Solution of Elasticity Problems using Collocation Techniques

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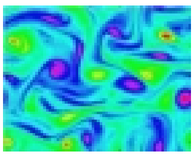
Ondelettes et équations aux dérivées partielles



Marseille, 9-12 April 2001

Outline

1. Motivation

2. Interpolating Wavelets

3. Collocation Techniques

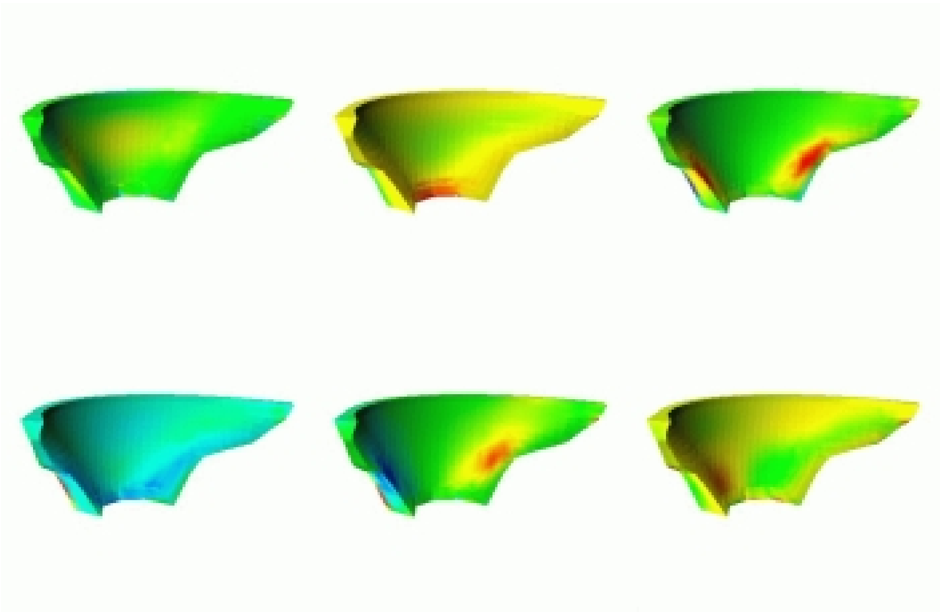
4. Plane elasticity problems

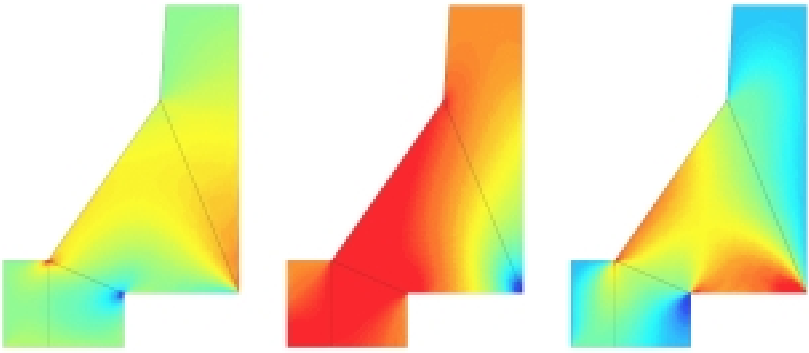
5. Numerical Applications

6. Conclusions and further developments

Motivation

Numerical Simulation of structural engineering problems





Structural Engineering

Some problems

- Definition of the geometry
- Definition of loading conditions
- Characterisation of material behaviour (concrete, soils, ...)
- "Size" of the problem

Objective

Collocation techniques

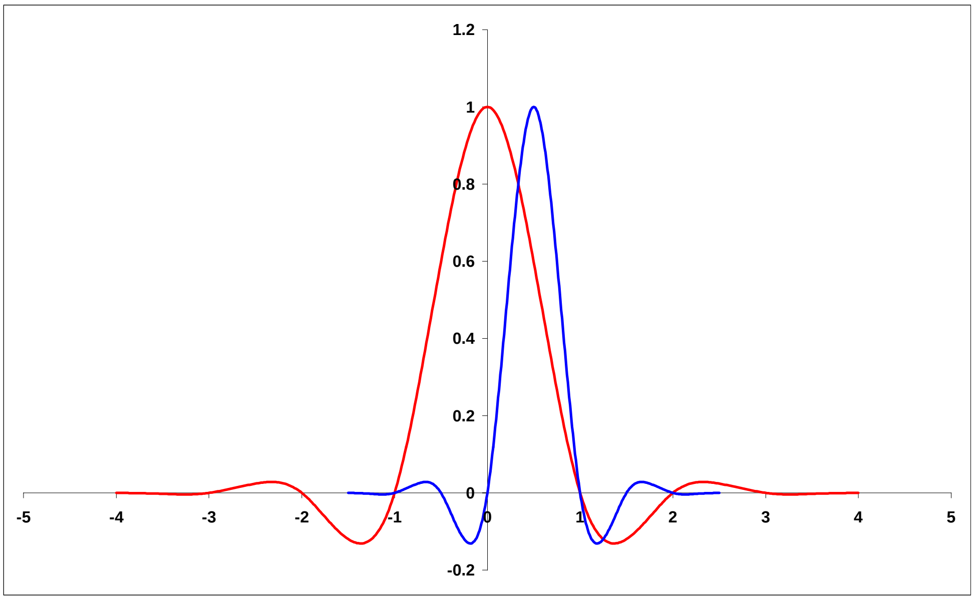
- Bertoluzza, S., "An Adaptive Collocation Method based on Interpolating Wavelets", in *Multiscale Wavelet Methods for Partial Differential Equations*, edited by Dahmen, Kurdila and Oswald, Academic Press, 1997.
- Bertoluzza, S. and Naldi, G., "A wavelet collocation method for the numerical solution of partial differential equations", ACHA, 3, 1996.
- Bertoluzza, S., "Adaptive wavelet collocation method for the solution of Burgers equation", *Transport Theory and Stat. Phys.*, 25, 1996.

Interpolating wavelets

- Deslaurier, G. and Dubuc, S., "Symmetric iterative interpolation processes, *Constructive Approximation*, 5, 1989.

