Elastoplastic Analysis in Structural Engineering

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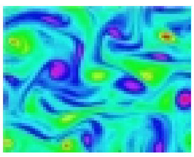
Wavelet Methods in Elastoplasticity

Training and Mobility of Researchers



Aachen, 21-22 February 2000

Outline

1. Classical Theory of Plasticity
2. Elastoplasticity in Structural Engineering
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   - 2D problems
   - 3D problems
3. *Standard* Numerical Techniques
4. Hybrid-Mixed Formulations
5. Approximation of plastic parameters
6. Perturbation Methods
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Classical Theory of Plasticity

Equilibrium conditions

$$\boldsymbol{D}\,\boldsymbol{\sigma} + \boldsymbol{b} = \mathbf{0} \ \text{ in } V$$

$$\boldsymbol{N}\,\boldsymbol{\sigma} = \boldsymbol{t}_\gamma \ \text{ on } \Gamma_\sigma$$

Compatibility conditions

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_e + \boldsymbol{\varepsilon}_p = \mathbf{D}^*\mathbf{u} \ \text{ in } V$$

$$\mathbf{u} = \mathbf{u}_\gamma \ \text{ on } \Gamma_u$$

Elasticity

$$\boldsymbol{\varepsilon}_e = \mathbf{f}\,\boldsymbol{\sigma} + \boldsymbol{\varepsilon}_\theta$$

Classical Theory of Plasticity

Plasticity

$$\phi_* = \phi_*(\sigma, \overline{\varepsilon}_p) - \overline{\sigma}_e$$

$$\phi_* \leq 0$$

$$\phi_* = \sqrt{\boldsymbol{\sigma}^t \mathbf{M}\, \boldsymbol{\sigma}} - \overline{\sigma}_e$$

$$d\phi_* = \left(\frac{\partial \phi}{\partial \sigma_{ij}} \right)^t d\sigma_{ij} + \frac{\partial \phi}{\partial \overline{\varepsilon}_p} d\overline{\varepsilon}_p$$

$$d\phi_* = \mathbf{n}_*^t\, d\sigma + \frac{\partial \phi}{\partial \overline{\varepsilon}_p} d\overline{\varepsilon}_p$$

$$d\phi_* = \mathbf{n}_*^t\, d\boldsymbol{\sigma} - h_*\, d\varepsilon_*$$

Classical Theory of Plasticity

Plasticity

$$d\varepsilon_{ij}^{(p)} = \frac{\partial \phi}{\partial \sigma_{ij}} d\varepsilon_*$$

$$d\varepsilon_* \geq 0$$

$$d\boldsymbol{\varepsilon}_p = \mathbf{n}_* \, d\varepsilon_*$$

$$\mathbf{n}_* = \frac{\mathbf{M}\,\boldsymbol{\sigma}}{\overline{\sigma}_e}$$

$$\phi_* \, d\varepsilon_* = 0 \quad , \quad d\phi_* \, d\varepsilon_* = 0$$

Classical Theory of Plasticity

Integration of constitutive relations

$$\Delta\boldsymbol{\varepsilon}_p = \mathbf{n}_* \, \Delta\varepsilon_* + \mathbf{R}_\varepsilon^*$$

$$\Delta\phi_* = \mathbf{n}_*^t \Delta\boldsymbol{\sigma} - h_* \Delta\varepsilon_* + R_\phi^*$$

$$\phi_* + \Delta\phi_* \leq 0$$

$$\Delta\varepsilon_* \geq 0$$

$$\begin{aligned} \phi_* \Delta\varepsilon_* &= 0 \\ \Delta\phi_* \Delta\varepsilon_* &= 0 \end{aligned}$$

Elastoplasticity in Structural Engineering

Some problems

- Definition of the geometry
- Definition of loading conditions
- Characterisation of material behaviour (concrete, soils, ...)

