Adaptive Wavelet Collocation for Elasticity

Luís Manuel Castro

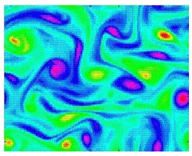
Silvia Bertoluzza

Instituto Superior Técnico, Lisbon
Istituto di Analisi Numerica del CNR, Pavia

luis@civil.ist.utl.pt
http://www.civil.ist.utl.pt/~luis

wavelet@dragon.ian.pv.cnr.it
http://dragon.ian.pv.cnr.it/~aivlis

Wavelets in Numerical Analysis and Simulation



Funchal, 11-12 March 2002

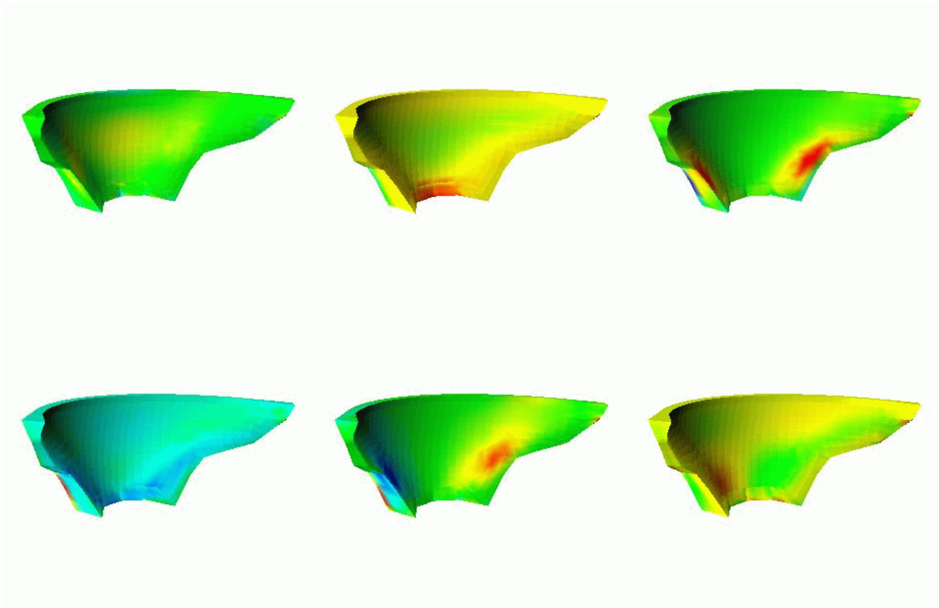
DECivil Núcleo de Análise de Estruturas - ICIST

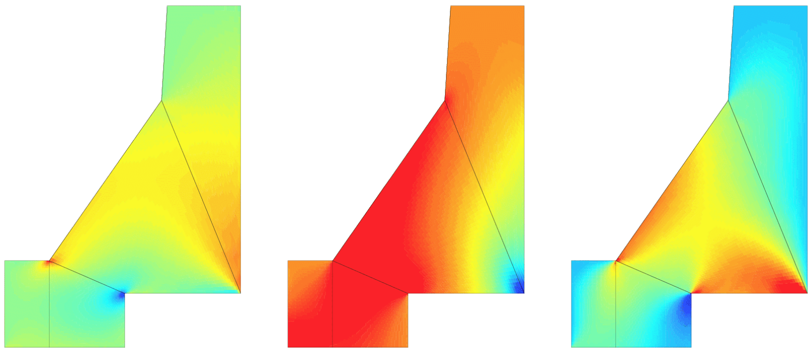
Outline

1. **Motivation**
2. **Adaptive collocation techniques**
3. **Plane elasticity problems**
4. **Numerical applications**
5. **Reissner-Mindlin plate bending problems**
6. **Numerical applications**
7. **Conclusions and further developments**

Motivation

Numerical Simulation of structural engineering problems





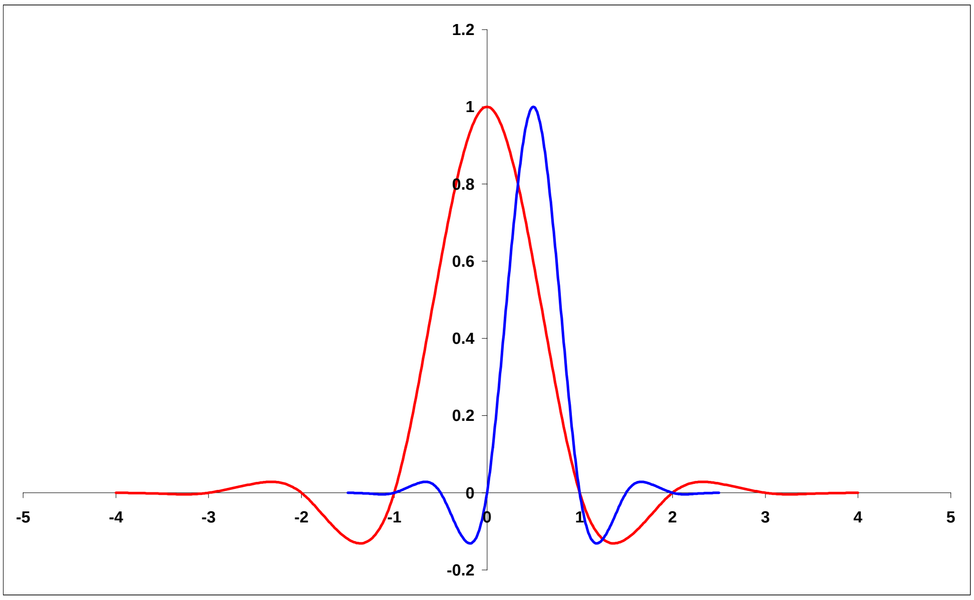
Adaptive Collocation techniques

- Bertoluzza, S., "An Adaptive Collocation Method based on Interpolating Wavelets", in *Multiscale Wavelet Methods for Partial Differential Equations*, edited by Dahmen, Kurdila and Oswald, Academic Press, 1997.
- Bertoluzza, S. and Naldi, G., "A wavelet collocation method for the numerical solution of partial differential equations", ACHA, 3, 1996.
- Bertoluzza, S., "Adaptive wavelet collocation method for the solution of Burgers equation", *Transport Theory and Stat. Phys.*, 25, 1996.

Interpolating wavelets

- Deslaurier, G. and Dubuc, S., "Symmetric iterative interpolation processes, *Constructive Approximation*, 5, 1989.

Deslaurier-Dubuc interpolating functions



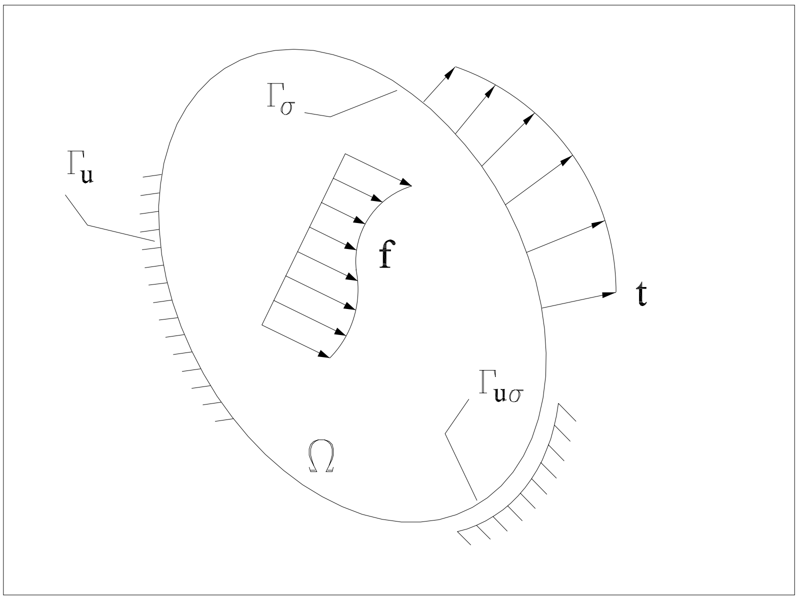
$$\theta_N(x) = \int \phi_L(y)\phi_L(y-x)\,dy$$

$$N = 2L + 1$$

Properties

- $supp\theta = [-N, N]$
- θ **is refinable**
- $\theta(n) = \int \phi_L(y)\phi_L(y-n)\,dy = \delta_{n0}$
- **Polynomials up to order** N **can be represented as a linear combination of the integer translates of** θ**.**

Classical Theory of Elasticity



$$\mathbf{f} = \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} \quad \mathbf{t} = \begin{Bmatrix} t_{x\gamma} \\ t_{y\gamma} \end{Bmatrix}$$

Displacements $\mathbf{u} = \begin{Bmatrix} u_x(x,y) \\ u_y(x,y) \end{Bmatrix}$

Strains $\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx}(x,y) \\ \varepsilon_{yy}(x,y) \\ \varepsilon_{xy}(x,y) \end{Bmatrix}$

Stresses $\boldsymbol{\sigma} = \begin{Bmatrix} \sigma_{xx}(x,y) \\ \sigma_{yy}(x,y) \\ \sigma_{xy}(x,y) \end{Bmatrix}$

Classical Theory of Elasticity

Definition of the stress field

$$
\begin{aligned}
\sigma_{xx} &= e_1 \, \frac{\partial \, u_x}{\partial \, x} + e_2 \, \frac{\partial \, u_y}{\partial \, y} \\
\sigma_{yy} &= e_2 \, \frac{\partial \, u_x}{\partial \, x} + e_1 \, \frac{\partial \, u_y}{\partial \, y} \\
\sigma_{xy} &= e_3 \, \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right)
\end{aligned}
$$

$$
\begin{aligned}
e_1 &= \frac{E}{1 - \nu^2} \\
e_2 &= \frac{\nu \, E}{1 - \nu^2} \\
e_3 &= \frac{E}{2 \, (1 + \nu)}
\end{aligned}
$$