Interpolating wavelets

Deslaurier-Dubuc interpolating functions



$$\theta_N(x) = \int \phi_L(y)\phi_L(y-x)\,dy$$

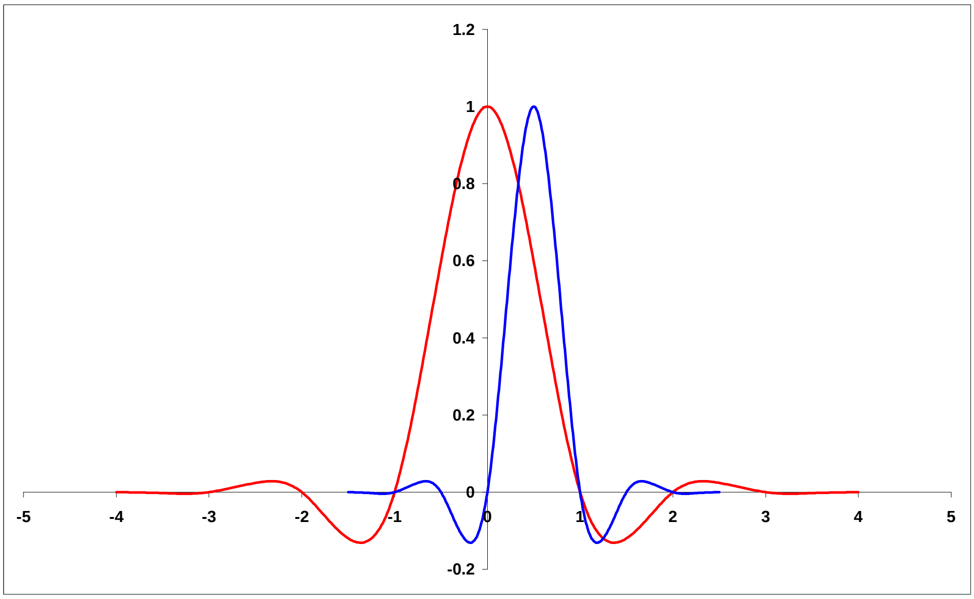
$$N = 2L + 1$$

Interpolating wavelets

Properties

- $supp\theta = [-N, N]$
- θ is refinable
- $\theta(n) = \int \phi_L(y)\phi_L(y-n)\,dy = \delta_{n0}$
- Polynomials up to order N can be represented as a linear combination of the integer translates of θ.

Interpolating wavelets



$$\lambda = x_k^j = k/2^j \in G_j(\Omega)$$

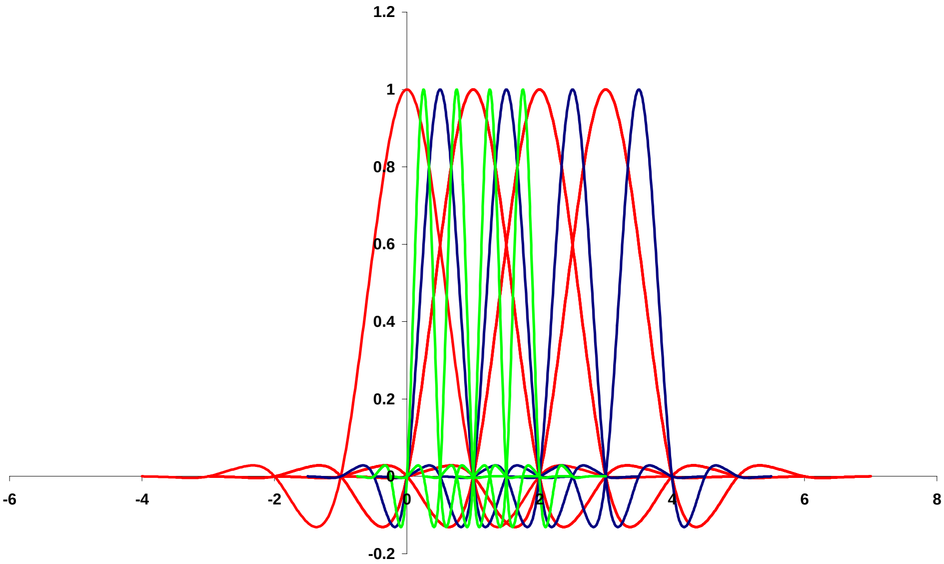
$$\theta_\lambda^j(\lambda) = 1, \quad \theta_\lambda^j(\nu) = 0, \quad \lambda \neq \nu \ (\lambda, \nu \in G_j(\Omega))$$

$$L_j(f) = \sum_{\lambda \in G_j(\Omega)} f(\lambda)\theta_\lambda^j$$

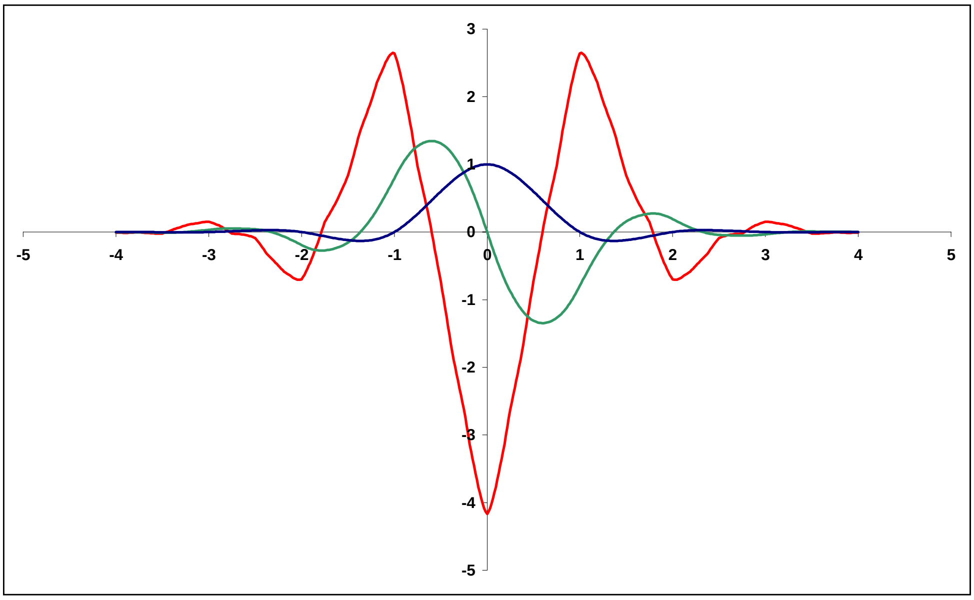
$$W_j(\Omega) = (L_{j+1} - L_j)V_{j+1}(\Omega)$$

$$\psi_\lambda = \theta_\lambda^{j+1}, \quad \lambda \in G_{j+1}(\Omega) \backslash G_j(\Omega)$$