Elastoplasticity in Structural Engineering

1D problems

Elastoplasticity in Structural Engineering

2D problems

Elastoplasticity in Structural Engineering

3D problems

Standard Numerical Techniques

Main features

- Use of displacement finite element formulations
- Control of plasticity on sets of selected points
- Use of Newton-Raphson algorithms to solve the non-linear governing system
- Use of a wide range of return-mapping algorithms

- Linearisation of the yield surface
- Use of Mathematical Programming Techniques
- Discretisation of plastic parameter field

Standard Numerical Techniques

Some limitations

- Insufficient quality of the stress field estimates
- Computation of *unsafe* estimates for the collapse load
- Position of control points
- Mesh dependency

Hybrid-Mixed Formulations

Approximation criteria

$$\boldsymbol{\sigma} = \boldsymbol{S}_v \, \boldsymbol{X} + \boldsymbol{\sigma}_p \ \text{ in V}$$

$$\boldsymbol{u} = \boldsymbol{U}_v \, \boldsymbol{q}_v + \boldsymbol{u}_p \ \text{ in V}$$

$$\boldsymbol{u} = \boldsymbol{U}_\gamma \, \boldsymbol{q}_\gamma \ \text{ on } \Gamma_\sigma$$

$$\Delta\varepsilon_* = \boldsymbol{P}_* \, \Delta \, \boldsymbol{e}_* \ \text{ in V}$$

Hybrid-Mixed Formulations

Energy conditions

$$\boldsymbol{e}^t\,\boldsymbol{X} = \int \boldsymbol{\varepsilon}^t\,(\boldsymbol{\sigma} - \boldsymbol{\sigma}_p)\,dV$$

$$\boldsymbol{q}_v^t\,\boldsymbol{Q}_v = \int \boldsymbol{b}^t\,(\boldsymbol{u} - \boldsymbol{u}_p)\,dV$$

$$\boldsymbol{q}_\gamma^t\,\boldsymbol{Q}_\gamma = \int \boldsymbol{u}^t\,\boldsymbol{t}_\gamma\;d\Gamma_\sigma$$

$$\Delta\,\boldsymbol{e}_*^t\,\Delta\boldsymbol{\Phi}_* = \int \Delta\varepsilon_*^t\,\Delta\phi_*\;dV$$

Hybrid-Mixed Formulations

Energy conditions

$$\boldsymbol{e} = \int \boldsymbol{S}_v^t \, \boldsymbol{\varepsilon} \, dV$$

$$\boldsymbol{Q}_v = \int \boldsymbol{U}_v^t \, \boldsymbol{b} \, dV$$

$$\Delta \boldsymbol{\Phi}_* = \int \boldsymbol{P}_*^t \, \Delta \phi_* \, dV$$

$$\boldsymbol{Q}_\gamma = \int \boldsymbol{U}_\gamma^t \, \boldsymbol{t}_\gamma \, d\Gamma_\sigma$$

Hybrid-Mixed Formulations

Equilibrium conditions

$$\boldsymbol{Q}_\gamma = \int \boldsymbol{U}_\gamma^t \boldsymbol{t}_\gamma \, d\Gamma\sigma = \int \boldsymbol{U}_\gamma^t \overline{\boldsymbol{t}}_\gamma \, \lambda \, d\Gamma\sigma = \boldsymbol{a}_e \lambda$$

$$\begin{bmatrix} -\boldsymbol{A}_v^t & 0 \\ \boldsymbol{A}_\gamma^t & -\boldsymbol{a}_e \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{X} \\ \Delta\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Compatibility conditions

$$\begin{bmatrix} -\boldsymbol{A}_v & \boldsymbol{A}_\gamma \\ 0 & -\boldsymbol{a}_e^t \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{q}_v \\ \Delta \boldsymbol{q}_\gamma \end{bmatrix} = \begin{bmatrix} \Delta \boldsymbol{e}_e + \Delta \boldsymbol{e}_p \\ -\Delta \overline{w} \end{bmatrix}$$

$$\Delta \overline{w} = \boldsymbol{a}_e^t \Delta \boldsymbol{q}_\gamma = \int \overline{\boldsymbol{t}}_\gamma^t \boldsymbol{U}_\gamma \Delta \boldsymbol{q}_\gamma \, d\Gamma\sigma = \int \overline{\boldsymbol{t}}_\gamma^t \, \Delta \boldsymbol{u} \, d\Gamma\sigma$$