Classical Theory of Elasticity

Problem 1

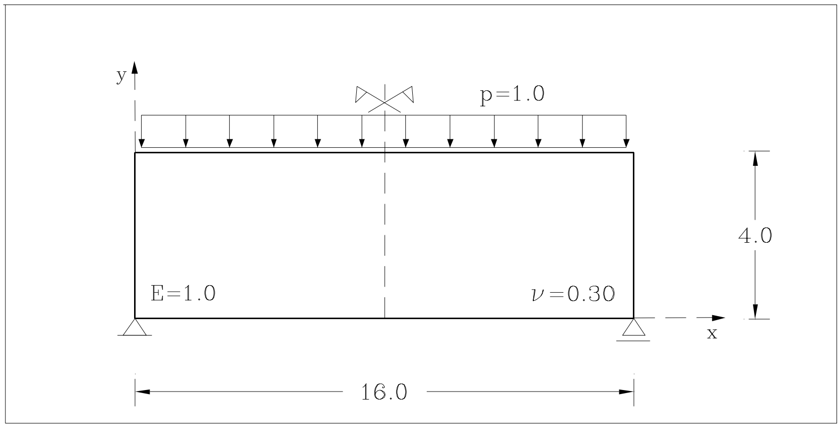
Find $\mathbf{u} = [u_x, u_y]^T$ such that

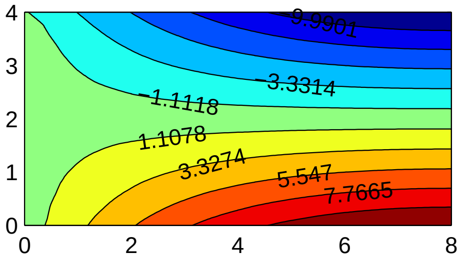
$$
\begin{aligned}
\mathcal{A}\,\mathbf{u} &= \mathbf{f} \qquad (\Omega) \\
\mathbf{u} &= \mathbf{g} \qquad (\Gamma_u) \\
\mathcal{B}\,\mathbf{u} &= \mathbf{t} \qquad (\Gamma_\sigma)
\end{aligned}
$$

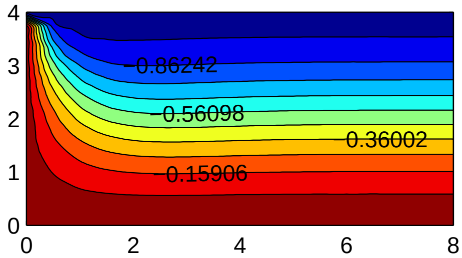
$$
\mathcal{A} = \begin{bmatrix} e_1 \dfrac{\partial^2}{\partial x^2} + e_3 \dfrac{\partial^2}{\partial y^2} & (e_2 + e_3) \dfrac{\partial^2}{\partial x \, \partial y} \\ (e_2 + e_3) \dfrac{\partial^2}{\partial x \, \partial y} & e_2 \dfrac{\partial^2}{\partial x^2} + e_1 \dfrac{\partial^2}{\partial y^2} \end{bmatrix}
$$

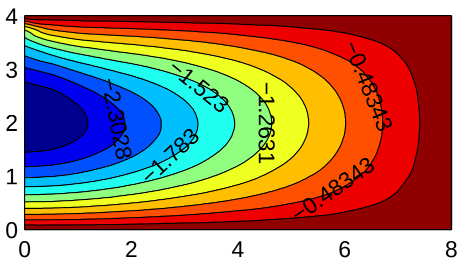
$$
\mathcal{B} = \begin{bmatrix} e_1 \, n_x \dfrac{\partial}{\partial x} + e_3 \, n_y \dfrac{\partial}{\partial y} & e_2 \, n_x \dfrac{\partial}{\partial y} + e_3 \, n_y \dfrac{\partial}{\partial x} \\ e_3 \, n_x \dfrac{\partial}{\partial y} + e_2 \, n_y \dfrac{\partial}{\partial x} & e_3 \, n_x \dfrac{\partial}{\partial x} + e_1 \, n_y \dfrac{\partial}{\partial y} \end{bmatrix}
$$

