by equations given in Åström and Wittenmark (1984). According to Safak (1991), the result in Eq. 3.14 can also be obtained using several other continuous-to-discrete conversion methodologies such as pole-zero mapping and covariance equivalence techniques.

The expression in Eq. 3.14 can be expanded as a rational polynomial (Safak, 1991),

\[ H(z) = \frac{b_1z^{-1} + b_2z^{-2} + \ldots + b_{2J}z^{-2J}}{1 + a_1z^{-1} + a_2z^{-2} + \ldots + a_{2J}z^{-2J}} \]

(3.15)

which represents a discrete time filter of order \( 2J \) (the filter order is the highest power in the denominator). Note that the order of the filter, and hence the order of the model, is twice the number of modes being included in the structural idealization.

The model represented by the discrete time transfer function in Eq. 3.15 can be described as a linear difference equation which relates the input \( x(t) \) and output \( y(t) \),

\[ y(t) + a_1y(t-1) + \ldots + a_{2J}y(t-2J) = b_1x(t-d-1) + b_2x(t-d-2) + \ldots + b_{2J}x(t-d-2J) \]

(3.16)
where \( d \) is the time delay between the input and output (i.e. an input at time \( t \) creates an output at time \( t+d \)). The specific representation in Eq. 3.16 of the general model in Eq. 3.15 is referred to as an ARX model, for autoregressive model with extra input (Ljung, 1987).

3.4.3 Solution Procedures

Least squares techniques were used to solve for the parameters \( a_j \) and \( b_j \) in Eq. 3.16 which describe the discrete time transfer function. Two approaches were generally used for these analyses. First, a single set of \( a_j \) and \( b_j \) parameters was determined which minimizes the sum of the errors between the model and recorded output over all \( N \) time steps. This procedure is referred to as the cumulative error method (CEM), and its results are only accurate if the system properties are time invariant. In the second approach, separate sets of \( a_j \) and \( b_j \) parameters are determined for each time step. Referred to as the recursive prediction error method (RPEM), this approach enables the time variation of linear system properties to be tracked.

The solution procedures for the CEM and RPEM are summarized in Parts (a) and (b) of this section. Procedures for extracting modal frequencies and damping ratios from the \( a_j \) and \( b_j \) parameters are presented in Part (c), while uncertainty in the estimated model is discussed in Part (d). The CEM as employed here was originally presented in Safak (1991), while the RPEM was presented in Safak (1988) and Ghanem et al. (1991).
Model parameter estimation by the cumulative error method (CEM)

To simplify the notation of the solution procedure, Eq. 3.16 is re-written by defining the following vectors:

\[
\Gamma(t) = [-y(t-1) \quad -y(t-2) \quad \cdots \quad -y(t-2J) \quad x(t-d-1) \quad x(t-d-2) \quad \cdots \quad x(t-d-2J)]^T
\]

and

\[
\Theta = (a_1 \quad a_2 \quad \cdots \quad a_{2J} \quad b_1 \quad b_2 \quad \cdots \quad b_{2J})^T
\]

With these substitutions, Eq. 3.16 can be re-written as

\[
y(t) = \Gamma^T(t)\Theta
\]

Since \(x(t)\) and \(y(t)\) are the recorded time histories, \(\Gamma(t)\) is known. The objective of the identification is then to determine the unknown vector \(\Theta\).

For a given \(\Theta\) and time \(t\), the error between the model and recorded output, \(\varepsilon(t, \Theta)\), can be written as

\[
\varepsilon(t, \Theta) = y(t) - \Gamma^T(t)\Theta
\]

A measure of the cumulative error \(V(\Theta)\) can then be taken as the sum of the squares of the errors for each time step as follows:

\[
V(\Theta) = \frac{1}{N} \sum_{t=2J+d+1}^{N} \varepsilon^2(t, \Theta) = \frac{1}{N} \sum_{t=2J+d+1}^{N} \left( y(t) - \Gamma^T\Theta \right)^2
\]

The summation starts from \(t=2J+d+1\) to prevent a negative time step from occurring in the formation of the \(\Gamma^T\) vector. The optimum set of parameters is determined by minimizing \(V(\Theta)\) as,
\[
\frac{dV(\Theta)}{d\Theta} = 0
\]  

(3.22)

which leads to the following equation for \( \Theta \):

\[
\Theta = \left[ \sum_{t=2J+d+1}^{N} \Gamma(t)\Gamma^T(t) \right]^{-1} \left[ \sum_{t=2J+d+1}^{N} \Gamma(t)y(t) \right]
\]  

(3.23)

With the vector \( \Theta \) known, the discrete time transfer function for the system is completely determined.

(b) \textit{Model parameter estimation by the recursive prediction error method (RPEM)}

The notation used here is similar to that in Part (a), namely, \( \Gamma(t) \) and \( \Theta \) are defined by Eqs. 3.17 and 3.18, respectively, and the error, \( \varepsilon(t, \Theta) \), at time step \( t \) is defined as

\[
\varepsilon(t, \Theta) = y(t) - \Gamma^T(t)\Theta(t-1)
\]  

(3.24)

Note that the vector of estimated parameters \( \Theta \) is now time dependent. The fundamental difference between the RPEM and CEM is that the total error is defined at each time step \( t \) by

\[
V(t, \Theta) = \frac{1}{2} \gamma(t) \sum_{\tau=2J+d+1}^{t} \beta(t, \tau)\varepsilon^2(\tau, \Theta)
\]  

(3.25)

where \( \beta(t, \tau) \) is a weighting factor and \( \gamma(t) \) is the normalization factor for \( \beta(t, \tau) \) defined by

\[
\gamma(t) = \frac{1}{\sum_{\tau=2J+d+1}^{t} \beta(t, \tau)}
\]  

(3.26)
The $\beta(t, \tau)$ and $\gamma(t)$ factors define a window of time within which incremental errors $\varepsilon(\tau, \Theta)$ are included in the computation of total error $V(t, \Theta)$ for time step $t$. In this study, an exponential window was used with a constant forgetting factor $\lambda$ (Ljung, 1987). For these conditions, the weighting factor is defined as,

$$\beta(t, \tau) = \lambda^{t-\tau} \quad (3.27)$$

The smaller the value of $\lambda$, the shorter is the window, and the more sensitive are the results to time dependent changes in the system properties. However, small data windows can be more susceptible to noise in the input and output signals. Hence, a tradeoff exists between the time-tracking ability of the analysis and the sensitivity of the solution to noise. In most cases, values of $\lambda=0.98$ to 0.99 were found to be appropriate.