Hybrid-Mixed Formulations

Elasticity

$$\boldsymbol{e}_e = \boldsymbol{F}\boldsymbol{X} + \boldsymbol{e}_\theta$$

Plasticity

$$\begin{bmatrix} \Delta\boldsymbol{e}_p \\ \Delta\boldsymbol{\Phi}_* \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \boldsymbol{N}_* \\ \boldsymbol{N}_*^t & -\boldsymbol{H}_* \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{X} \\ \Delta\boldsymbol{e}_* \end{bmatrix} + \begin{bmatrix} \boldsymbol{R}_e^* \\ \boldsymbol{R}_\Phi^* \end{bmatrix}$$

$$\boldsymbol{\Phi}_* + \Delta\boldsymbol{\Phi}_* \leq 0$$

$$\begin{cases} \boldsymbol{\Phi}_*^t\,\Delta\boldsymbol{e}_* = 0 \\ \Delta\boldsymbol{\Phi}_*^t\,\Delta\boldsymbol{e}_* = 0 \end{cases}$$

$$\Delta\boldsymbol{e}_* \geq 0$$

Hybrid-Mixed Formulations

Governing system - elastoplastic analysis

$$\begin{bmatrix} \boldsymbol{F} & \boldsymbol{A}_v & -\boldsymbol{A}_\gamma & \boldsymbol{N}_* & 0 \\ \boldsymbol{A}_v^t & 0 & 0 & 0 & 0 \\ -\boldsymbol{A}_\gamma^t & 0 & 0 & 0 & \boldsymbol{a}_e \\ \boldsymbol{N}_*^t & 0 & & -\boldsymbol{H}_* & 0 \\ 0 & 0 & \boldsymbol{a}_e^t & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\boldsymbol{X} \\ \Delta\boldsymbol{q}_v \\ \Delta\boldsymbol{q}_\gamma \\ \Delta\boldsymbol{e}_* \\ \Delta\lambda \end{bmatrix} = \begin{bmatrix} -\boldsymbol{R}_e^* \\ 0 \\ 0 \\ \Delta\boldsymbol{\Phi}_* - \boldsymbol{R}_\Phi^* \\ \Delta\overline{w} \end{bmatrix}$$

$$\boldsymbol{\Phi}_* + \Delta\boldsymbol{\Phi}_* \leq 0 \; ; \; \begin{cases} \boldsymbol{\Phi}_*^t \, \Delta\boldsymbol{e}_* = 0 \\ \Delta\boldsymbol{\Phi}_*^t \, \Delta\boldsymbol{e}_* = 0 \; . \end{cases} \; ; \; \Delta\boldsymbol{e}_* \geq 0$$

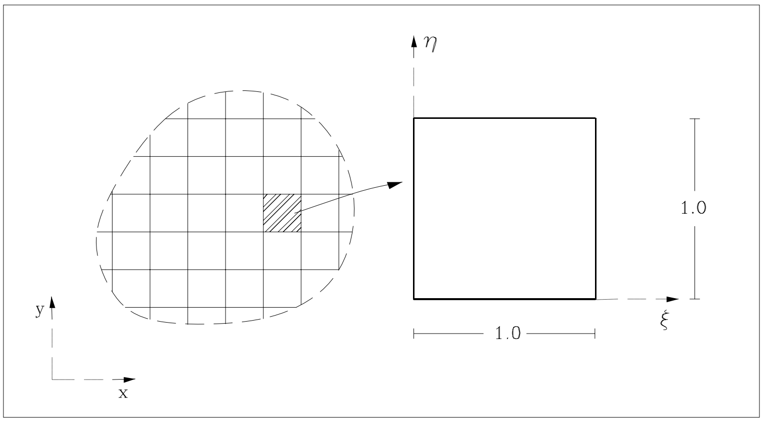
Hybrid-Mixed Formulations

Governing system - elastic analysis

$$\begin{bmatrix} \boldsymbol{F} & \boldsymbol{A}_v & -\boldsymbol{A}_\gamma \\ \boldsymbol{A}_v^t & 0 & 0 \\ -\boldsymbol{A}_\gamma^t & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{q}_v \\ \boldsymbol{q}_\gamma \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}_\gamma - \boldsymbol{e}_\theta \\ -\boldsymbol{Q}_v \\ -\boldsymbol{Q}_\gamma \end{bmatrix}$$