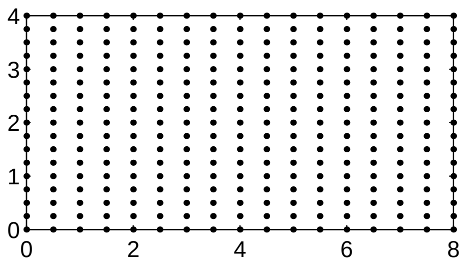
Numerical Applications



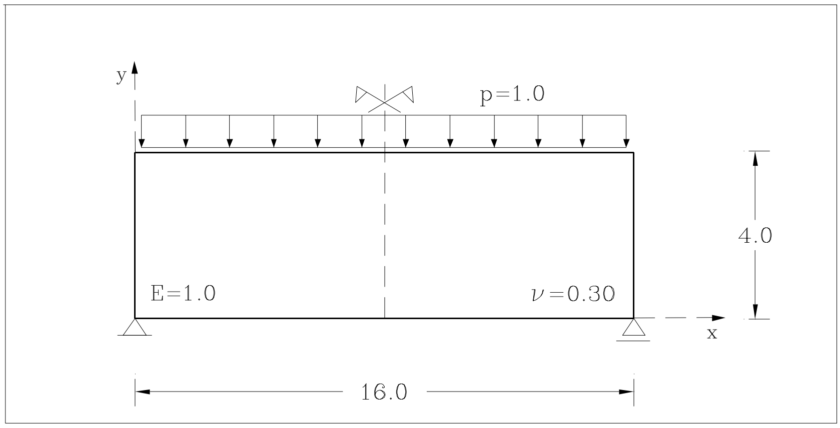


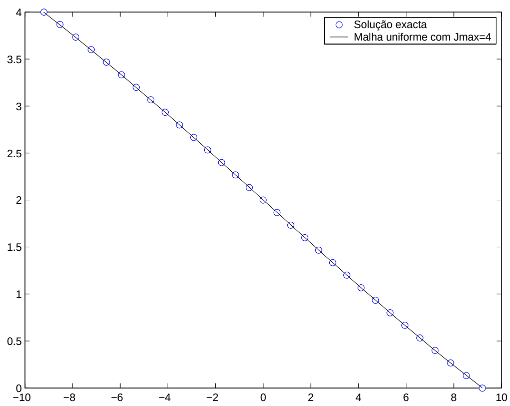


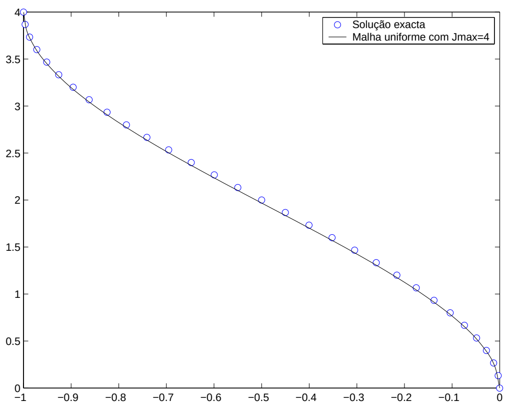


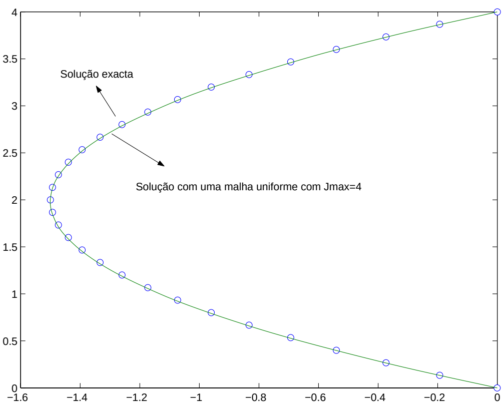


Numerical Applications

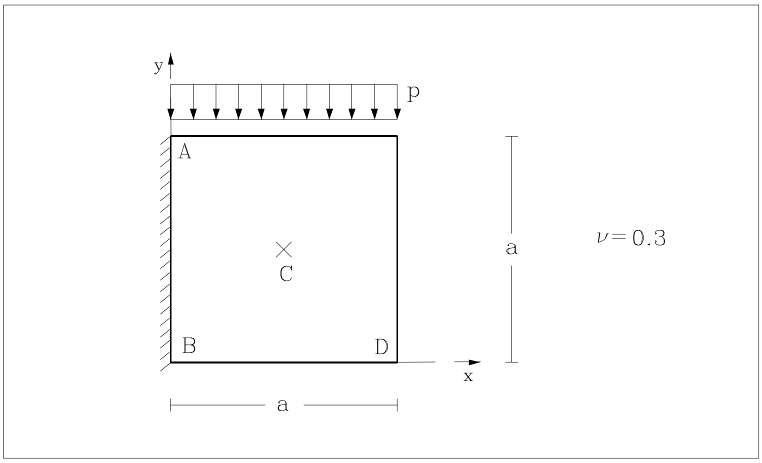






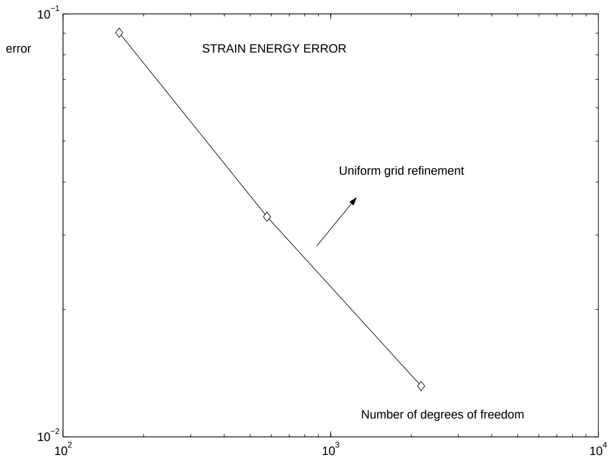


Numerical Applications

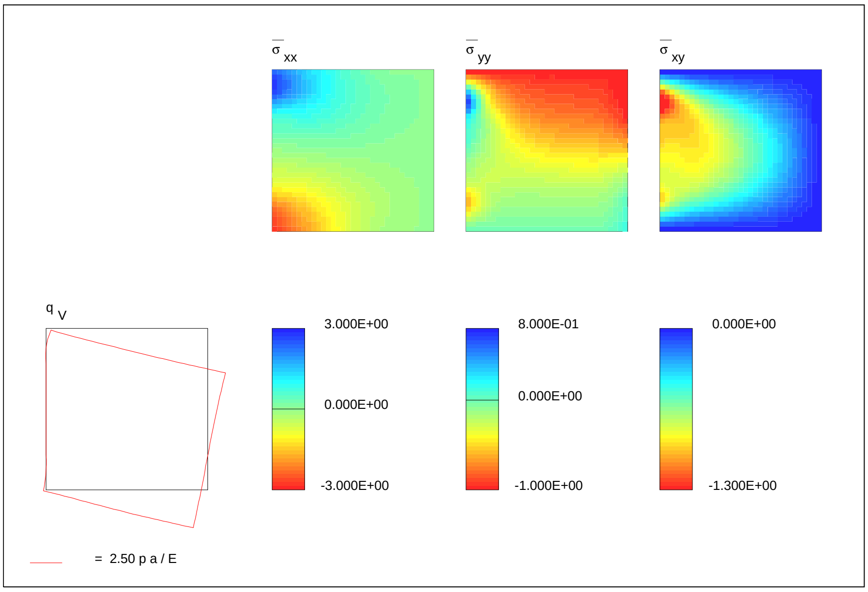


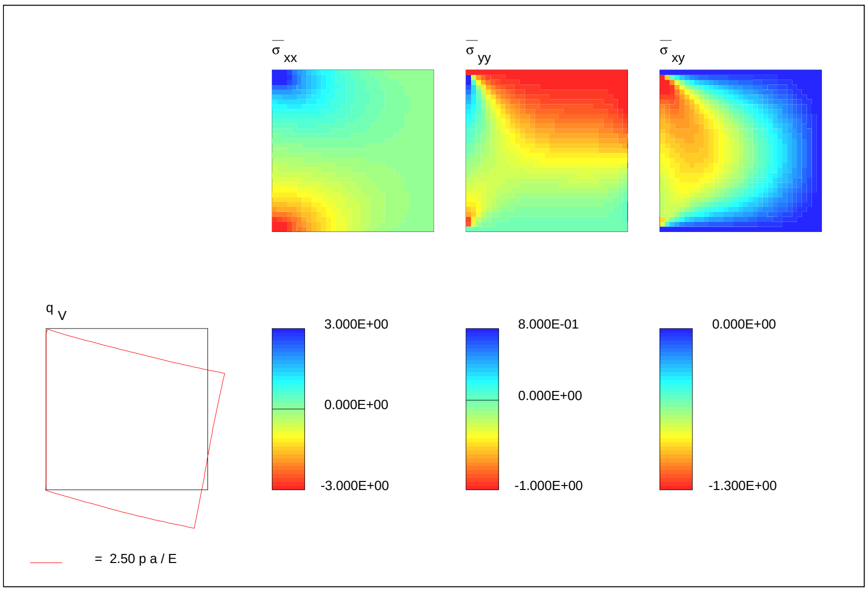
Discretisation	N	j_0	j_{max}	n_{grid}	n_{dof}	T_{CPU}
A	4	3	3	81	162	0.04
B	4	3	4	289	578	0.72
C	4	3	5	1089	2178	52.63