As with the CEM, it is required with the RPEM to minimize the total error according to least squares criteria,

$$\frac{\partial V(t, \Theta)}{\partial \Theta} = V'(t, \Theta) = 0 \quad (3.28a)$$

where $V'$ is a vector of length $4J$. Due to the randomness of the signals resulting from the noise in the system, the condition in Eq. 3.28(a) is met in an average sense by requiring,

$$E[V'(t, \Theta)] = 0 \quad (3.28b)$$

where $E$ denotes the expected value operator. The solution of Eq. 3.28(b) is obtained using stochastic approximation techniques (Safak, 1988). The result is the following recursive relationship for $\Theta(t)$:

$$\Theta(t) = \Theta(t-1) + \alpha_t [V'(t, \Theta(t-1))]^{-1} V'(t, \Theta(t-1)) \quad (3.29)$$

where $\alpha_t$ is a series of positive constants generally taken as $\alpha_t=1$ (Safak, 1988).
The solution of Eq. 3.29 requires the estimation of the total error derivatives $V''(t, \Theta)$ and $V'(t, \Theta)$. The development of recursive relations for these derivatives is detailed in Safak (1988) and will not be repeated here. The final form of the recursion relation for the RPEM algorithm is (Safak, 1988 and Ljung, 1987),

$$\Theta(t) = \Theta(t - 1) + \gamma(t)R^{-1}(t)\Psi(t)\varepsilon(t, \Theta)$$

(3.30)

where $\alpha_t$ was taken as 1, $\Psi(t)$ is defined for the case of an ARX model as

$$\Psi(t) = [-y(t - 1) \cdots -y(t - 2J) \ y(t - d - 1) \cdots y(t - d - 2J)]^T$$

(3.31)

and $R(t, \Theta)$, a 4Jx4J matrix, is estimated as follows:

$$R(t) = R(t - 1) + \gamma(t)[\Psi(t)\Psi^T(t) - R(t - 1)]$$

(3.32)

Numerical inversion procedures for $R^{-1}(t)$ are discussed in Safak (1988). In order to start the recursion, initial values of the vectors $\Theta(0)$ and $\Psi(0)$ and matrix $\gamma(0)R^{-1}(0)$ are taken as

$$\Theta(0) = 0 \quad \Psi(0) = 0 \quad \gamma(0)R^{-1}(t) = 10^4 \times I$$

(3.33)

The initial conditions cause the results for $\Theta(t)$ early in the time history to be erratic and unreliable, though a stable solution is usually achieved subsequently. When erratic results extend through more than a few seconds of the time history as a result of the initial conditions in Eq. 3.33, a second RPEM analysis can be performed using initial conditions derived from parameters associated with an appropriate time step from the first analysis.
(c) **Evaluation of modal frequencies and damping ratios**

Once the parameters describing the discrete time transfer function (i.e. the $\Theta$ or $\Theta(t)$ vectors) have been determined, the frequencies and damping ratios corresponding to the $J$ modes included in the analysis can be estimated. The poles of the discrete time transfer function are first identified as the roots of the denominator from Eq. 3.15,

$$1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_{2J} z^{-2J} = 0$$  \hspace{1cm} (3.34)

The roots $z_1, z_2, \ldots, z_{2J}$ are complex numbers which lie within a unit circle in the complex plane for stable systems.

The poles of the discrete time transfer function are related to the poles in the Laplace domain by (Safak, 1991),

$$s_j = \frac{1}{\Delta t} \ln z_j$$  \hspace{1cm} (3.35)

where $\Delta t$ = the data sampling interval. The poles $s_j$ are the roots of the transfer function denominator in Eq. 3.7(b), which can be expressed as the complex conjugate pair (Fenves and Desroches, 1994),

$$s_j, s_j^* = -\zeta_j \omega_j \pm i \omega_j \sqrt{1 - \zeta_j^2}$$  \hspace{1cm} (3.36)

from which the modal frequencies and damping ratios can be computed as follows:

$$\omega_j = \sqrt{s_j s_j^*}$$

$$\zeta_j = -\frac{\text{Re}(s_j)}{\omega_j}$$  \hspace{1cm} (3.37)

where $\text{Re}(s_j)$ means to take the real part of the complex number $s_j$. 

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(d) *Uncertainty in the estimated model*

There is always uncertainty in models identified from parametric analyses due to imperfect model structures and disturbances in the output data (Ljung, 1995). Systematic errors can result from inadequate model structure (i.e. poor selection of the $d$ or $J$ parameters) which cannot be readily quantified. In this study, such errors were minimized through careful selection of the $d$ and $J$ parameters as described in Section 3.5.2(c). A second type of model variability results from random disturbances in the data. This variability is associated with how the model would change if the identification were repeated with the same model structure and input, but with a different realization of the output. This uncertainty can be readily quantified as part of the estimation of an ARX model.

3.5 **Summary of System Identification Analysis Procedures**

3.5.1 Data Preprocessing

Strong motion data must be preprocessed to provide satisfactory identifications of soil-structure systems. Preprocessing of data consists of baseline correction, removal of outliers, filtering, decimation, and synchronization and alignment of input and output. Many of these operations were performed by the agencies which provided much of the strong motion data used in this study (California Strong Motion Instrumentation Program, CSMIP, and United States Geological Survey, USGS).