Interpolating wavelets

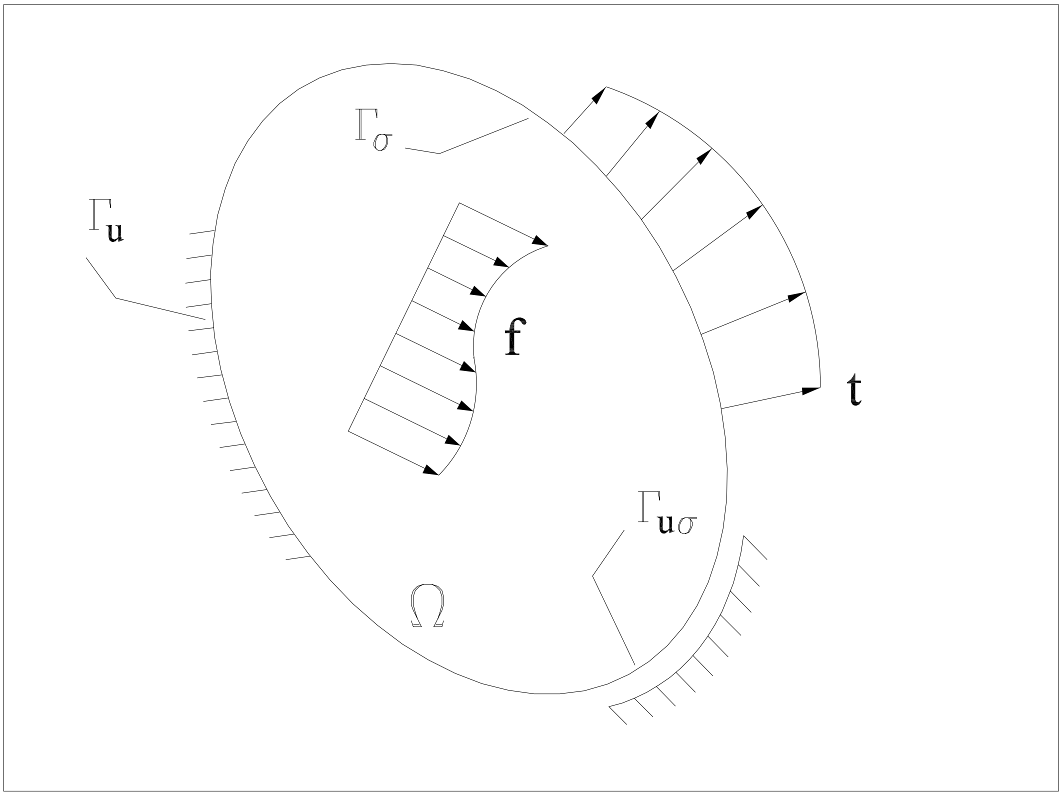


Interpolating wavelets



Derivatives of $\theta(x)$

Classical Theory of Elasticity



$$\boldsymbol{f} = \left\{ \begin{array}{c} f_x \\ f_y \end{array} \right\} \quad \boldsymbol{t} = \left\{ \begin{array}{c} t_{x\gamma} \\ t_{y\gamma} \end{array} \right\}$$

Classical Theory of Elasticity

Displacements

$$\boldsymbol{u} = \left\{ \begin{array}{c} u_x(x,y) \\ u_y(x,y) \end{array} \right\}$$

Strains

$$\boldsymbol{\varepsilon} = \left\{ \begin{array}{c} \varepsilon_{xx}(x,y) \\ \varepsilon_{yy}(x,y) \\ \varepsilon_{xy}(x,y) \end{array} \right\}$$

Stresses

$$\boldsymbol{\sigma} = \left\{ \begin{array}{c} \sigma_{xx}(x,y) \\ \sigma_{yy}(x,y) \\ \sigma_{xy}(x,y) \end{array} \right\}$$

Classical Theory of Elasticity

Equilibrium conditions

$$\boldsymbol{D}\,\boldsymbol{\sigma} + \boldsymbol{b} = \mathbf{0} \quad \text{in } V$$

$$\boldsymbol{N}\,\boldsymbol{\sigma} = \boldsymbol{t}_\gamma \quad \text{on } \Gamma_\sigma$$

Compatibility conditions

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_e + \boldsymbol{\varepsilon}_p = \mathbf{D}^*\mathbf{u} \quad \text{in } V$$

$$\mathbf{u} = \mathbf{u}_\gamma \quad \text{on } \Gamma_u$$

Elasticity

$$\boldsymbol{\varepsilon}_e = \mathbf{f}\,\boldsymbol{\sigma} + \boldsymbol{\varepsilon}_\theta$$

Classical Theory of Elasticity

Definition of the stress field

$$\sigma_{xx} = e_1 \frac{\partial u_x}{\partial x} + e_2 \frac{\partial u_y}{\partial y}$$

$$\sigma_{yy} = e_2 \frac{\partial u_x}{\partial x} + e_1 \frac{\partial u_y}{\partial y}$$

$$\sigma_{xy} = e_3 \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right)$$

$$e_1 = \frac{E}{1 - \nu^2}$$

$$e_2 = \frac{\nu \, E}{1 - \nu^2}$$

$$e_3 = \frac{E}{2\,(1 + \nu)}$$

Classical Theory of Elasticity

Field equation

$$\begin{bmatrix} \boldsymbol{K}_{11} & \boldsymbol{K}_{12} \\ \boldsymbol{K}_{21} & \boldsymbol{K}_{22} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} + \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \boldsymbol{K}_{11} &= e_1 \frac{\partial^2}{\partial x^2} + e_3 \frac{\partial^2}{\partial y^2} \\ \boldsymbol{K}_{22} &= e_2 \frac{\partial^2}{\partial x^2} + e_1 \frac{\partial^2}{\partial y^2} \\ \boldsymbol{K}_{12} &= \boldsymbol{K}_{21} = (e_2 + e_3) \frac{\partial^2}{\partial x \, \partial y} \end{aligned}$$

Classical Theory of Elasticity

Boundary conditions (Dirichlet)