Table 1: Discretisations involved in the analysis of the square cantilever

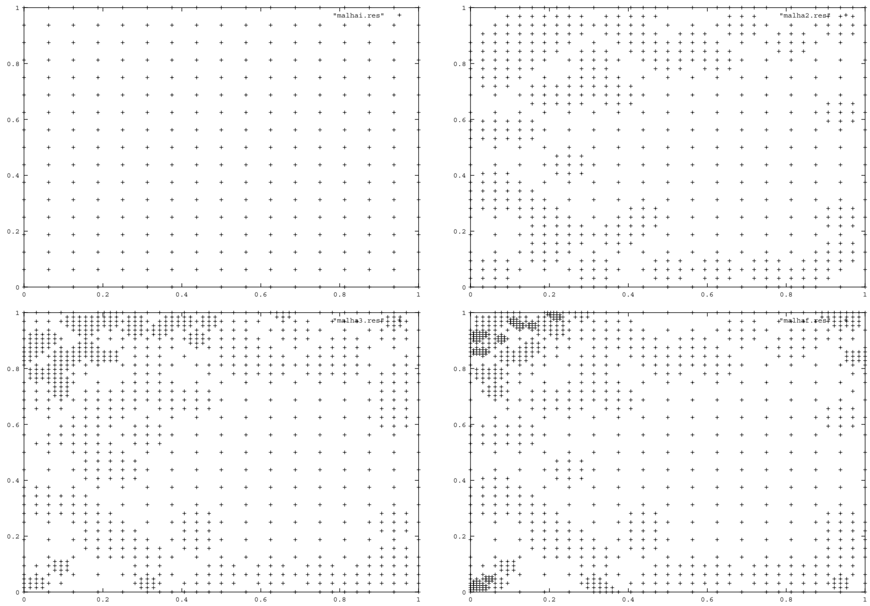


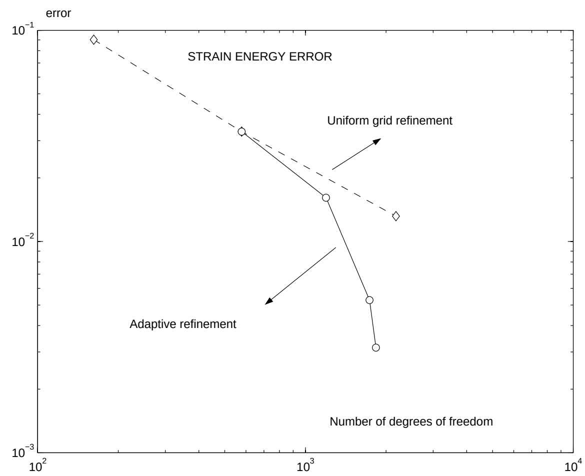
Square cantilever





Square cantilever

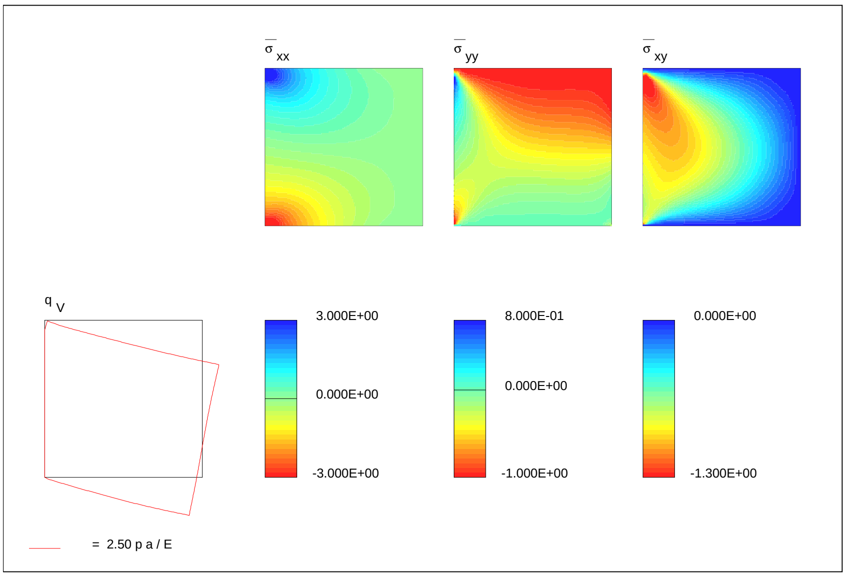




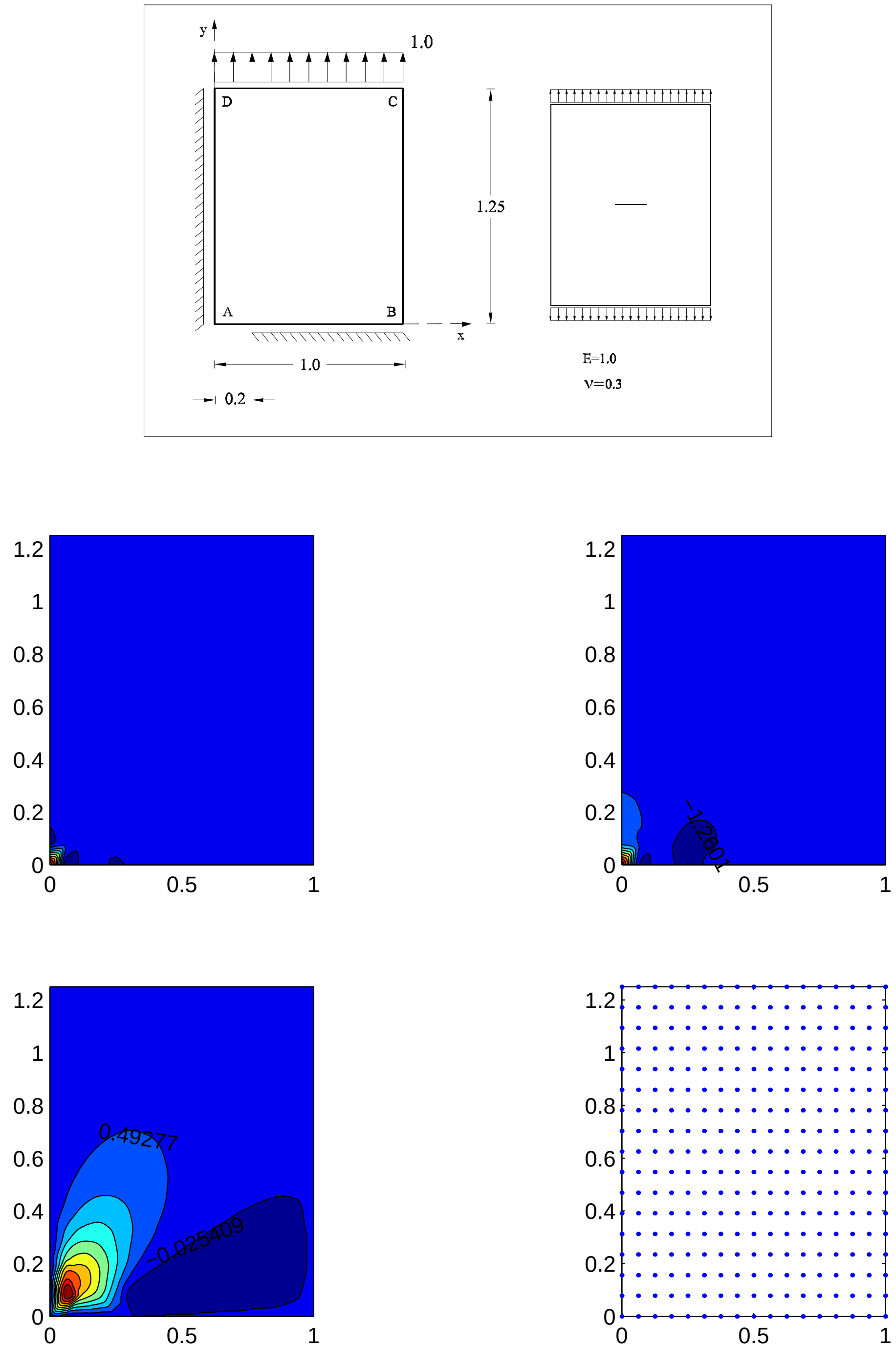
Square cantilever

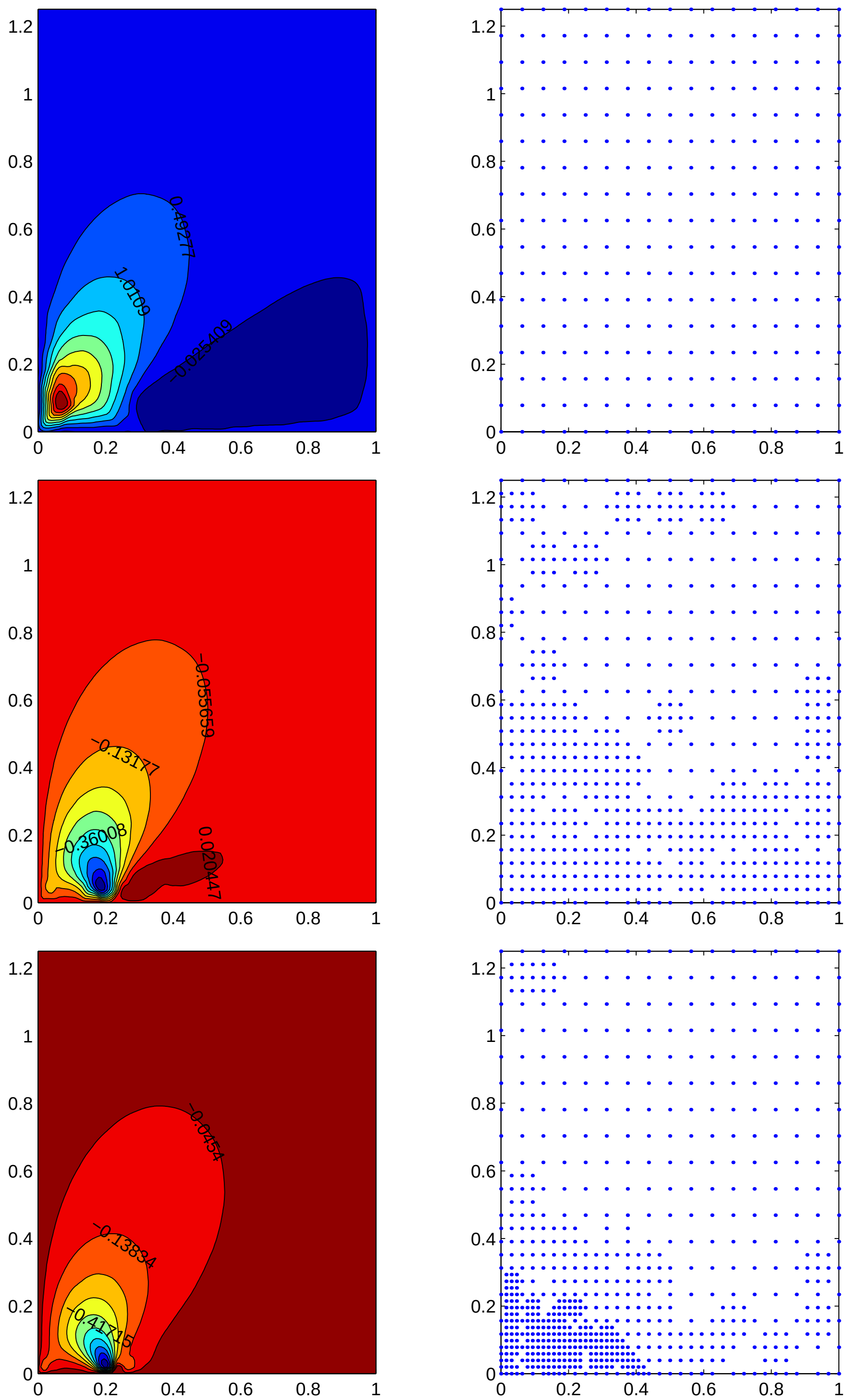
Mesh	j_{max}	n_{grid}	n_{dof}
initial	4	289	578
2	5	597	1194
3	6	868	1736
final	7	917	1834

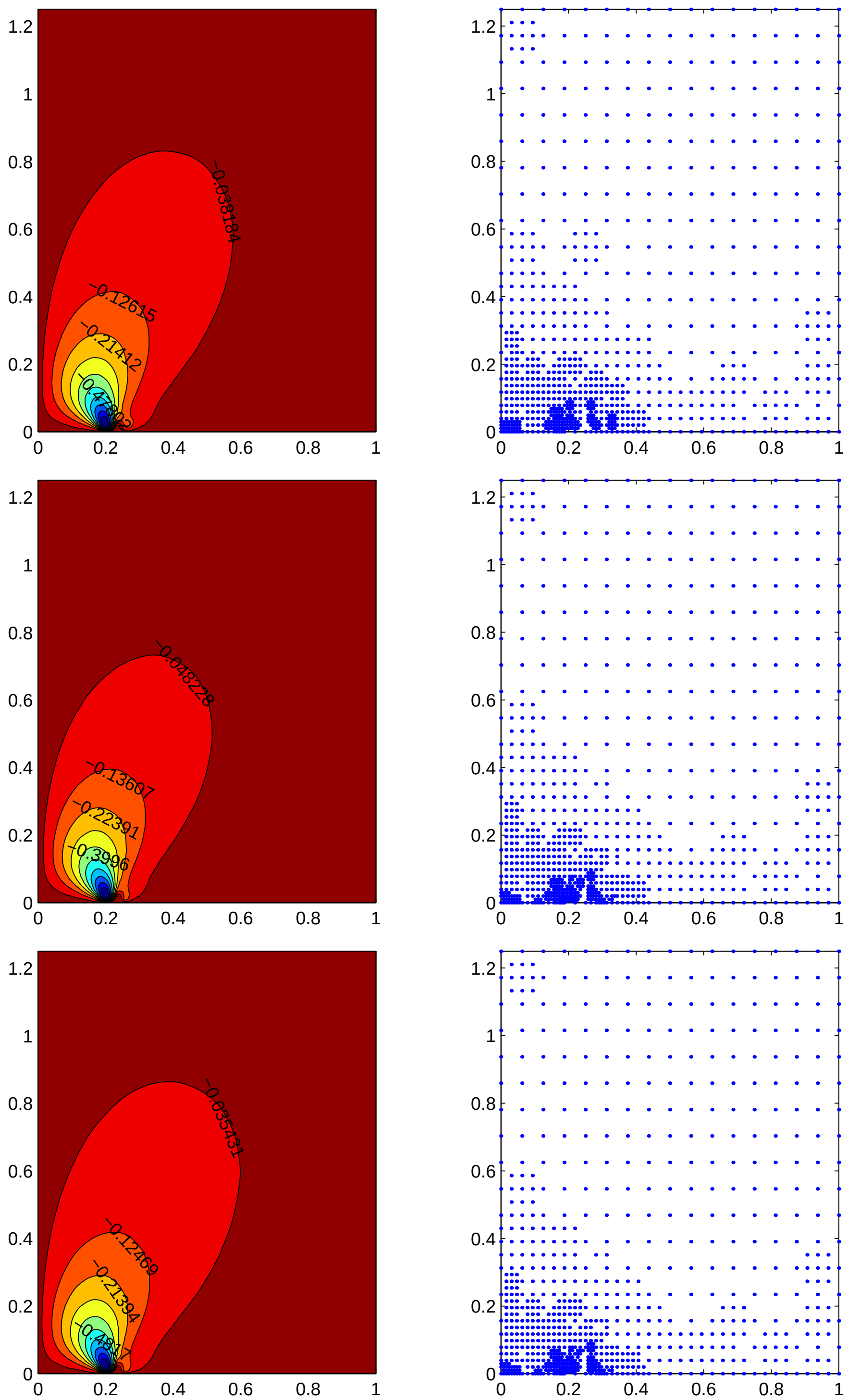
Table 2: Adaptive non-uniform grids used in the solution of the square plate