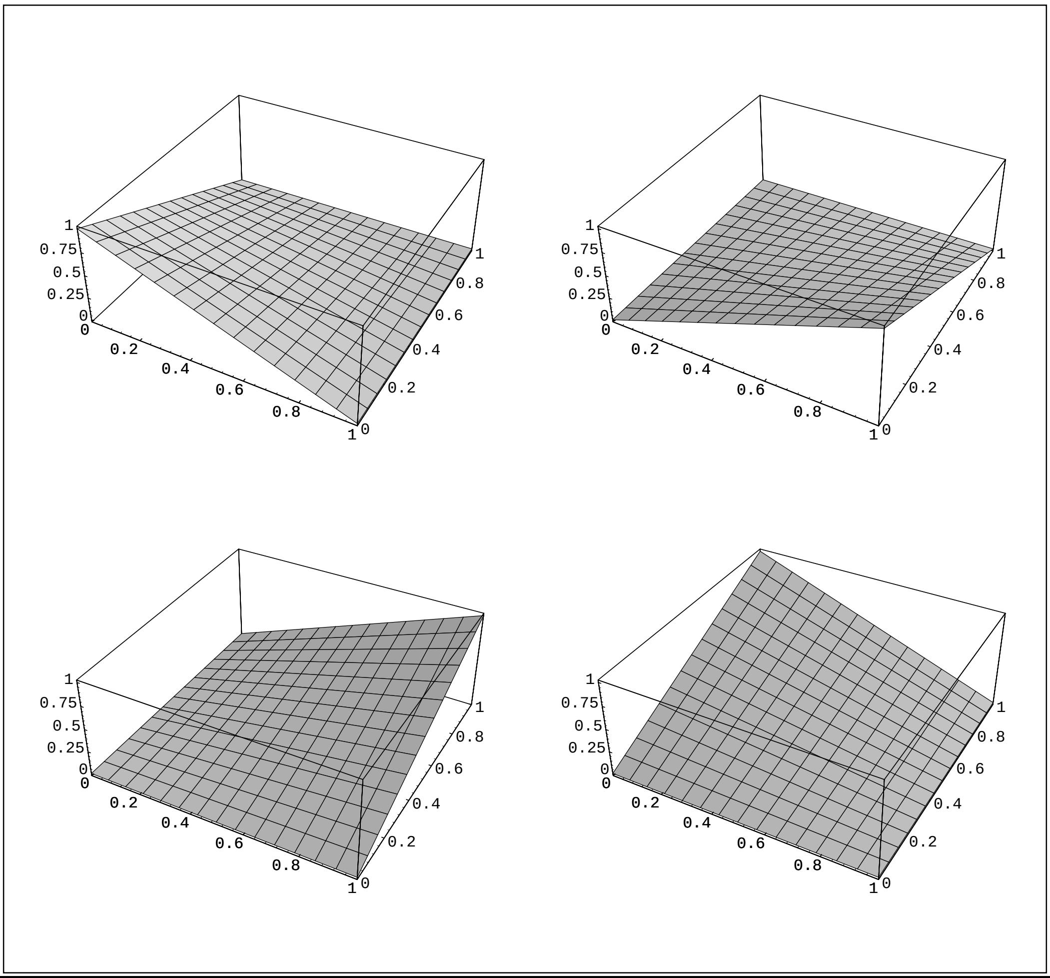
Approximation of plastic parameters

$$\Delta\varepsilon_* = \boldsymbol{P}_* \, \Delta \, \boldsymbol{e}_* \ \text{ in V}$$