Non-zero mean values in strong motion data represent static components of the system and low frequency drifts (Safak, 1991). These are removed via baseline
correction, which is typically accomplished by subtracting the mean or using high-pass filters. Outliers in the data are erroneous peaks typically associated with instrument failure or accidental impact against the accelerograph. Baseline correction and removal of outliers from accelerograms is performed as part of the digitization process by CSMIP and USGS.

The objective of filtering is to remove the frequency components of data dominated by noise. These frequencies are at the low and high end of the spectrum, and are usually not of interest in system identification because they are far removed from the modal frequencies of typical soil-structure systems. The cutoff frequencies used in filtering vary somewhat from site to site due to variable accelerograph sensitivities, though typical high-pass cutoff frequencies are about 0.1 to 0.5 Hz and typical low-pass cutoff frequencies are about 15 to 50 Hz. Both CSMIP and USGS typically perform low- and high-pass filtering on accelerograms during processing (though much of the USGS data from the Northridge Earthquake was not low-pass filtered). The frequency cutoffs used for each site are indicated in Stewart (1997).

Decimation refers to a process by which the sampling rate of the time histories is decreased. Most accelerograms are originally digitized at a sampling interval of $\Delta t=0.005$ sec, and are decimated during processing to $\Delta t=0.01$ or 0.02 sec. An accelerogram contains information up to its Nyquist frequency, which is half the sampling rate in Hz (e.g. for $\Delta t=0.01$ sec, $f_{\text{Nyq}}=50$ Hz). In most buildings, the highest modal frequency of interest is much less than the Nyquist frequency, and hence the accelerograms can be further decimated without losing relevant information. In such cases, decimation was performed by first low-pass filtering the data with a corner at the desired new $f_{\text{Nyq}}$ (8th
order low-pass Chebyshev type I filter), and then re-sampling the resulting smoothed signal at the specified lower rate. The analysis summaries in Stewart (1997) indicate whether decimation was performed for a given set of records and the order of decimation (i.e. the factor by which the sampling rate was decreased).

Synchronization of accelerograms used in system identification analysis was necessary to ensure a constant start time. Sensors within a structure are usually connected to a central recorder, and hence are triggered simultaneously. At some recently instrumented sites, free-field instruments are also connected to the central recorder, so no synchronization was necessary. If any synchronization was required, it was generally between free-field and structural data. The required time shift was typically determined by maximizing the cross correlation between the vertical free-field and foundation accelerations. The cross correlations and associated time shifts are provided with the site data in Stewart (1997) for sites where synchronization was necessary.

At some sites, the azimuths of free-field and structural recordings were different. In such cases, the horizontal free-field data was rotated to align it with structural sensors.

3.5.2 Analysis Procedures

Key aspects of the analyses performed for each site are outlined in this section. Considerations in the selection of structural instruments are described, and the nonparametric and parametric system identification techniques are summarized. In general, the analyses were performed using routines written for the Matlab programming environment (Krauss et al., 1994 and Ljung, 1995). The results compiled for each site are not listed here, but are provided in Stewart (1997).
(a)  \textit{Instrument selection}

A given instrumented level of a structure typically had several sensors at different locations. Sensors were selected for use in system identification analysis by considering the following: (1) sensors should be located within the planes of lateral force resisting elements such as frames or shear walls, (2) for irregular structures, sensors should be located near the centers of mass and rigidity so that torsional contributions to the recorded motions are minimized, and (3) sensors at different elevations should be located directly over each other. Specific instruments used for each structure are listed in Stewart (1997).

(b)  \textit{Nonparametric system identification}

Transfer and coherence functions were computed for the transverse and longitudinal directions of the structure using roof/free-field, roof/foundation, and foundation/free-field output/input pairs. The amplitude and phase of the transmissibility function were examined to provide rough estimates of modal frequencies. Coherence functions were also computed to assess the reliability of the transmissibility function amplitudes. Each of these functions was smoothed according to the criteria in Section 3.3.2.

(c)  \textit{Parametric system identification}

The steps below were performed for input-output pairs of interest. The procedure is discussed using the example of the roof/free-field pair at Site A23 (Northridge Earthquake). These time histories are shown in the top two plots of Fig. 3.2.
Fig. 3.2: Time variation of first-mode, flexible-base parameters for Site A23, transverse direction, 1994 Northridge Earthquake
1. Two user-defined parameters are needed to define the parametric model: the time delay, \( d \), between the input and output motions, and the number of modes necessary to optimize the response, \( J \). The delay is evaluated by examining the variation of cumulative error between the recorded output and model output as a function of \( d \) using a single mode, i.e. \( J=1 \). The value that minimizes the error for the example structure is \( d=2 \) time steps, as shown in Fig. 3.3. Using this delay, the model order is estimated by calculating the variation of error with \( J \). Fig. 3.4 shows that the error initially decreases rapidly with \( J \), but stabilizes beyond a value of \( J=4 \), which is the selected model order.

2. Using these \( d \) and \( J \) values, parameters describing the transfer function surface are calculated by the CEM. These parameters are used to define a transfer function surface with “poles” (high points) and “zeros” (low points). Fig. 3.9 presents the roof/free-field transfer function surface for the example structure with the axes on the horizontal plane scaled according to Eq. 3.37 to match the modal frequency and damping at the poles. The modal frequencies (\( \omega_i \)) and damping ratios (\( \zeta_i \)) computed from the complex-valued pole locations are indicated in Table 3.1. Standard deviations arising from random disturbances in the data are reported along with the mean values in Table 3.1. Coefficients of variation for first-mode parameters are usually about 0.5 to 1.5% for frequency and 5 to 15% for damping.