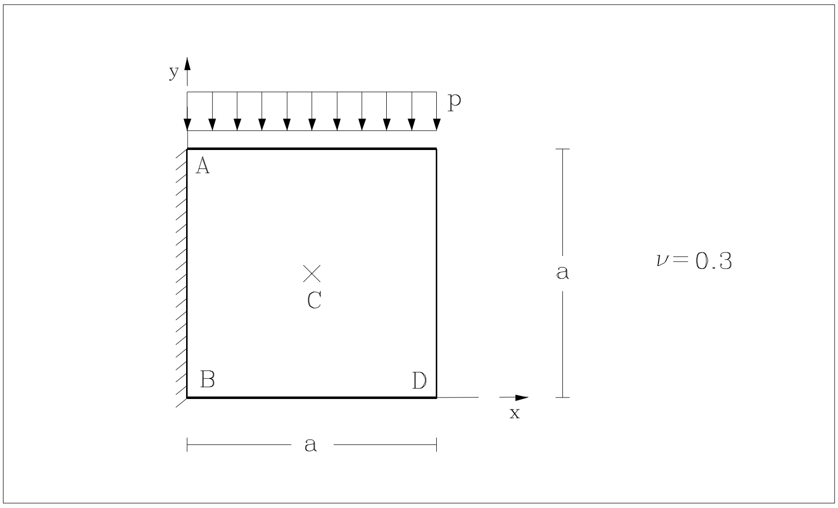
$$\begin{aligned} u_x &= \overline{u}_x \quad on \quad \Gamma_u \\ u_y &= \overline{u}_y \quad on \quad \Gamma_u \end{aligned}$$

Boundary conditions (Neumann)

$$\begin{bmatrix} \boldsymbol{t}_{11} & \boldsymbol{t}_{12} \\ \boldsymbol{t}_{21} & \boldsymbol{t}_{22} \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

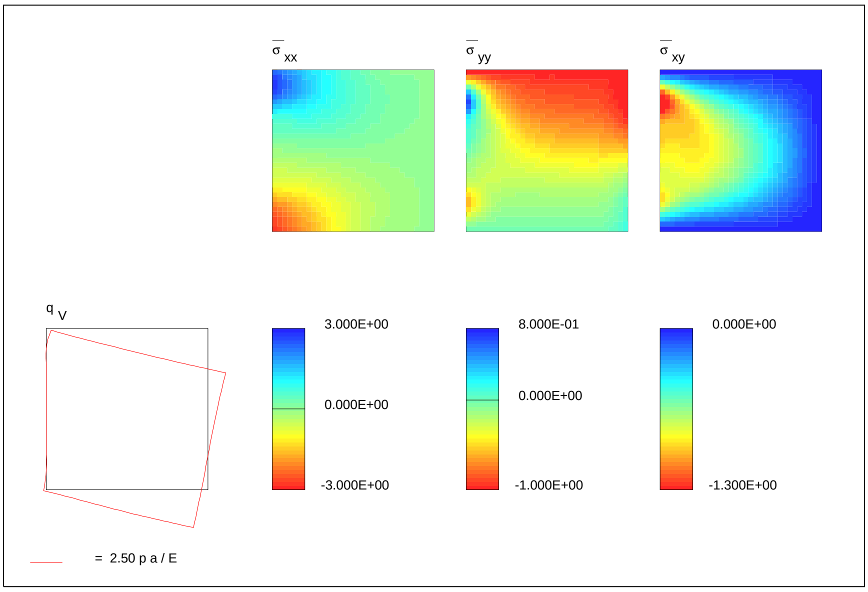
$$\begin{aligned} \boldsymbol{t}_{11} &= e_1\, n_x \frac{\partial}{\partial\, x} + e_3\, n_y \frac{\partial}{\partial\, y} \\ \boldsymbol{t}_{22} &= e_3\, n_x \frac{\partial}{\partial\, x} + e_1\, n_y \frac{\partial}{\partial\, y} \\ \boldsymbol{t}_{12} &= e_2\, n_x \frac{\partial}{\partial\, y} + e_3\, n_y \frac{\partial}{\partial\, x} \\ \boldsymbol{t}_{21} &= e_3\, n_x \frac{\partial}{\partial\, y} + e_2\, n_y \frac{\partial}{\partial\, x} \end{aligned}$$