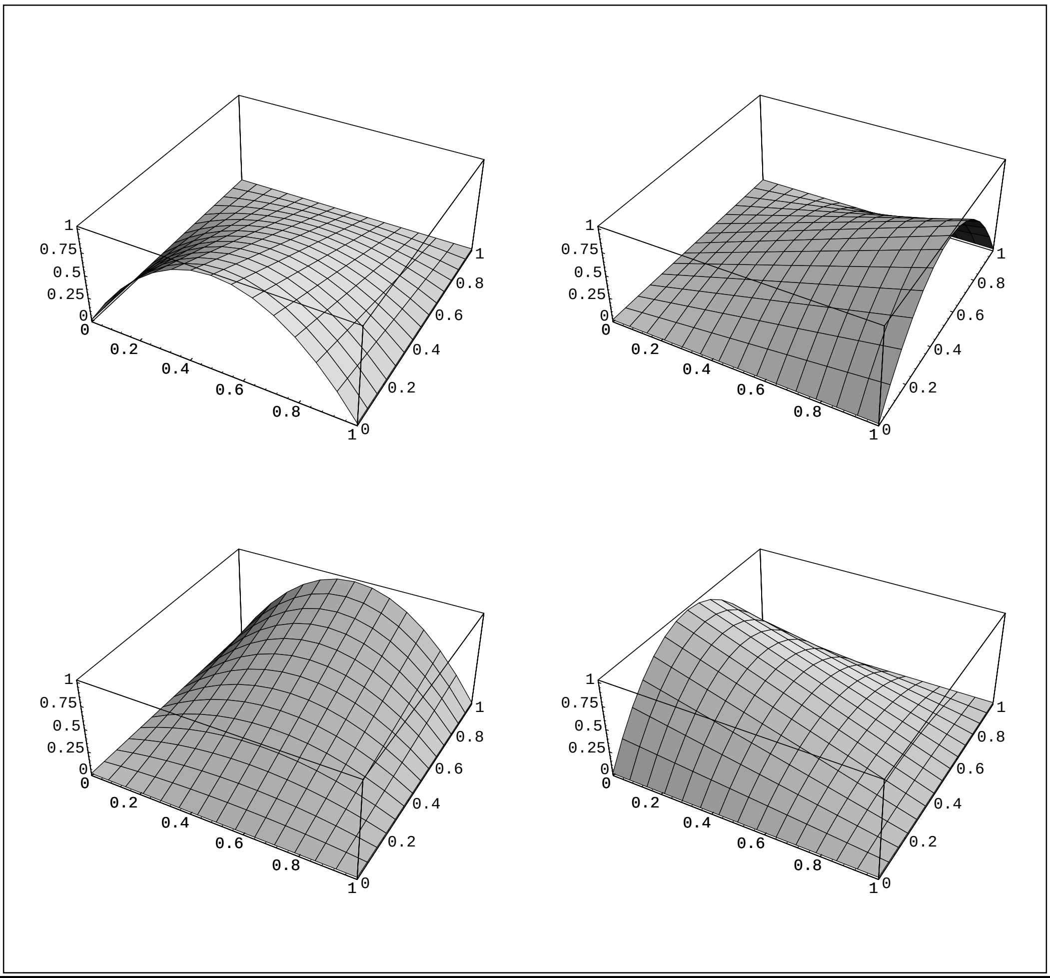
Approximation of plastic parameters

$$f_*^{(1)} = (1-\xi)(1-\eta)$$
$$f_*^{(2)} = \xi(1-\eta)$$
$$f_*^{(3)} = \xi\eta$$
$$f_*^{(4)} = (1-\xi)\eta$$