Numerical Applications

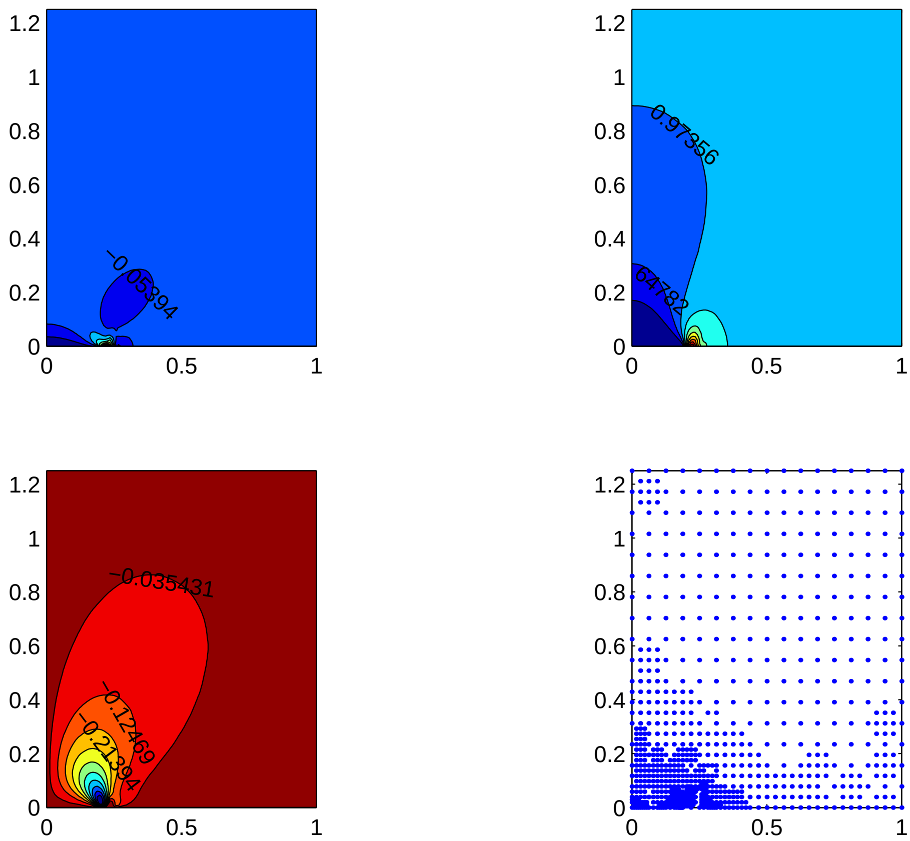



0.49277
1.0109
-0.035409
-0.055659
-0.13177
-0.36008
0.020447
-0.0454
-0.13834
-0.41715
0
0.2
0.4
0.6
0.8
1
1.2

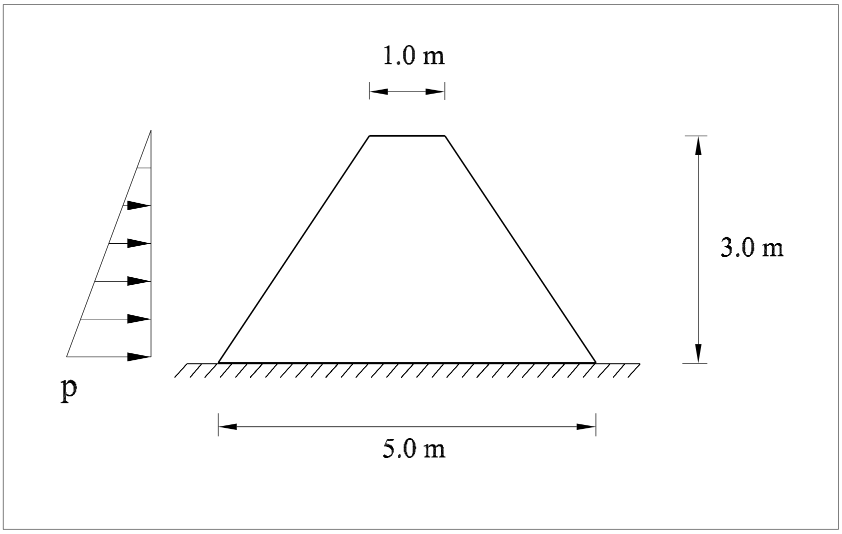

-0.038184
-0.12615
-0.21412
-0.048228
-0.13607
-0.22391
-0.3996
-0.035431
-0.12469
-0.21394

Square plate with a central crack

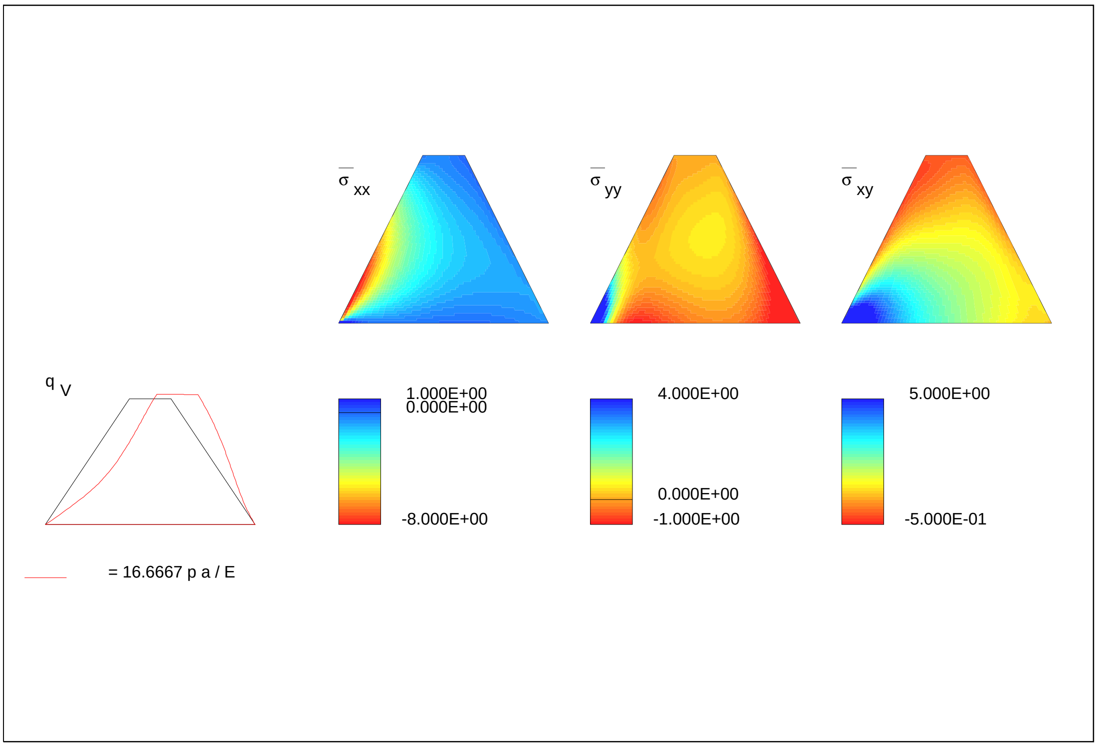
Mesh	j_{max}	n_{grid}	n_{dof}	
1	4	289	578	8.1355
2	5	625	1250	0.6666
3	6	670	1340	0.6591
4	7	812	1624	0.6530
5	8	823	1646	0.6529
6	9	1082	2164	0.6529