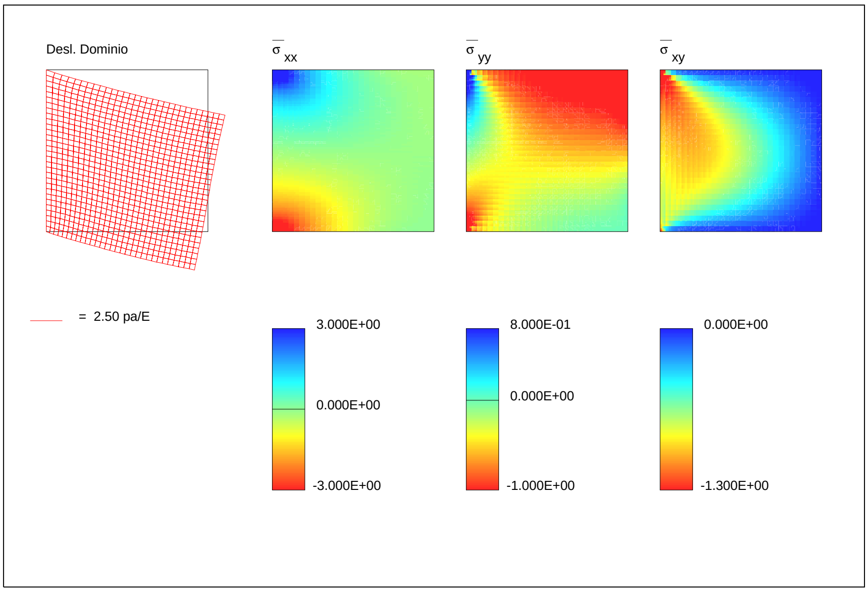
Numerical Applications



Discretisation	N	j_0	j_{max}	n_{dof}
A	4	2	3	162
B	4	2	4	578
C	4	2	5	2178

Square cantilever



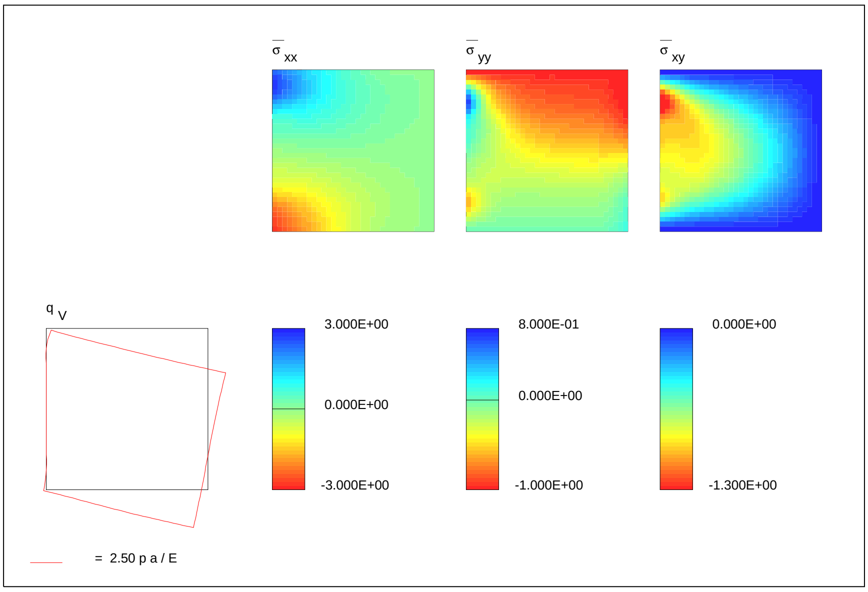


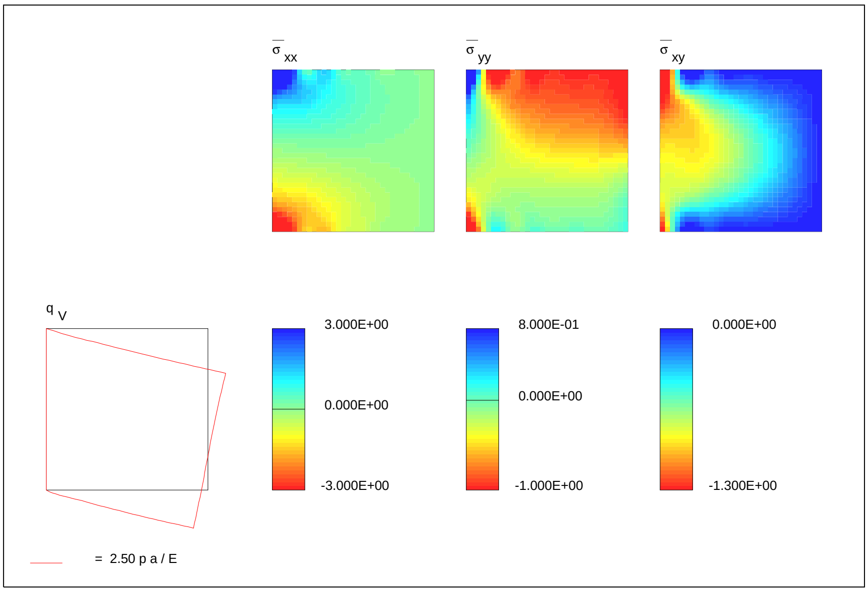
Square cantilever

	A	B	C	COSMOS
$\overline{d}_x$	*-1.114*	*-1.057*	*-1.044*	*-1.039*
$\overline{d}_y$	*-2.930*	*-2.941*	*-2.963*	*-2.979*

$$\overline{d}_x = \frac{E}{p\,a} d_x \ ; \ \overline{d}_y = \frac{E}{p\,a} d_y$$

Square cantilever





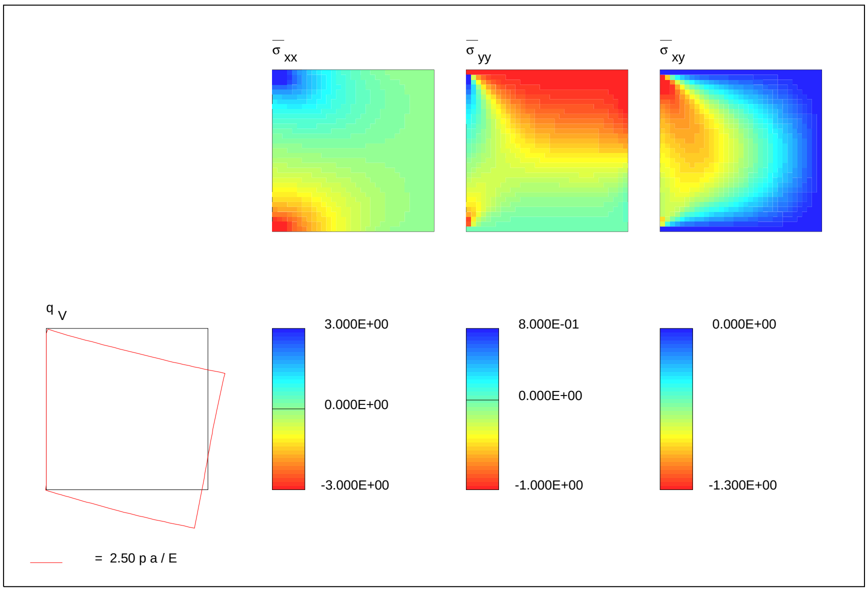
Square cantilever

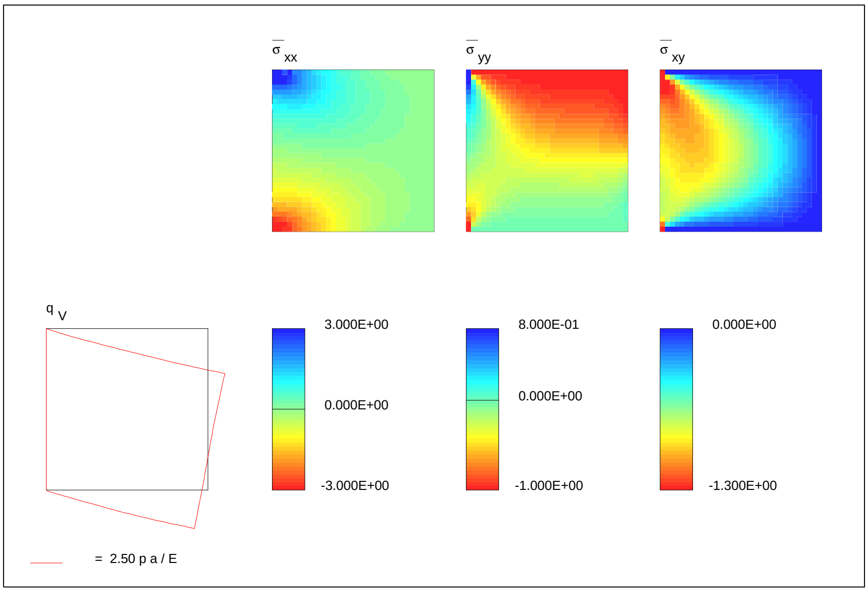
P		A	B	C	COSMOS
A	$\overline{\sigma}_{xx}$	2.402	3.463	4.726	7.623
	$\overline{\sigma}_{yy}$	-1.000	-1.000	-1.000	2.287
	$\overline{\sigma}_{xy}$	0.000	0.000	0.000	-1.656
B	$\overline{\sigma}_{xx}$	-2.708	-3.295	0.029	-5.281
	$\overline{\sigma}_{yy}$	0.000	0.000	0.000	-1.584
	$\overline{\sigma}_{xy}$	0.000	0.000	0.000	-0.814
C	$\overline{\sigma}_{xx}$	0.026	0.029	0.031	0.031
	$\overline{\sigma}_{yy}$	-0.473	-0.472	-0.473	-0.473
	$\overline{\sigma}_{xy}$	-0.670	-0.716	-0.736	-0.747