<table>
<thead>
<tr>
<th>Mode</th>
<th>( \tilde{f} ) (Hz)</th>
<th>( \tilde{\zeta} ) (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.12 ± 0.01</td>
<td>5.5 ± 0.5</td>
</tr>
<tr>
<td>2</td>
<td>3.85 ± 0.05</td>
<td>8.4 ± 1.0</td>
</tr>
</tbody>
</table>

Table 3.1: Results of CEM parametric analyses for roof/free-field pair at Site A23, transverse direction, 1994 Northridge Earthquake
Fig. 3.3: Variation of Error with Time Delay

Fig. 3.4: Variation of Error with Number of Modes

Fig. 3.5: Comparison of transmissibility functions from nonparametric analysis (light line) and parametric model (heavy line)

Fig. 3.6: Zeros (o) and poles (+) of the discrete-time transfer function

Fig. 3.7: (a) Comparison of model and recorded output, and (b) residual of identification for roof motions

Fig. 3.8: Cross-correlation function between input and residual and 99% confidence limits of independence

80
Flexible-Base Transfer Function

axis 'a' is associated with damping, 'b' with frequency

Fig. 3.9: Flexible-base transfer function surface identified by parametric system identification, Site A23, 1994 Northridge earthquake
3. The intersection of the model transfer function surface with the imaginary plane is compared to the nonparametric transmissibility function amplitude to check the model. Major peaks of the curves should occur at similar frequencies, but the amplitude match is not always good because nonparametric transmissibility functions are somewhat arbitrary due to their dependence on the smoothing technique used and the number of points in the FFT. As shown in Fig. 3.5, a good match is obtained near the first-mode frequency. The match is poor at higher frequencies, indicating the limited ability of transmissibility functions to capture higher mode responses containing relatively little seismic energy.

4. Additional checks on the parametric identification are performed as follows: (a) The unscaled poles and zeros of the transfer function are plotted in the complex plane to check for pole-zero cancellation (Fig. 3.6). The pole locations (s_j) in Fig. 3.6 are the unscaled counterparts to the pole locations on the horizontal plane in Fig. 3.9. The unscaled poles should always plot inside the unit circle, whereas zeros can be inside or outside the circle. If poles and zeros are found to overlap, the model is over-constrained and J is decreased. (b) The model and recorded outputs are compared, and the residual is computed (e.g. Fig. 3.7). This check is made to confirm that the residual is small compared to the recorded output, and that the residual has no dominant frequencies. (c) The cross-correlation of the residual with the input is computed to determine if there are components common to these time series (e.g. Fig. 3.8). The dashed lines in Fig. 3.8 are the 99% confidence intervals of independence, meaning that there is a 99% probability that the cross correlation will be contained within these limits if the residual and input are truly independent. Significant cross-correlation indicates that the model order should be
increased to better define the transfer function (Safak, 1988). However, high cross-correlation at negative lags is common and indicates output feedback in the input (Ljung, 1987). This is a product of SSI, and does not imply a problem with the model.

5. The nonlinearity of the structural response can be investigated using the time variability of first-mode parameters calculated by recursive parametric identification (Safak, 1988). These analyses are performed using the $d$ and $J$ values from Step 1. Plots of the time dependent first-mode frequencies and damping ratios for the example pair are presented in Fig. 3.2, from which an essentially time invariant first-mode response is observed.

3.6 Interpretation of Results

In system identification analysis of structural systems, the physical meaning of the results depends on the input and output motions used. The purpose of this section is to derive the input and output pairs used to evaluate fixed- and flexible-base modal parameters, and to describe how these parameters can be estimated when a direct identification is impossible as a result of insufficient instrumentation at a site.

3.6.1 Base Fixity Conditions for Different Input-Output Pairs

In this section, transfer functions and corresponding pole descriptions for different input-output pairs are derived in terms of soil, foundation, and structure properties. Expressions for the frequencies and damping ratios associated with the poles are then
derived so that relationships between these modal parameters and the system properties can be defined.

The interpretation of the modal parameters that are identified from different input-output pairs is made with respect to the simple SSI model shown in Fig. 2.2. Single degree-of-freedom structural models are commonly employed in SSI analyses because inertial interaction effects are most pronounced in the first mode. As noted in Section 2.2.1, if this simple system represents an approximate model of a multi-mode, multi-storey structure, the height $h$ is the distance from the base to the centroid of the inertial forces associated with the first vibration mode, and displacement $u$ is that of the centroid. The effective displacement in Fig. 2.2 is different than the roof displacement used for single-output system identification. This use of different displacements in the single degree-of-freedom model and the system identification does not affect the location of the poles, and fundamental mode parameters derived from the system identification procedures in Section 3.4 can be used in conjunction with the simple model in Fig. 2.2.

As shown in Fig. 2.2, the displacement of mass $m$ has contributions from structural deformation $u$, free-field motion $u_g$, and foundation translation and rocking, $u_f$ and $h\theta$, respectively. The equations of motion describing the simple system in Fig. 2.2 are as follows (Chopra and Gutierrez, 1973),

\begin{align*}
\text{lateral at m:} & \quad m(\ddot{u}_f + h\ddot{\theta} + \ddot{u}) + c\ddot{u} + ku = -m\ddot{u}_g \quad (3.38a) \\
\text{total lateral:} & \quad m(\ddot{u}_f + h\ddot{\theta} + \ddot{u}) + m_f\ddot{u}_f + c_u\ddot{u}_f + k_uu_f = -(m + m_f)\ddot{u}_g \quad (3.38b) \\
\text{total rotation:} & \quad mh(\ddot{u}_f + h\ddot{\theta} + \ddot{u}) + l\dddot{\theta} + c_\theta\ddot{\theta} + k_\theta\theta = -mh\dddot{u}_g \quad (3.38c)
\end{align*}
where $I$ is the rotational moment of inertia of the structure, and $k_u$, $c_u$, $k_\theta$, and $c_\theta$ are foundation impedance values (Eq. 2.4) evaluated at the frequency of the soil-structure system. Eq. 3.38a-c are divided through by $m$, $m+m_f$, and $m_h$, respectively, and the time-dependent functions are transformed to the Laplace domain according to $f(t) = \hat{f}e^{st}$.