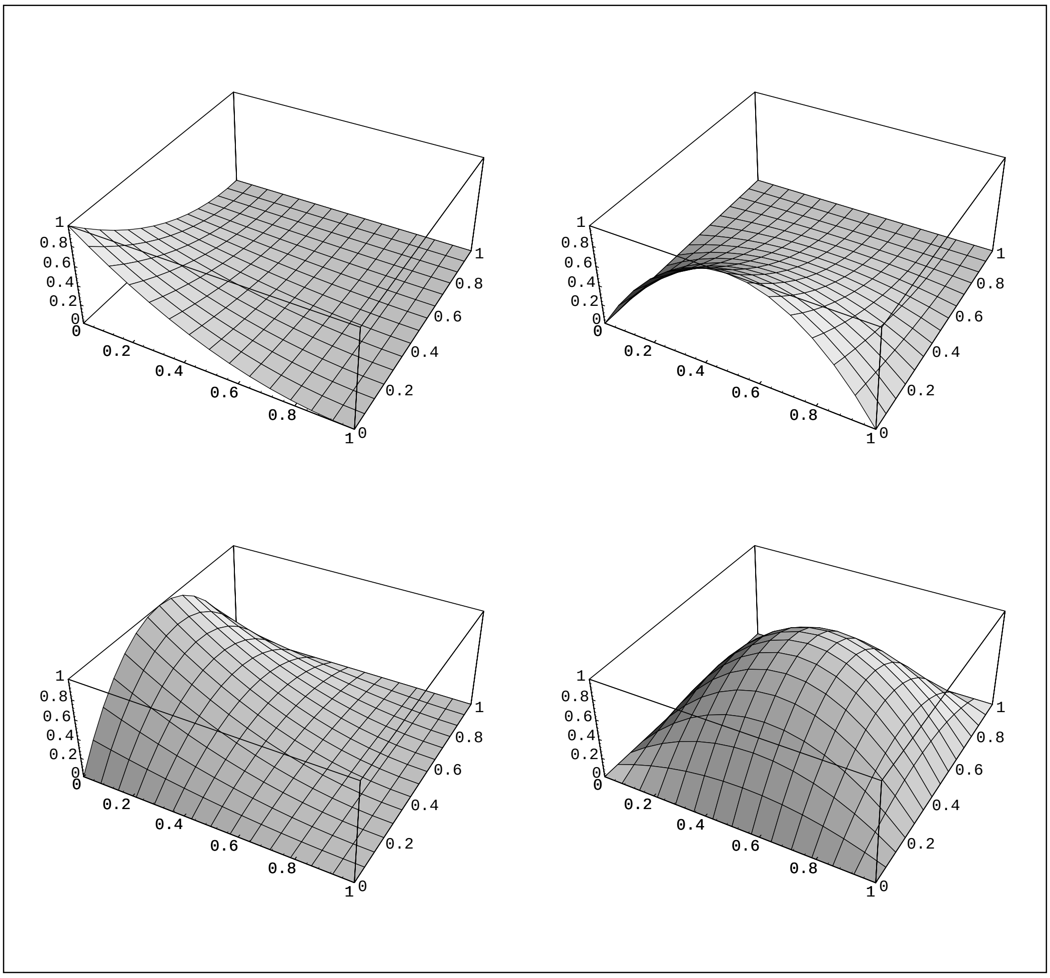
Approximation of plastic parameters

$$f_*^{(5)} = 4\xi(\xi - 1)(\eta - 1)$$
$$f_*^{(6)} = 4\xi\eta(1 - \eta)$$
$$f_*^{(7)} = 4\xi\eta(1 - \xi)$$
$$f_*^{(8)} = 4\eta(\xi - 1)(\eta - 1)$$