Square cantilever

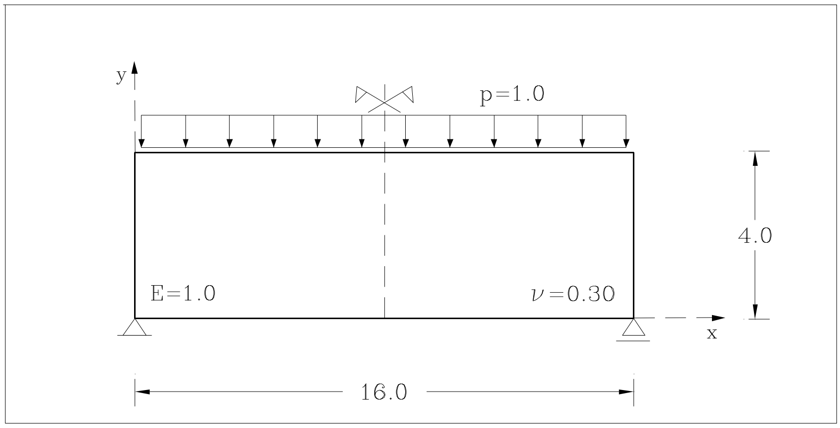
P		A	B	C	COSMOS
A	$\overline{\sigma}_{xx}$	8.387	10.588	13.180	7.623
	$\overline{\sigma}_{yy}$	2.516	3.176	3.954	2.287
	$\overline{\sigma}_{xy}$	-2.527	-3.009	-3.571	-1.656
B	$\overline{\sigma}_{xx}$	-5.934	-6.913	-8.088	-5.281
	$\overline{\sigma}_{yy}$	-1.780	-2.074	-2.426	-1.584
	$\overline{\sigma}_{xy}$	-1.357	-1.524	-1.753	-0.814
C	$\overline{\sigma}_{xx}$	0.027	0.029	0.031	0.031
	$\overline{\sigma}_{yy}$	-0.473	-0.472	-0.473	-0.473
	$\overline{\sigma}_{xy}$	-0.670	-0.716	-0.736	-0.747