With this transformation, the quantities $\hat{k}_u$, $\hat{c}_u$, $\hat{k}_\theta$, and $\hat{c}_\theta$ are interpreted as foundation impedance values in the Laplace domain evaluated at the pole of the transfer function being sought. With these substitutions, Eqs. 3.38a-c can be re-written as,

$$s^2\hat{u}_f + s^2\hat{\theta} + A\hat{u} = -s^2\hat{u}_g$$ (3.39a)

$$A_u\hat{u}_f + s^2\mu\hat{\theta} + s^2\mu\hat{u} = -s^2\hat{u}_g$$ (3.39b)

$$s^2\hat{u}_f + A_\theta\hat{\theta} + s^2\hat{u} = -s^2\hat{u}_g$$ (3.39c)

where $\mu = m/(m+m_f)$. Neglecting the rotational inertia of the structure and the mass of the foundation (i.e. $I = 0$, $\mu = 1$), the $A$ coefficients are defined as

$$A = s^2 + 2\zeta\omega s + \omega^2$$ (3.40a)

$$A_u = s^2 + 2\zeta_u\omega_u s + \omega^2_u$$ (3.40b)

$$A_\theta = s^2 + 2\zeta_\theta\omega_\theta s + \omega^2_\theta$$ (3.40c)

where the frequencies and damping ratios in Eqs. 3.40a-c describe the dynamic behavior of the structure ($\omega$, $\zeta$) or soil-foundation system ($\omega_u$, $\zeta_u$ and $\omega_\theta$, $\zeta_\theta$). These parameters are related to the system properties in Fig. 2.2 as follows,

$$\omega^2 = \frac{\hat{k}}{m} \quad \zeta = \frac{\hat{c}}{2m\omega}$$ (3.41a)
\[ \omega^2_u = \frac{\hat{k}_u}{m} \quad \zeta_u = \frac{\hat{\zeta}_u}{2m\omega_u} \]  

\[ \omega^2_{\theta} = \frac{\hat{k}_{\theta}}{mh^2} \quad \zeta_{\theta} = \frac{\hat{\zeta}_{\theta}}{2mh^2\omega_{\theta}} \]  

In Eqs. 3.39a-c, there are three unknown response functions (\( u, u_f, \) and \( \theta \)) and three equations. Hence, the deformations can be solved for directly in terms of the system properties, with the results that follow:

\[ \frac{\dot{u}}{\dot{u}_g} = \frac{-B_\theta B_u s^2}{s^2(B_u B + B_u B_{\theta} + B_{\theta} B) + B_u B B_{\theta}} = \frac{-B_\theta B_u s^2}{C_s} \]  

\[ \frac{\dot{u}_f}{\dot{u}_g} = -\frac{B_\theta B s^2}{C_s} \]  

\[ \frac{\dot{h}}{\dot{u}_g} = -\frac{B B_u s^2}{C_s} \]  

where \( B = A - s^2, B_u = A_u - s^2, B_{\theta} = A_{\theta} - s^2, \) and \( C_s = s^2(B_u B + B_u B_{\theta} + B_{\theta} B) + B_u B B_{\theta}. \)

Eqs. 3.42a-c represent the complete solution for the simple structure in Fig. 2.2, so any transfer function of interest can be directly evaluated from these results. Three specific input-output pairs are considered in the following Parts (a) to (c).

(a) **Flexible-base**

For the flexible-base case, the input is the free-field ground motion (\( u_g \)) and the output is the total motion at the roof level (\( u_g + u_f + h\theta + u \)). The transfer function is defined as the ratio of these two motions,
Substituting Eqs. 3.42 into Eq. 3.43 and simplifying,

\[ H_a(s) = \frac{B_BB_\theta}{C_s} \]  \hspace{1cm} (3.44)

The poles for the flexible-base case are defined as those values of \( s \) for which \( C_s = 0 \).

Expanding \( C_s \) in terms of the system properties and ignoring small terms which are the product of two damping ratios, a 3rd order polynomial in \( s \) is found as follows,

\[ C_s = 2s^3\left[\zeta_\omega\left(\omega_u^2 + \omega_\theta^2\right) + \zeta_u\omega_u\left(\omega^2 + \omega_\theta^2\right) + \zeta_\theta\omega_\theta\left(\omega_u^2 + \omega^2\right)\right] + \cdots \]
\[ + s^2\left(\omega_u^2\omega_u^2 + \omega_\theta^2\omega_\theta^2 + 2\omega^2\omega_\theta^2\right) + \cdots \]
\[ + 2s\left(\zeta_\omega\omega_u^2\omega_\theta^2 + \zeta_u\omega_u\omega^2\omega_\theta^2 + \zeta_\theta\omega_\theta\omega_u^2\omega^2\right) + \omega_\theta^2\omega_u^2\omega^2 \]  \hspace{1cm} (3.45a)

which can be re-written as

\[ C_s = A_1s^3 + A_2s^2 + A_3s + A_4 \]  \hspace{1cm} (3.45b)

where

\[ A_1 = 2\left[\zeta_\omega\left(\omega_u^2 + \omega_\theta^2\right) + \zeta_u\omega_u\left(\omega^2 + \omega_\theta^2\right) + \zeta_\theta\omega_\theta\left(\omega_u^2 + \omega^2\right)\right] \]
\[ A_2 = \left(\omega_u^2\omega_u^2 + \omega_\theta^2\omega_\theta^2 + 2\omega^2\omega_\theta^2\right) \]
\[ A_3 = 2\left(\zeta_\omega\omega_u^2\omega_\theta^2 + \zeta_u\omega_u\omega^2\omega_\theta^2 + \zeta_\theta\omega_\theta\omega_u^2\omega^2\right) \]
\[ A_4 = \omega_\theta^2\omega_u^2\omega^2 \]  \hspace{1cm} (3.45c-f)