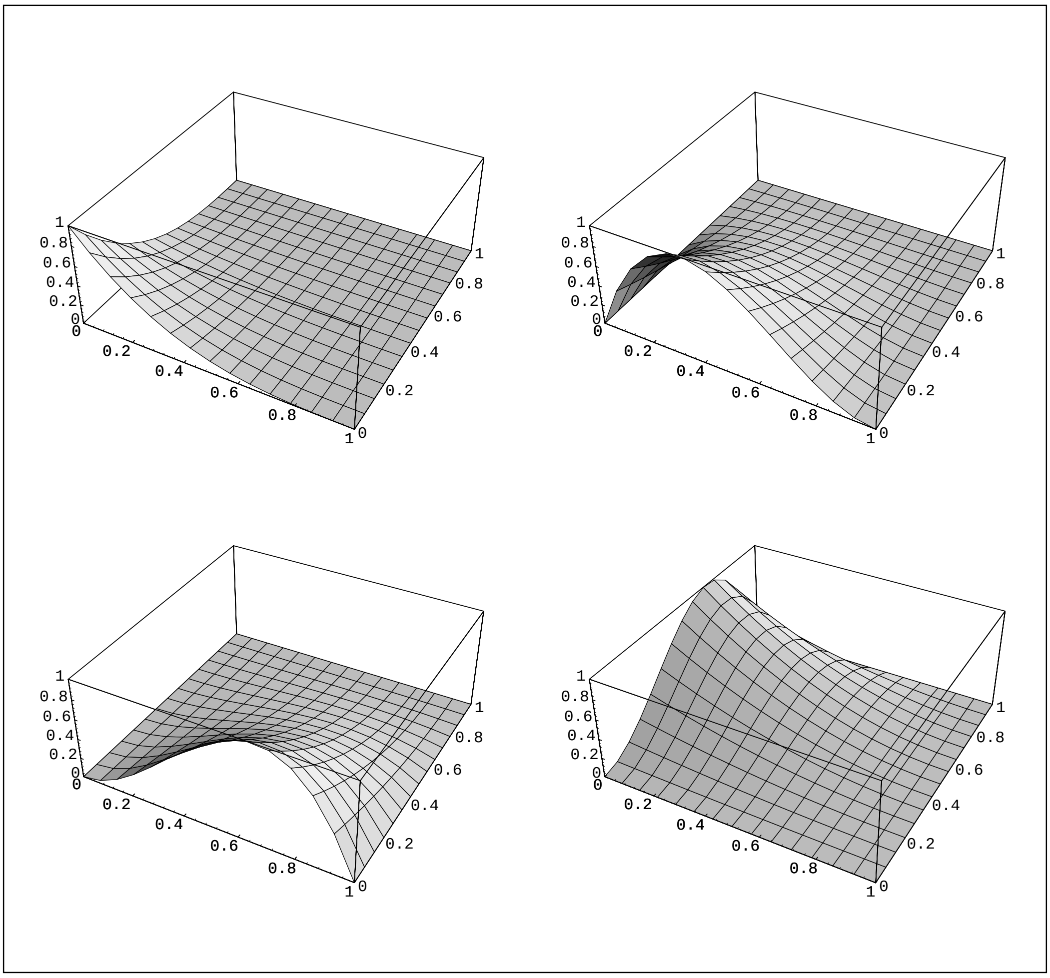
Approximation of plastic parameters

Quadratic approximation



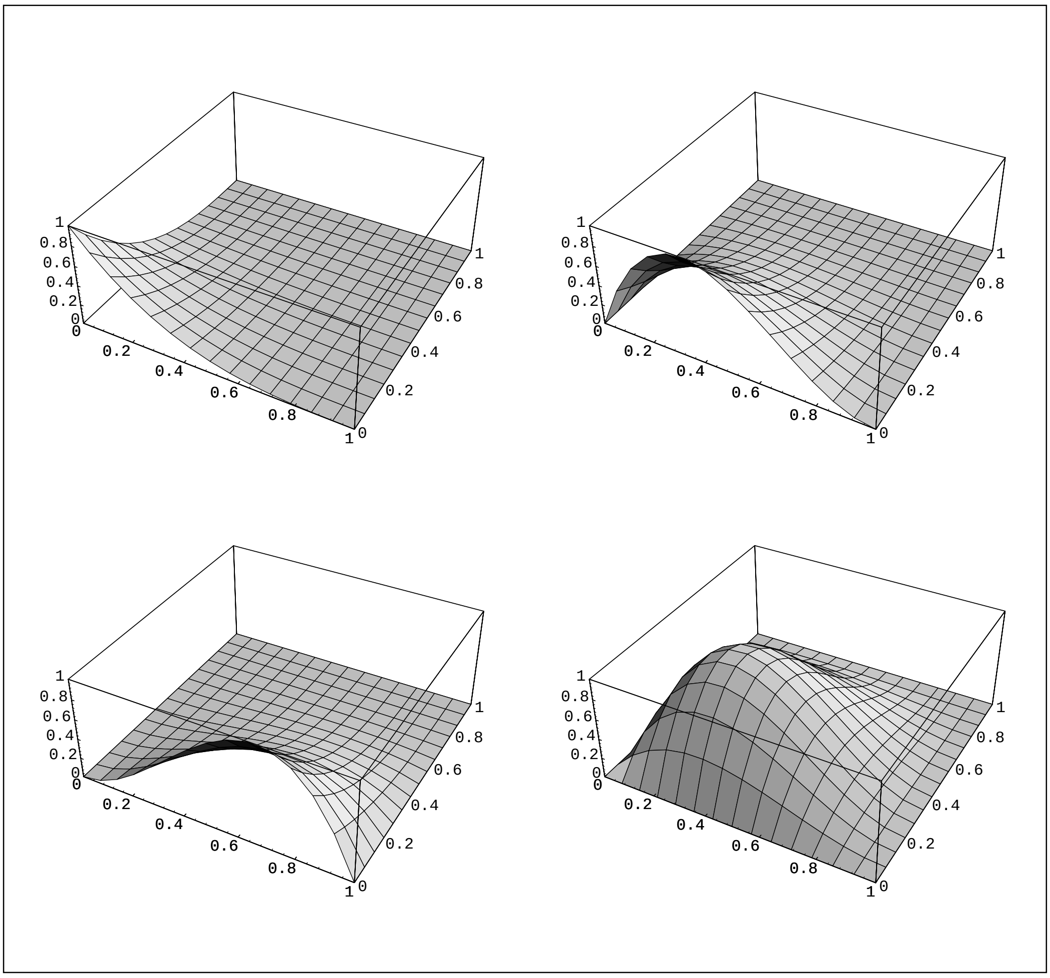
Approximation of plastic parameters

Cubic approximation



Approximation of plastic parameters

Quartic approximation



Perturbation Methods

Basic idea

$$\Delta v = \sum_{n=1}^{\infty} \frac{v^{(n)}\tau^n}{n!}$$

$$\begin{bmatrix} \boldsymbol{F} & \boldsymbol{A}_v & -\boldsymbol{A}_\gamma & \boldsymbol{N}_* & 0 \\ \boldsymbol{A}_v^t & 0 & 0 & 0 & 0 \\ -\boldsymbol{A}_\gamma^t & 0 & 0 & 0 & \boldsymbol{a}_e \\ \boldsymbol{N}_*^t & 0 & 0 & -\boldsymbol{H}_* & 0 \\ 0 & 0 & \boldsymbol{a}_e^t & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{X}^{(n)} \\ \boldsymbol{q}_v^{(n)} \\ \boldsymbol{q}_\gamma^{(n)} \\ \boldsymbol{e}_*^{(n)} \\ \lambda^{(n)} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{R}_e^{*(n)} \\ 0 \\ 0 \\ -\boldsymbol{R}_\Phi^{*(n)} \\ \overline{w}^{(n)} \end{bmatrix}$$