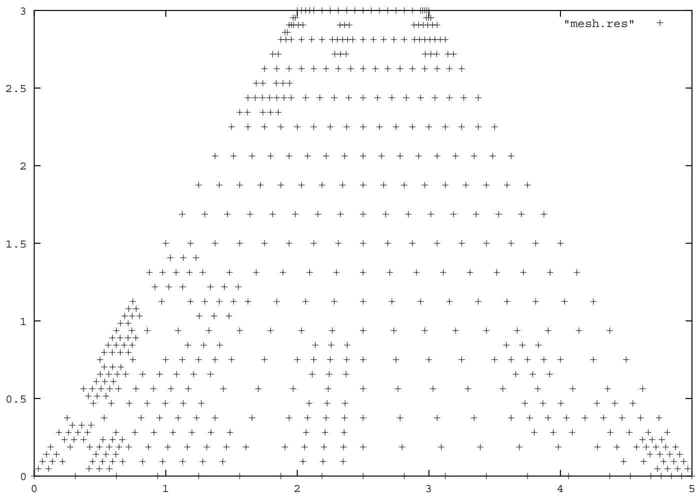


Non-rectangular elements

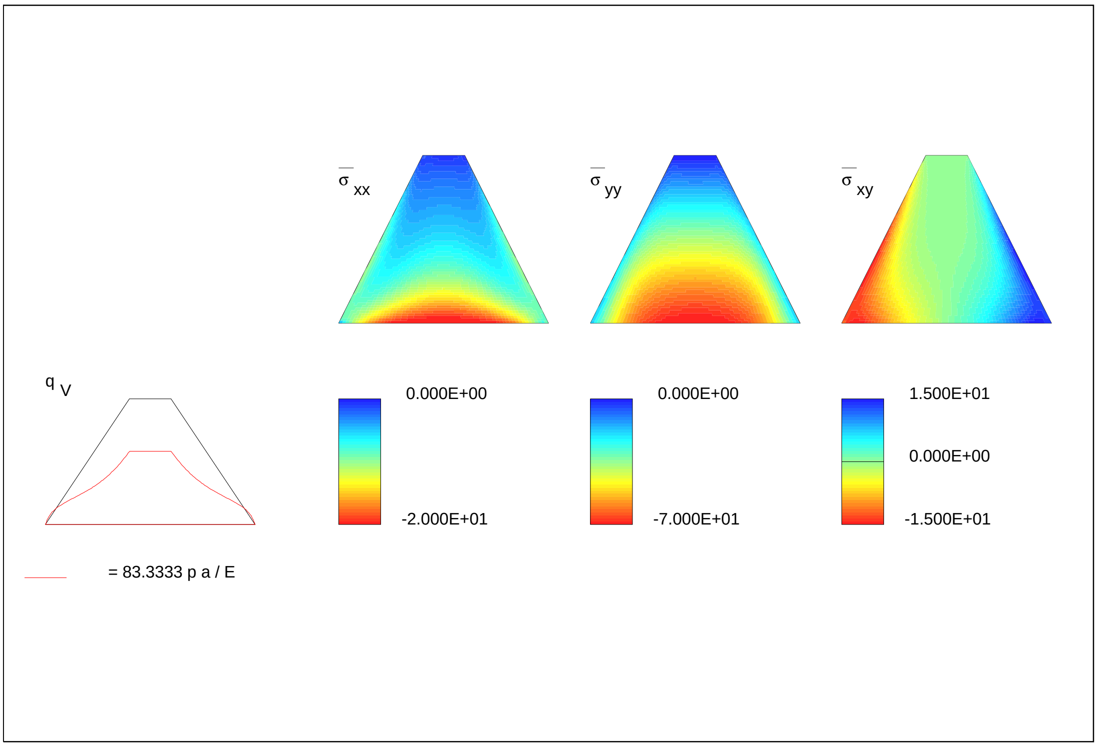


Gravity dam

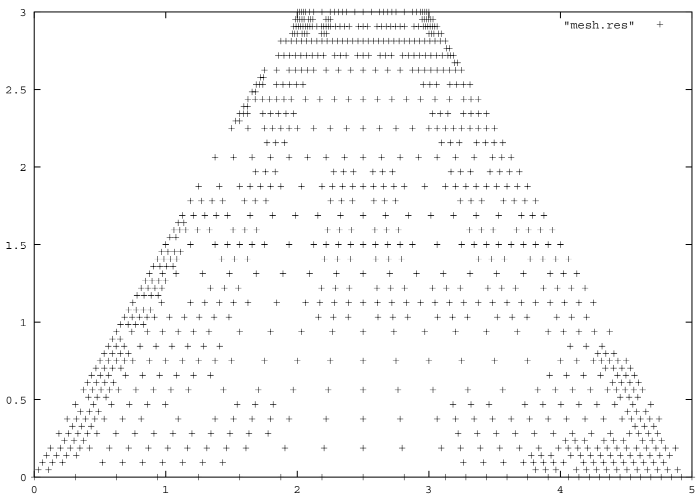

σ xx
σ yy
σ xy
q V
1.000E+00
0.000E+00
-8.000E+00
4.000E+00
0.000E+00
-1.000E+00
5.000E+00
-5.000E-01
= 16.6667 p a / E


"mesh.res"
0
0.5
1
1.5
2
2.5
3
0
1
2
3
4
5

Gravity dam

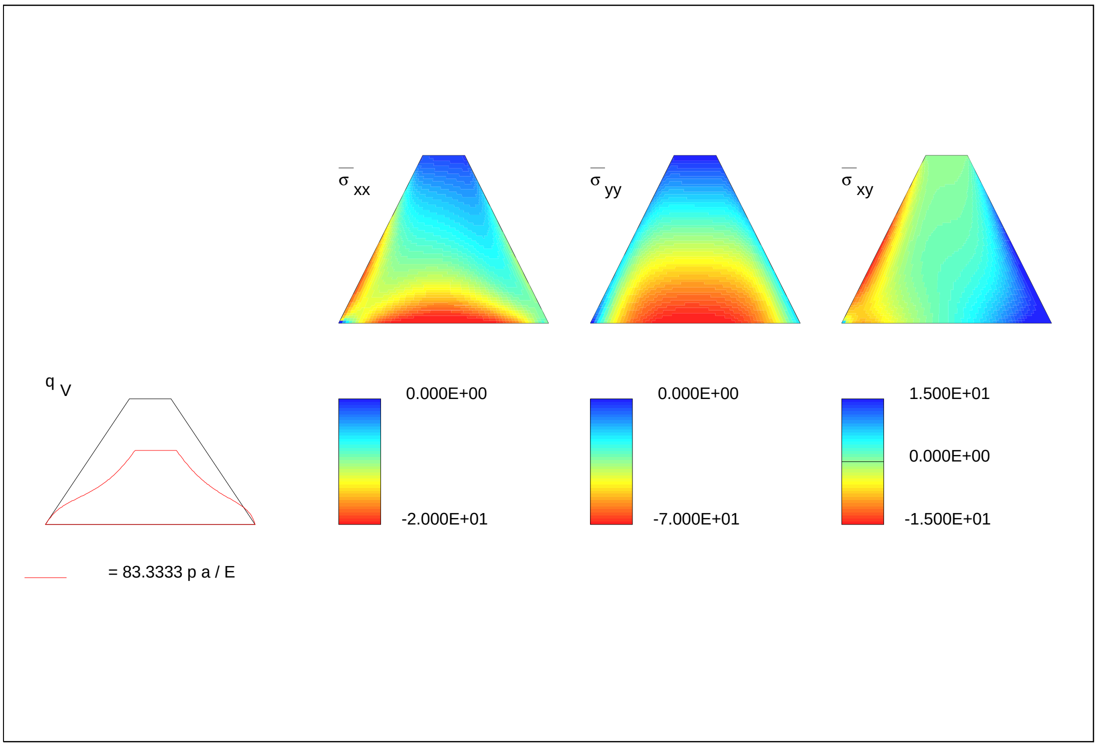


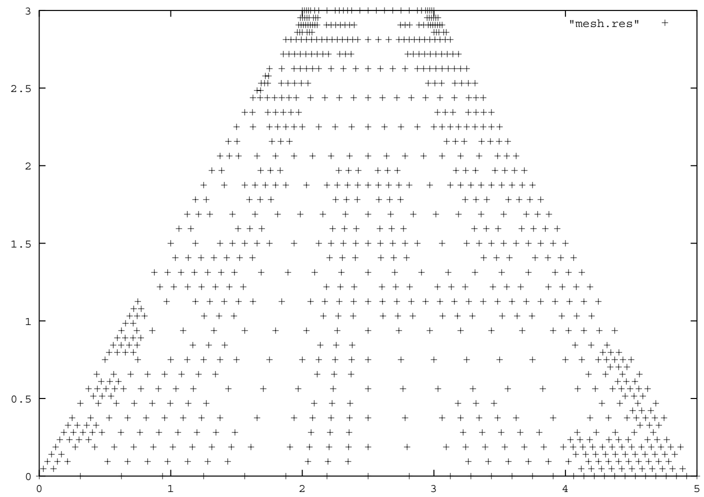
$\bar{\sigma}_{xx}$
$\bar{\sigma}_{yy}$
$\bar{\sigma}_{xy}$
q_V
0.000E+00
-2.000E+01
0.000E+00
-7.000E+01
1.500E+01
0.000E+00
-1.500E+01
= 83.3333 p a / E



"mesh.res"

Gravity dam


σ xx
σ yy
σ xy
q V
0.000E+00
-2.000E+01
0.000E+00
-7.000E+01
1.500E+01
0.000E+00
-1.500E+01
= 83.3333 p a / E


"mesh.res"
3
2.5
2
1.5
1
0.5
0
0
1
2
3
4
5

Reissner-Mindlin plate bending theory

$$\mathbf{f} = \begin{Bmatrix} 0 \\ 0 \\ q \end{Bmatrix} \quad \mathbf{t} = \begin{Bmatrix} m_{x\gamma} \\ m_{y\gamma} \\ q_\gamma \end{Bmatrix}$$

$$\textbf{\textit{Displacements}} \quad \mathbf{u} = \begin{Bmatrix} \theta_x(x,y) \\ \theta_y(x,y) \\ w(x,y) \end{Bmatrix}$$

$$\textbf{\textit{Strains}} \quad \boldsymbol{\varepsilon} = \begin{Bmatrix} \chi_{xx}(x,y) \\ \chi_{yy}(x,y) \\ \chi_{xy}(x,y) \\ \gamma_x(x,y) \\ \gamma_y(x,y) \end{Bmatrix}$$

$$\textbf{\textit{Stress resultants}} \quad \boldsymbol{\sigma} = \begin{Bmatrix} m_{xx}(x,y) \\ m_{yy}(x,y) \\ m_{xy}(x,y) \\ v_x(x,y) \\ v_y(x,y) \end{Bmatrix}$$

Reissner-Mindlin plate bending theory

Definition of the stress resultant fields

$$
\begin{aligned}
m_{xx} &= D_f \left[\frac{\partial \theta_x}{\partial x} + \nu \frac{\partial \theta_y}{\partial y} \right] \\
m_{yy} &= D_f \left[\nu \frac{\partial \theta_x}{\partial x} + \frac{\partial \theta_y}{\partial y} \right] \\
m_{xy} &= D_1 \left[\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x} \right] \\
v_x &= D_2 \left[\theta_x + \frac{\partial w}{\partial x} \right] \\
v_y &= D_2 \left[\theta_y + \frac{\partial w}{\partial y} \right]
\end{aligned}
$$

$$
\begin{aligned}
D_f &= \frac{E\,h^3}{12(1-\nu^2)} \\
D_1 &= \frac{G\,h^3}{12} \\
D_2 &= \frac{5}{6}\,G\,h
\end{aligned}
$$

Reissner-Mindlin plate bending theory

Problem 2

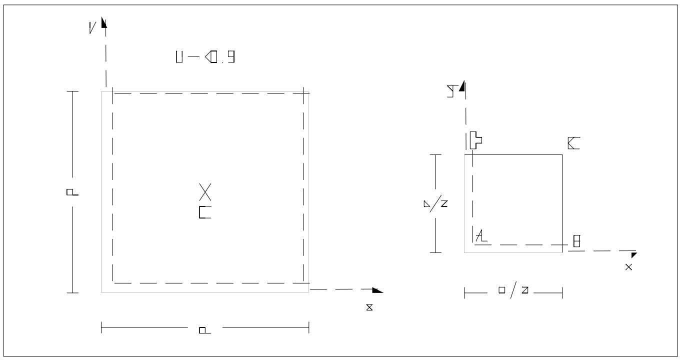
Find $\mathbf{u} = [\theta_x, \theta_y, w]^T$ such that