Finding the roots of the third-order polynomial in Eq. 3.45(a) is a non-trivial algebraic problem, though it is known from the factors in Eqs. 3.45c-f that the discriminant is positive and hence there are one real and two complex conjugate roots. Of these, only the complex conjugate roots are physically meaningful. The complex conjugate roots were
found to be well approximated by an expression with the same form as Eq. 3.36, but with

system frequency and damping ($\tilde{\omega}$ and $\tilde{\zeta}$) defined as,

$$\tilde{\omega}^2 = \frac{A_4}{A_1} = \frac{\omega_0^2 \omega_u^2 \omega^2}{\omega^2 \omega_u^2 + \omega_0^2 \omega_u^2 + \omega^2 \omega_0^2} = \frac{1}{\frac{1}{\omega_0^2} + \frac{1}{\omega^2} + \frac{1}{\omega_u^2}}$$  (3.46a)

and

$$\frac{\tilde{\zeta}}{\tilde{\omega}} = \frac{A_2 A_3 - A_4 A_1}{2 A_2 A_4}$$  (3.46b)

which simplifies to,

$$\tilde{\zeta} = \left(\frac{\tilde{\omega}}{\omega_u}\right)^3 \zeta_u + \left(\frac{\tilde{\omega}}{\omega}\right)^3 \zeta + \left(\frac{\tilde{\omega}}{\omega_0}\right)^3 \zeta_0$$  (3.46c)

Eqs. 3.46a-c are an approximate representation of the roots to Eq. 3.45(a) because terms

which are the product of damping ratios are neglected.

These results indicate that flexible-base parameters are dependent on the foundation

impedance in translation and rocking and the structural parameters.

**(b) Pseudo flexible-base**

The pseudo flexible-base case applies for a condition of partial base flexibility,

representing base rocking only. This condition is important because actual flexible-base

parameters are often well-approximated by pseudo flexible-base parameters.

Furthermore, pseudo flexible-base parameters can be used in procedures to estimate

either fixed- or flexible-base parameters (see Section 3.6.2). For the pseudo flexible-base
case, the input is the total base translation \((u_g + u_f)\) and the output is the total motion at the roof level \((u_g + u_f + h\theta + u)\). Proceeding as in Part (a), the transfer function is,

\[
H_b(s) = \frac{\ddot{u}_g + \ddot{u}_f + h\dot{\theta} + \ddot{u}}{\ddot{u}_g + \ddot{u}_f} = \frac{\ddot{u}_g \left(1 + \frac{\ddot{u}_f}{\ddot{u}_g} + \frac{h\dot{\theta}}{\ddot{u}_g} + \frac{\ddot{u}}{\ddot{u}_g}\right)}{\ddot{u}_g \left(1 + \frac{\ddot{u}_f}{\ddot{u}_g}\right)} = \frac{B_uB_B\theta}{C_s - s^2B_\theta B}
\]

(3.47)

The denominator is again a 3rd order polynomial in \(s\),

\[
C_s - s^2B_\theta B = 2s^3\left[\zeta_\theta \omega_\theta^2 + \zeta_u \omega_u(\omega^2 + \omega_\theta^2) + \zeta_\theta \omega_\theta \omega_u^2\right] + \cdots
\]

\[
s^2(\omega^2 \omega_u^2 + \omega_\theta^2 \omega_u^2) + \cdots
\]

\[
2s(\zeta_\theta \omega_\theta^2 \omega_\theta^2 + \zeta_u \omega_u \omega^2 \omega_\theta^2 + \zeta_\theta \omega_\theta \omega_u^2 \omega^2) + \omega_\theta^2 \omega_u^2 \omega^2
\]

from which the system frequency and damping ratio can be evaluated as,

\[
(\bar{\omega}^2)^* = \frac{\omega_\theta^2 \omega_u^2 \omega^2}{\omega^2 \omega_u^2 + \omega_\theta^2 \omega_u^2} = \frac{1}{\frac{1}{\omega_\theta^2} + \frac{1}{\omega^2}}
\]

(3.49a)

\[
\left(\bar{\zeta}\right)^* = \left(\frac{\bar{\omega}}{\omega}\right)^3 \zeta + \left(\frac{\bar{\omega}}{\omega_\theta}\right)^3 \zeta_\theta
\]

(3.49b)

From Eqs. 3.49a-b, pseudo flexible-base parameters are seen to be dependent on the rocking impedance of the foundation and the structural properties. It may be noted that flexible- and pseudo flexible-base parameters are numerically similar when base rocking dominates the SSI (which, excepting broad, short structures, is often true).
(c) **Fixed-base**

For the fixed-base case, the input is the sum of the total base translation and the contribution of base rocking to roof translation \((u_g + u_f + h\dot{\theta})\), and the output is the total motion at the roof level \((u_g + u_f + h + u)\). Proceeding as with Parts (a) and (b), the transfer function is,

\[
H_c(s) = \frac{\ddot{u}_g + \ddot{u}_f + h\ddot{\theta} + \ddot{u}}{\ddot{u}_g + \ddot{u}_f + h\ddot{\theta}} = \frac{\ddot{u}_g \left(1 + \frac{\ddot{u}_f}{\ddot{u}_g} + \frac{h\ddot{\theta}}{\ddot{u}_g} + \frac{\ddot{u}}{\ddot{u}_g}\right)}{\ddot{u}_g \left(1 + \frac{\ddot{u}_f}{\ddot{u}_g} + \frac{h\ddot{\theta}}{\ddot{u}_g}\right)} = \frac{B_u B\theta}{C_s - s^2 (B\theta + B_B)}
\]

(3.50)

The denominator in this case can be reduced to the product of a first- and second-order polynomial in \(s\),

\[
C_s - s^2 (B\theta + B_B) = \left[2s(\zeta_u \omega_u \omega_\theta^2 + \zeta_\theta \omega_\theta \omega_u^2) + \omega_\theta^2 \omega_u^2\right] s^2 + 2\zeta_\omega s + \omega_\omega^2
\]

(3.51)

The second-order polynomial in Eq. 3.51 is the same as the denominator of Eq. 3.7(b). Hence, by analogy to that solution, the fixed-base parameters identified from this analysis are simply \(\omega\) and \(\zeta\), respectively.