Square cantilever

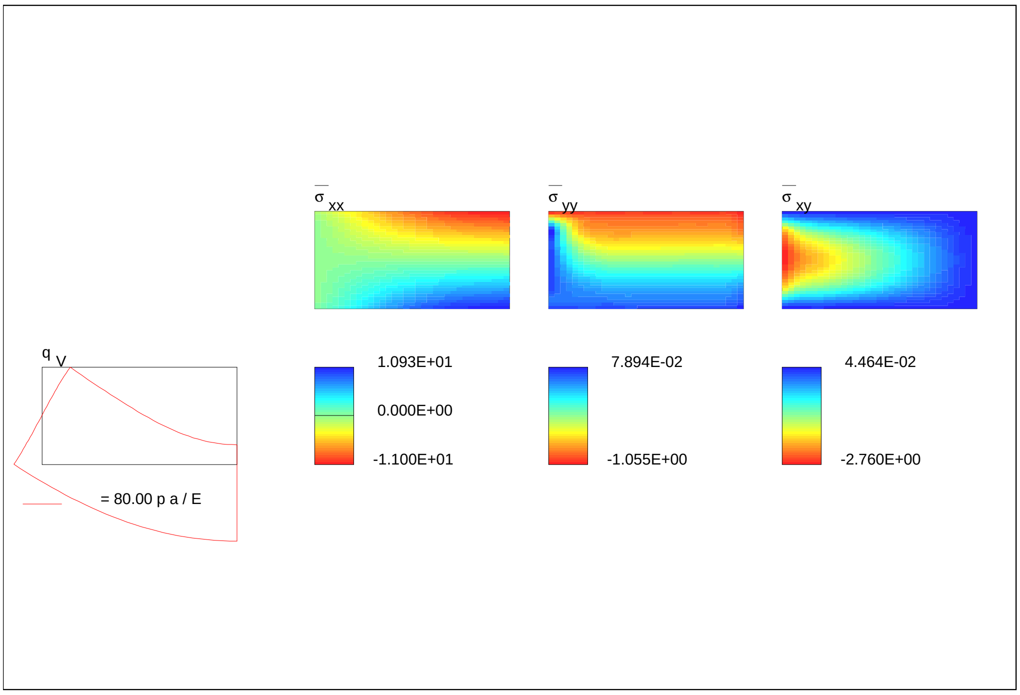


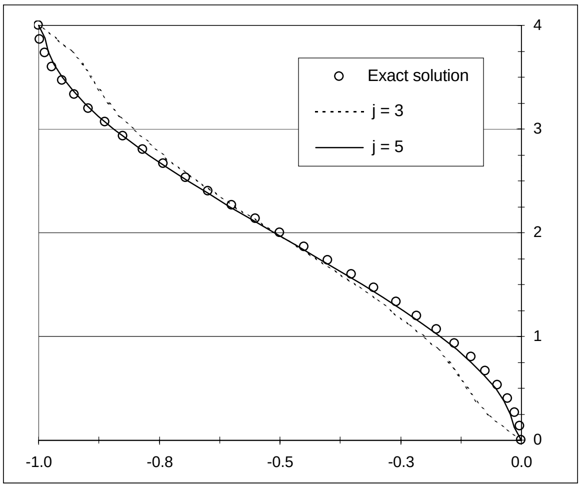


Numerical Applications

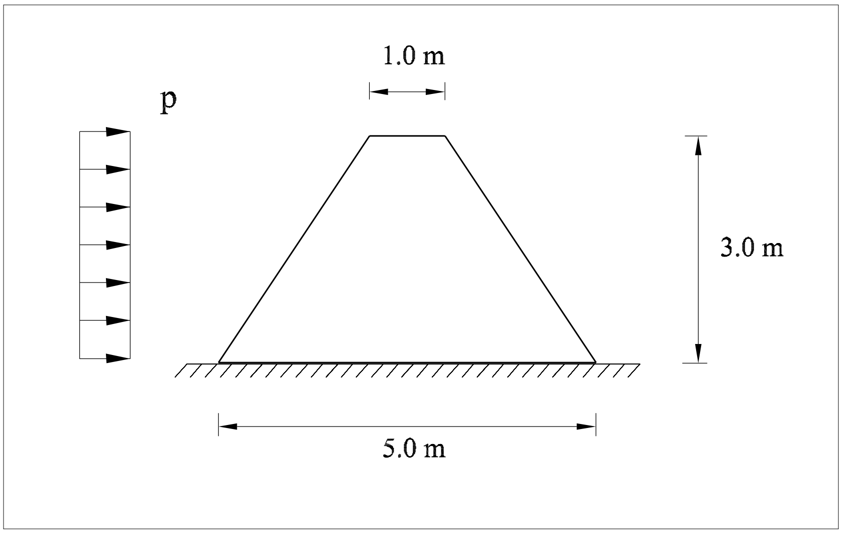


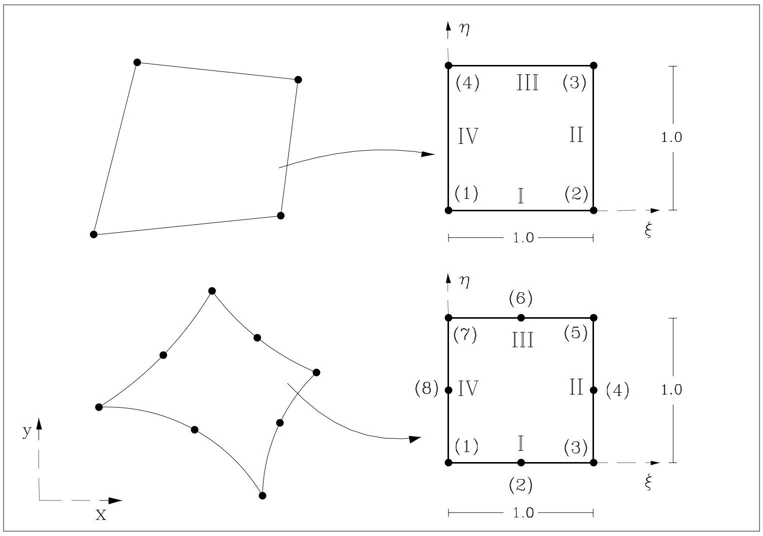
Numerical Applications



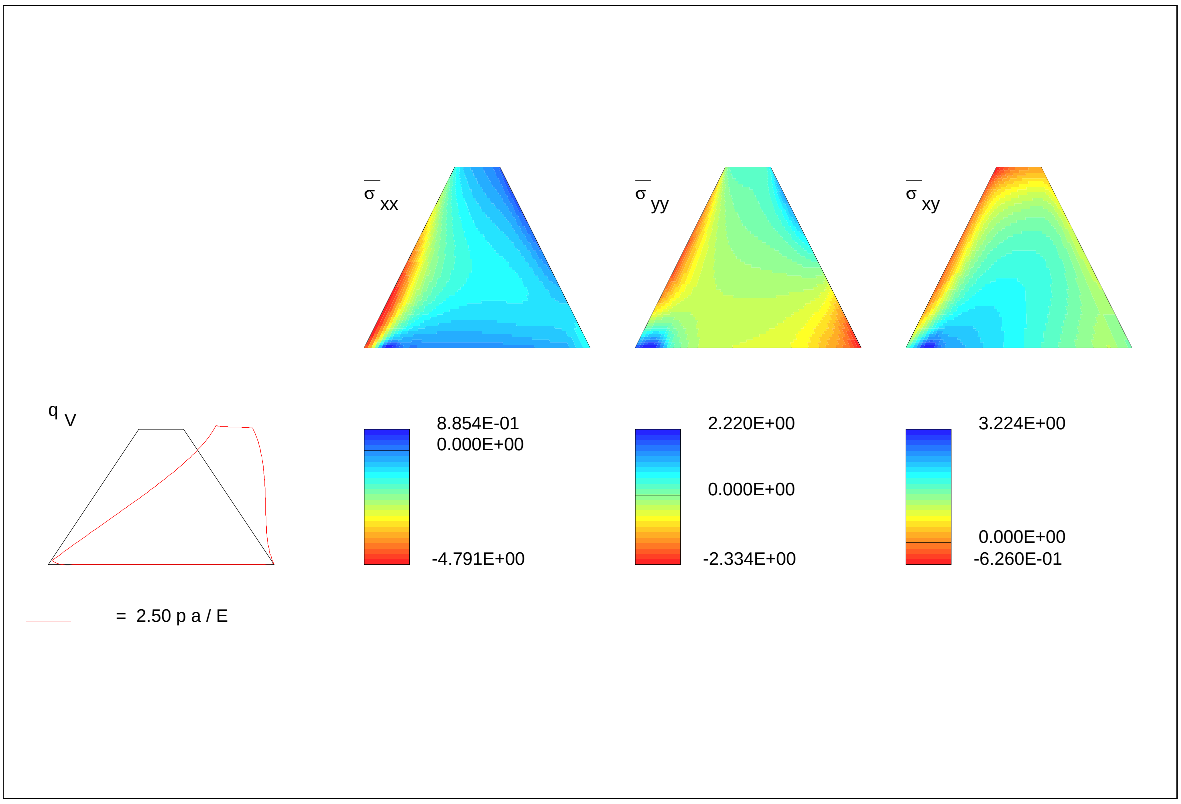


Non-rectangular elements

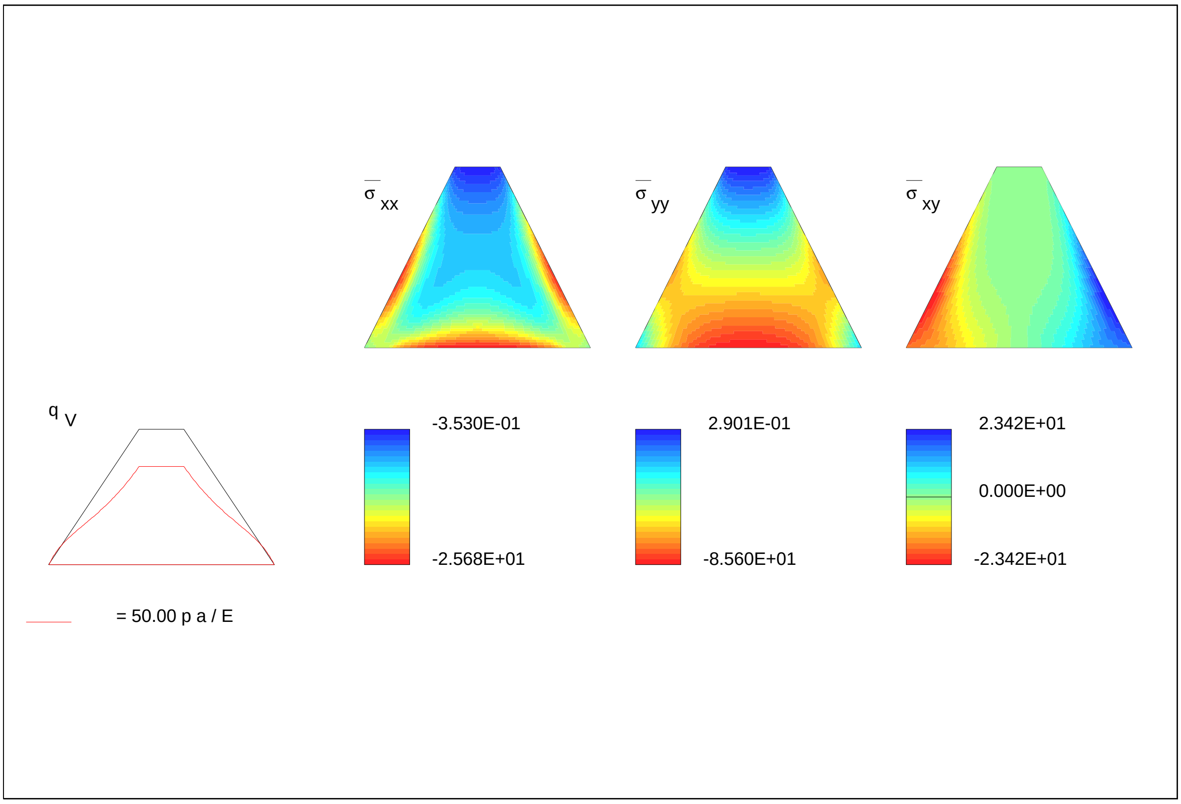