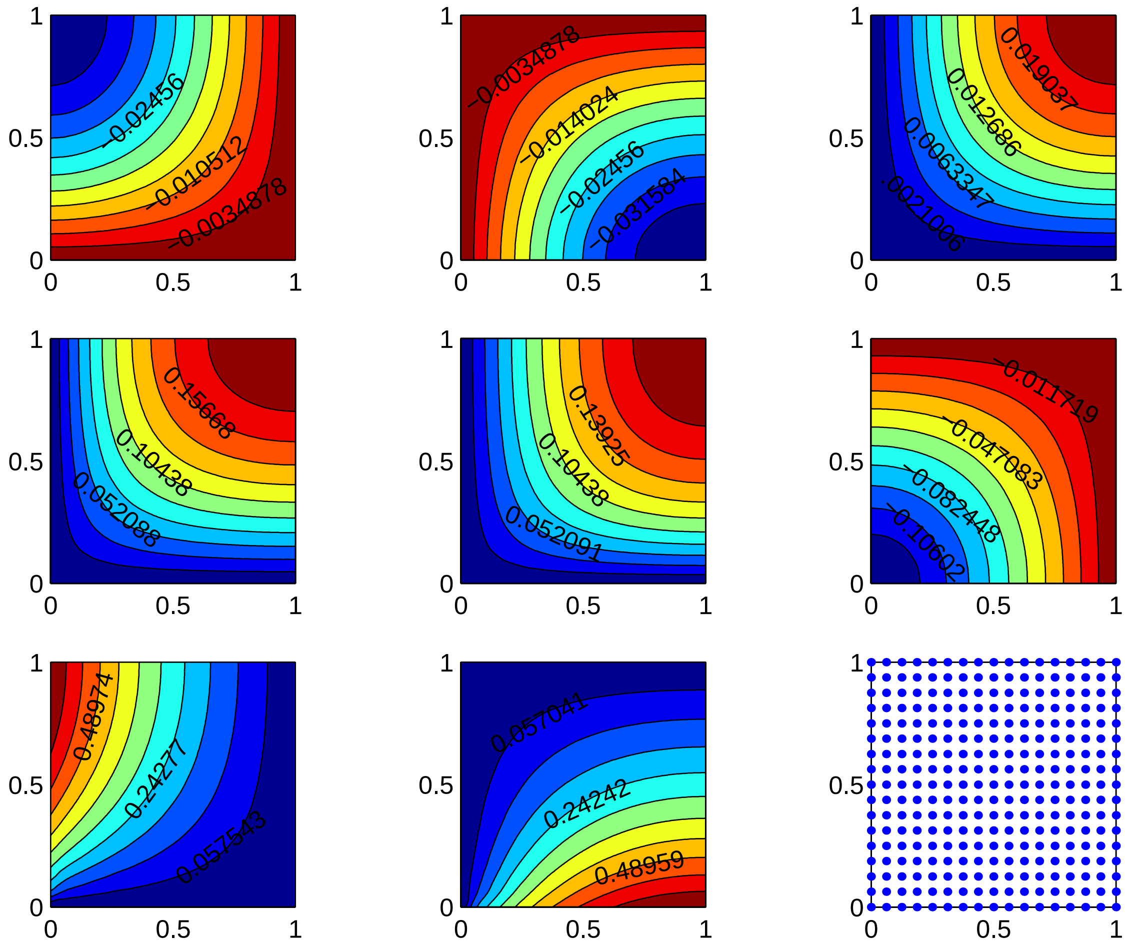
$$
\begin{aligned}
\mathcal{A}\,\mathbf{u} &= \mathbf{f} \qquad (\Omega)\\
\mathbf{u} &= \mathbf{g} \qquad (\Gamma_u)\\
\mathcal{B}\,\mathbf{u} &= \mathbf{t} \qquad (\Gamma_\sigma)
\end{aligned}
$$

$$
\mathcal{A} = \begin{bmatrix}
D_2 \dfrac{\partial}{\partial x} & D_2 \dfrac{\partial}{\partial y} & D_2 \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} \right) \\[2ex]
D_f \dfrac{\partial^2}{\partial x^2} + D_1 \dfrac{\partial^2}{\partial y^2} - D_2 & \nu\, D_f \dfrac{\partial^2}{\partial x\, \partial y} + D_1 \dfrac{\partial^2}{\partial x\, \partial y} & -D_2 \dfrac{\partial}{\partial x} \\[2ex]
\nu\, D_f \dfrac{\partial^2}{\partial x\, \partial y} + D_1 \dfrac{\partial^2}{\partial x\, \partial y} & D_f \dfrac{\partial^2}{\partial y^2} + D_1 \dfrac{\partial^2}{\partial x^2} - D_2 & -D_2 \dfrac{\partial}{\partial y}
\end{bmatrix}
$$

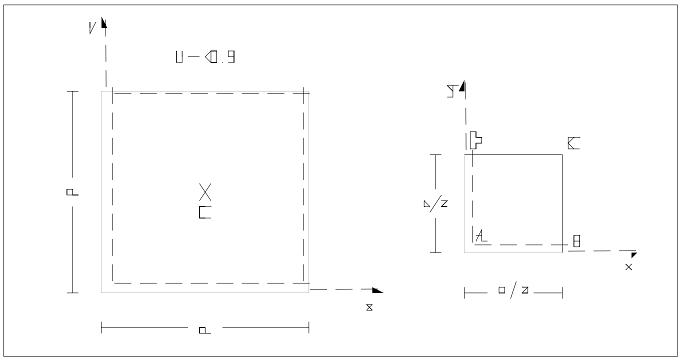
$$
\mathcal{B} = \begin{bmatrix}
D_2\, n_x & D_2\, n_y & D_2 \dfrac{\partial}{\partial x}\, n_x + D_2 \dfrac{\partial}{\partial y}\, n_y \\[2ex]
D_f \dfrac{\partial}{\partial x}\, n_x + D_1 \dfrac{\partial}{\partial y}\, n_y & \nu\, D_f \dfrac{\partial}{\partial y}\, n_x + D_1 \dfrac{\partial}{\partial x}\, n_y & 0 \\[2ex]
\nu\, D_f \dfrac{\partial}{\partial x}\, n_y + D_1 \dfrac{\partial}{\partial y}\, n_x & D_f \dfrac{\partial}{\partial y}\, n_y + D_1 \dfrac{\partial}{\partial x}\, n_x & 0
\end{bmatrix}
$$