(d) **Summary**

The results presented in parts (a)-(c) show that a component of system flexibility is absent from the parametric system identification results when its associated motion is added to \(u_g\) in the input. For example, in Part (b), when the base translation motion is added to \(u_g\) in the input, the results represent only the structural flexibility and rocking foundation flexibility (i.e. base translation effects are “removed”). Similarly, when base
rocking and translation are added to $u_g$ for the input in Part (c), the only remaining system flexibility is that of the structure.

The input and output required to evaluate system parameters for various conditions of base fixity are summarized in Table 3.2. These same results were derived by Luco (1980a), who solved the equations of motion (i.e. Eqs. 3.38a-c) in the frequency domain to determine the input-output pairs necessary for evaluations of flexible-, pseudo flexible-, and fixed-base modal parameters using nonparametric identification procedures. The advantage of the present approach is that the more accurate estimates of transfer functions from parametric identification can be used to derive fundamental mode vibration parameters for various conditions of base fixity.

<table>
<thead>
<tr>
<th>System</th>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>a Flexible-Base</td>
<td>$u_g$</td>
<td>$u_g + u_f + h\theta + u$</td>
</tr>
<tr>
<td>b Pseudo Flexible-Base</td>
<td>$u_g + u_f$</td>
<td>$u_g + u_f + h\theta + u$</td>
</tr>
<tr>
<td>c Fixed-Base</td>
<td>$u_g + u_f + h\theta$</td>
<td>$u_g + u_f + h\theta + u$</td>
</tr>
</tbody>
</table>

3.6.2 Estimation of Fixed and Flexible-Base Modal Parameters

Based on the results in Table 3.2, it is necessary to have recordings of free-field, foundation and roof translations as well as base rocking to evaluate directly both fixed- and flexible-base modal parameters of structures. If no recordings of roof translation are available, no modal parameters can be identified. However, if one of the other three motions is missing, the set of modal parameters not directly evaluated can be estimated. The two specific cases that will be considered here are missing base rocking motions (in
which case fixed-base parameters are estimated), and missing free-field motions (in which case flexible-base parameters are estimated). The derivations in this section are made with respect to the single degree-of-freedom model in Fig. 2.2. For multi-mode structures, the quantities $m$ and $h$ are the effective mass and height, respectively.

Although $m$ and $h$ are functions of participation factors (which are not identified), the fundamental frequency and damping ratio are generally insensitive to reasonable estimates of $m$ and $h$.

Verification of these parameter estimation procedures is presented in Section 5.3.2 using results from sites where all three sets of modal parameters were directly evaluated from system identification analyses.

(a) Estimation of fixed-base modal parameters (missing base rocking motions)

It will be shown in this section that fixed-base modal parameters for a structure can be estimated from “known” flexible- and pseudo flexible-base parameters. Hence, it is assumed that $\tilde{\omega}$, $\tilde{\zeta}$, $\tilde{\omega}^*$, and $\tilde{\zeta}^*$ have been determined from system identification analyses [Cases (a) and (b) in Table 3.2]. Considering frequency first, Eqs. 3.46(a) and 3.49(a) are re-written as,

$$\frac{1}{\tilde{\omega}^2} = \frac{1}{\omega^2} + \frac{1}{\omega_\theta^2} + \frac{1}{\omega_\theta^2}$$

$$\frac{1}{(\tilde{\omega}^2)^*} = \frac{1}{\omega^2} + \frac{1}{\omega_\theta^2}$$

from which $\omega_\theta$ can be readily determined as
The ratio of $\omega_b$ to $\omega_a$ is then taken using Eq. 3.41 to evaluate $\omega_b$ as follows,

$$\left(\frac{\omega_b}{\omega_a}\right)^2 = \frac{\hat{k}_\theta}{m h^2} = \frac{k_\theta \cdot \hat{\alpha}_\theta(s)}{K_u \cdot \hat{\alpha}_u(s)} \frac{1}{h^2}$$

(3.54)

where $\hat{\alpha}_u$ and $\hat{\alpha}_\theta$ are modifiers to the static foundation impedance defined in the Laplace domain (similar to the frequency dependent $\alpha$ factors defined in Eq. 2.4).

For surface foundations, Eq. 3.54 can be simplified using the impedance for a rigid circular disk foundation on the surface of a homogeneous, isotropic halfspace. In this case, the static stiffnesses are as indicated in Eq. 2.5, and the ratio of $\omega_b$ to $\omega_a$ can be expressed as,

$$\left(\frac{\omega_b}{\omega_a}\right)^2 = \left(\frac{r_2^3}{r_1 h^2}\right)^2 \frac{2 - \nu}{3(1 - \nu)} \left(\frac{\hat{\alpha}_\theta(s)}{\hat{\alpha}_u(s)}\right)$$

(3.55)

where $r_2$ and $r_1$ are defined in Eq. 2.3. Assuming the effective structure height, foundation geometry, and soil Poisson's ratio are known, the only unknown quantity in Eq. 3.55 is the ratio of dynamic factors $\hat{\alpha}_\theta / \hat{\alpha}_u$. In the frequency domain, the factors $\alpha_u$ and $\alpha_\theta$ are commonly evaluated as functions of both frequency and soil hysteretic damping. Hence, such solutions may also be considered valid in the Laplace domain. As an approximation, the ratio $\hat{\alpha}_\theta / \hat{\alpha}_u$ can be computed for surface foundations using the frequency dependent impedance factors for a circular foundation on a uniform viscoelastic half-space derived by Veletsos and Verbic (1973). As shown in Fig. 2.3,
both factors are nearly unity for the low frequencies of most structures \((a_0 < 1)\), so the \(\hat{\alpha}_q / \hat{\alpha}_u\) ratio should not significantly affect the results.