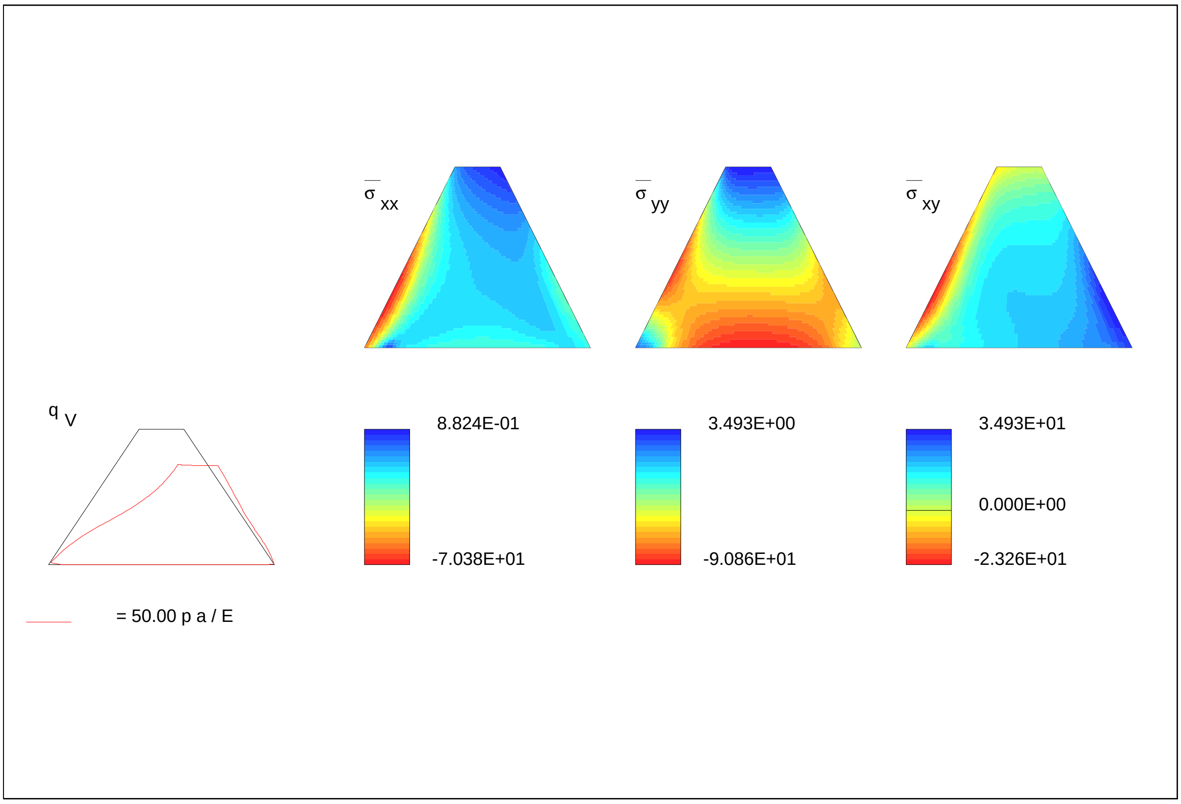
Non-rectangular elements



Non-rectangular elements



Non-rectangular elements



Further developments

- *General domains (domain decompositon)*
- *Plate bending problems*
- *3D problems*
- *Preconditioning - Solution of the governing system*
- *Material behaviour*