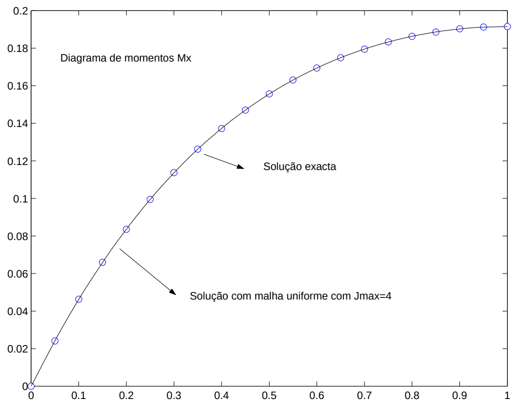
Simply supported plate

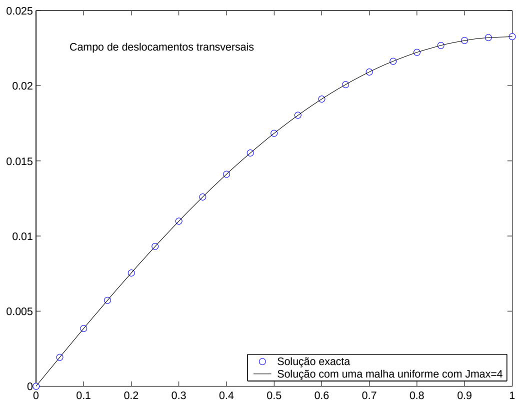


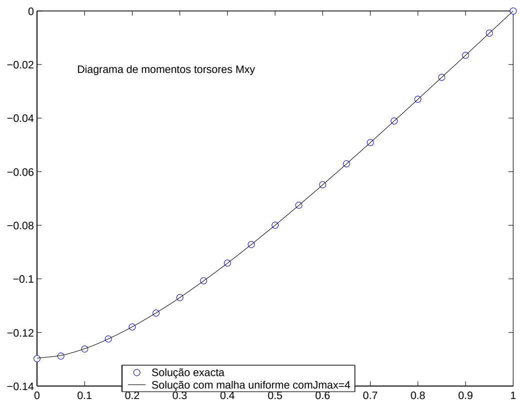


Simply supported rectangular plate

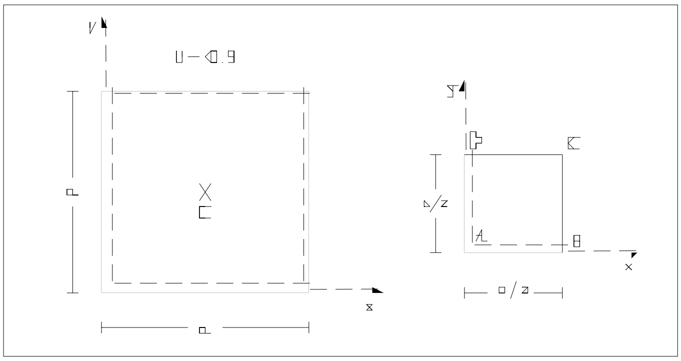


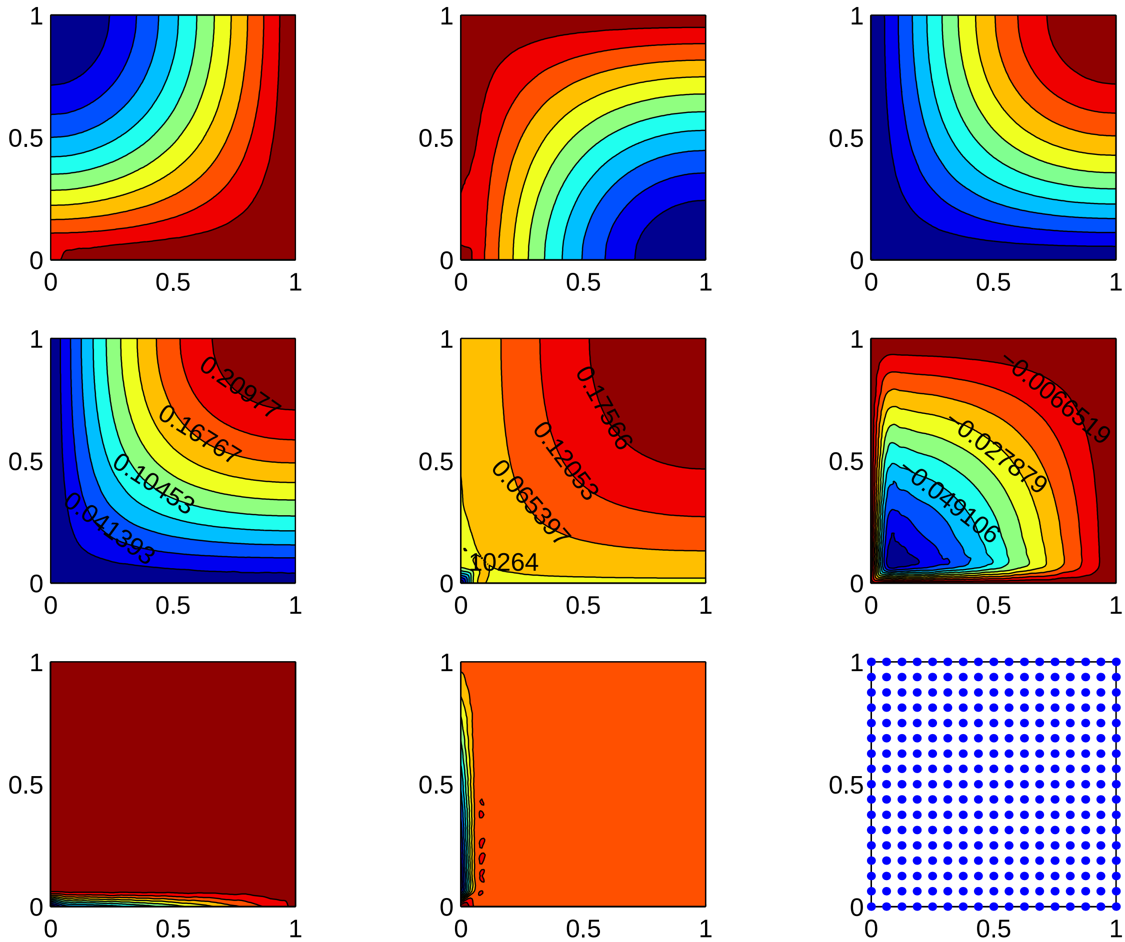




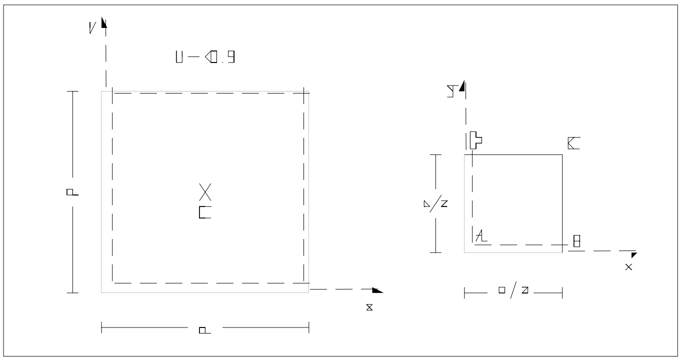


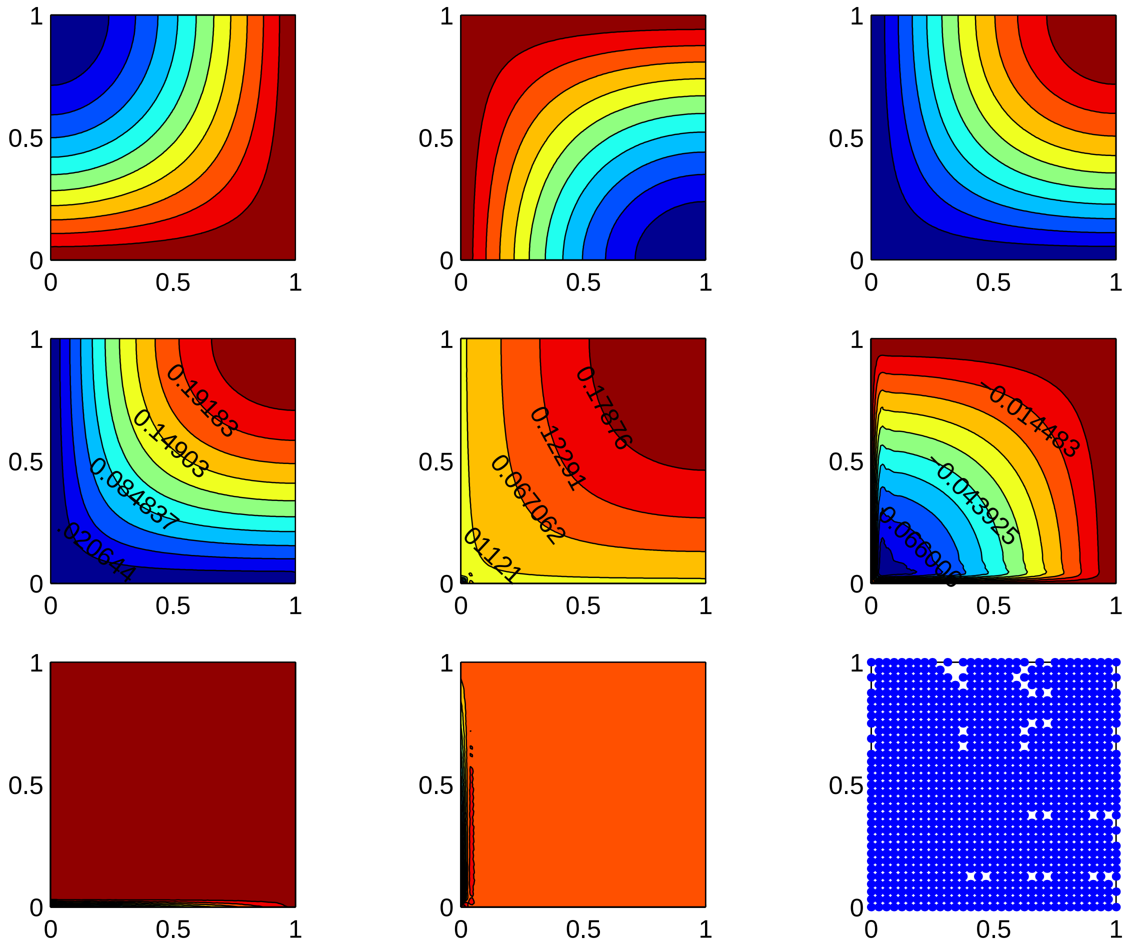
Simply supported rectangular plate



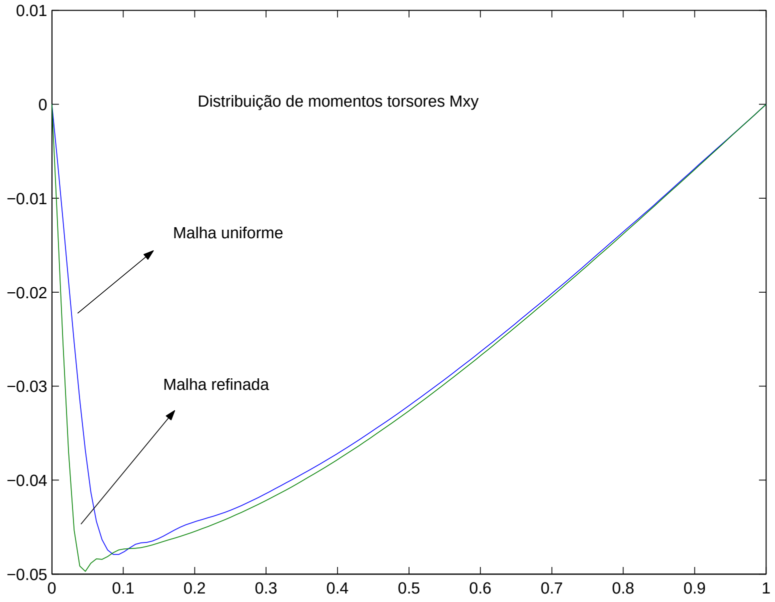


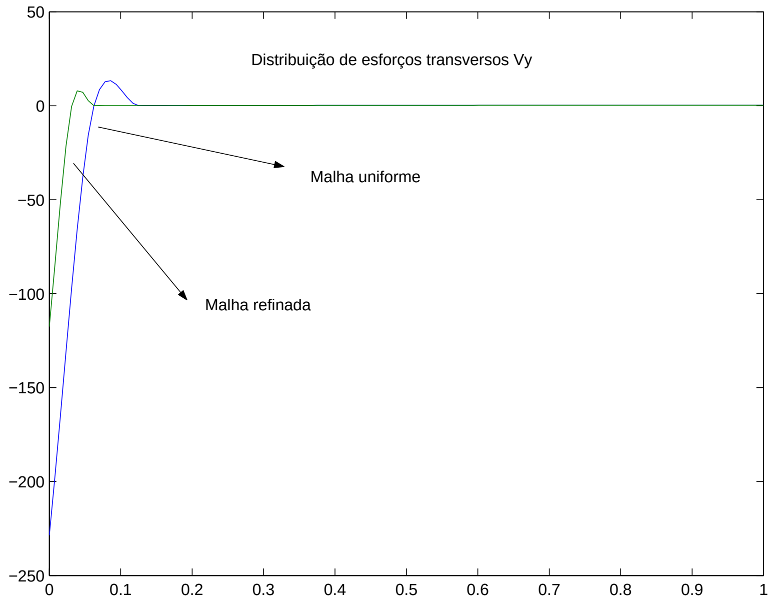
Simply supported rectangular plate





Simply supported rectangular plate





Further developments

- **General domains (domain decompositon)**
- **3D problems**
- **Material behaviour**