For embedded foundations, Eq. 3.54 can be simplified using the impedance for a rigid circular disk foundation embedded into a homogeneous, isotropic halfspace. Using the solution by Elsabee and Morray (1977) for shallowly embedded \((e/r < 1)\) foundations discussed in Section 2.2.2(d), the solution of Eq. 3.54 is reduced to

\[
\left(\omega_u / \omega_0\right)^2 = \left(\frac{r^3}{r_h^2}\right) \left(\frac{2 - v}{3(1 - v)}\right) \left(\frac{\hat{\alpha}_q(s)}{\hat{\alpha}_u(s)}\right) \left(\frac{1 + 2(e/r)}{1 + 0.67(e/r)}\right)
\]

(3.56)

where \(\hat{\alpha}_u\) and \(\hat{\alpha}_q\) are derived for surface foundations.

Obviously, use of Elsabee and Morray’s approximate solution for embedded foundation impedance increases the expected error in estimates of the ratio \(\omega_u / \omega_0\).

However, for the moderate embedment ratios of many building structures \((e/r < 0.5)\), this error is expected to be small.

Once \(\omega_0\) is computed from Eqs. 3.55 or 3.56, the fixed-base frequency can be readily computed from Eq. 3.49(a) as,

\[
\frac{1}{\omega^2} = \frac{1}{(\omega_0^2) - \omega_0^2}
\]

(3.57)

The evaluation of fixed-base damping involves slightly more algebra, but the same principles apply. Eqs. 3.46(c) and 3.49(b) are used in conjunction with the ratio of the damping definitions for \(\zeta_u\) and \(\zeta_\theta\) (Eq. 3.41) to evaluate the unknown damping quantities \(\zeta_u\), \(\zeta_\theta\), and \(\zeta\). The result is as follows,
\[ \zeta = \frac{1}{C_1} \cdot \zeta - \frac{C_2}{C_1} \zeta^* \]  

(3.58a)

where

\[ C_1 = \left( \frac{\tilde{\omega}}{\omega} \right)^3 - C_3 \left( \frac{\tilde{\omega}^*}{\omega} \right)^3 \]  
\[ C_2 = \frac{C_3}{\left( \frac{\tilde{\omega}^*}{\omega} \right)^3} \]  
\[ C_3 = C_4 \left( \frac{\tilde{\omega}}{\omega_u} \right)^3 + \left( \frac{\tilde{\omega}}{\omega_\theta} \right)^3 \]  
\[ C_4 = \frac{\omega_\theta}{\omega_u} \frac{\hat{\beta}_u}{\hat{\beta}_\theta} \frac{r^2 h^2}{r^2} \cdot \frac{3(1 - \nu)}{2 - \nu} \]  

(3.58b-e)

The terms \( \hat{\beta}_u \) and \( \hat{\beta}_\theta \) in \( C_4 \) are dimensionless dashpot coefficients similar to the \( \beta_u \) and \( \beta_\theta \) terms in Eq. 2.4. The only assumption made in this analysis is that the ratio of soil damping factors \( \hat{\beta}_u / \hat{\beta}_\theta \) can be computed from the formulation in Veletsos and Verbic (1973), with appropriate corrections for foundation embedment, shape, and non-rigidity effects. Even for surface foundations and low frequencies, these factors can be quite sensitive to frequency and hysteretic soil damping (e.g. Fig. 2.3), so the evaluation of \( \hat{\beta}_u / \hat{\beta}_\theta \) may be subject to significant errors. These errors are compounded for embedded foundations because radiation damping from basement-wall/soil interaction is not rigorously accounted for in the estimation of \( \hat{\beta}_u \) and \( \hat{\beta}_\theta \). Hence, estimates of fixed-base damping are subject to greater uncertainty than estimates of fixed-base frequency.
(b) Estimation of flexible-base modal parameters (missing free-field motions)

For the derivation of flexible-base modal parameters, it is assumed that fixed- and pseudo flexible-base system identification analyses have been performed [Cases (b) and (c) in Table 3.2]. Hence, parameters \( \tilde{\omega}^* \), \( \tilde{\zeta}^* \), \( \omega \), and \( \zeta \) are assumed known. The derivation follows the same steps as in Part (a). Using Eq. 3.49(a), \( \omega_\theta \) is evaluated as

\[
\frac{1}{\omega_\theta^2} = \frac{1}{{(\tilde{\omega}^*)^2}} - \frac{1}{\omega^2}
\]

Frequency \( \omega_\theta \) is computed from the ratio \( \omega_\theta/\omega_\delta \) in Eqs. 3.55 or 3.56, and the flexible-base frequency is determined directly from Eq. 3.46(a).

For the case of damping, the algebra is less lengthy than Part (a). The first step is the calculation of \( \zeta_\theta \) from Eq. 3.49(b),

\[
\zeta_\theta = \frac{\zeta^* - \left(\frac{\tilde{\omega}^*}{\omega}\right)^3 \zeta}{\left(\frac{\tilde{\omega}^*}{\omega_\theta}\right)^3}
\]

Damping \( \zeta_u \) is then evaluated from the following,

\[
\zeta_u = \zeta_\theta \frac{\omega_\theta}{\omega_u} \frac{\hat{\beta}_u}{\hat{\beta}_\theta} \frac{r_1^2 h^2}{r_2^4} \cdot \frac{3(1 - \nu)}{2 - \nu}
\]

and the flexible-base damping is determined directly from Eq. 3.46(c).

The same limitations on solution accuracy that were stated in Part (a) apply here as well. Namely, the accuracy of estimated flexible-base frequency is generally better than the accuracy of flexible-base damping, especially for embedded foundations.