Perturbation Methods

Definition of residual vectors

$$\phi_*^{(n)} = \boldsymbol{n}_*^t \, \boldsymbol{\sigma}^{(n)} + R_\phi^{*(n)}$$

$$\begin{aligned}
R_\phi^{*(1)} &= 0 \\
R_\phi^{*(2)} &= \boldsymbol{\sigma}^{(1)\,t} \boldsymbol{M} \, \boldsymbol{\sigma}^{(1)} - (\phi^{(1)})^2 \\
R_\phi^{*(3)} &= 3\boldsymbol{\sigma}^{(1)\,t} \boldsymbol{M} \, \boldsymbol{\sigma}^{(2)} - 3\phi^{(1)}\phi^{(2)} \\
R_\phi^{*(4)} &= 4\boldsymbol{\sigma}^{(1)\,t} \boldsymbol{M} \, \boldsymbol{\sigma}^{(3)} \\
&+ 3\boldsymbol{\sigma}^{(2)\,t} \boldsymbol{M} \, \boldsymbol{\sigma}^{(2)} - 3(\phi^{(2)})^2 - 4\phi^{(1)}\phi^{(3)}
\end{aligned}$$

Perturbation Methods

Definition of residual vectors

$$\boldsymbol{\varepsilon}_p^{(n)} = \boldsymbol{n}_* \, \varepsilon_*^{(n)} + \boldsymbol{R}_\varepsilon^{*(n)}$$

$$\begin{aligned}
\boldsymbol{R}_\varepsilon^{*(1)} &= 0 \\
\boldsymbol{R}_\varepsilon^{*(2)} &= \boldsymbol{n}_*^{(1)} \, \varepsilon_*^{(1)} \\
\boldsymbol{R}_\varepsilon^{*(3)} &= 2\boldsymbol{n}_*^{(1)} \, \varepsilon_*^{(2)} + \boldsymbol{n}_*^{(2)} \, \varepsilon_*^{(1)} \\
\boldsymbol{R}_\varepsilon^{*(4)} &= 3\boldsymbol{n}_*^{(1)} \, \varepsilon_*^{(3)} \\
&+ 3\boldsymbol{n}_*^{(2)} \, \varepsilon_*^{(2)} + \boldsymbol{n}_*^{(3)} \, \varepsilon_*^{(1)}
\end{aligned}$$

Perturbation Methods

Step length

1.

$$\Phi_*^{(i)} + \Phi_*^{(1,i)}\tau + \Phi_*^{(2,i)}\frac{\tau^2}{2} + \Phi_*^{(3,i)}\frac{\tau^3}{6} = 0$$

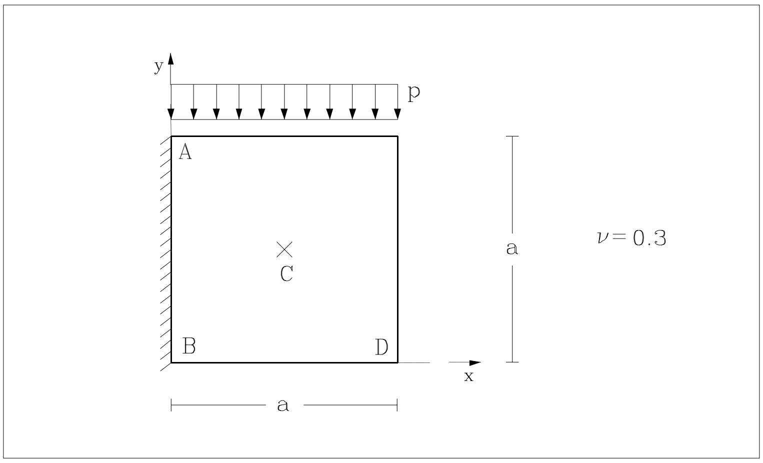
2.

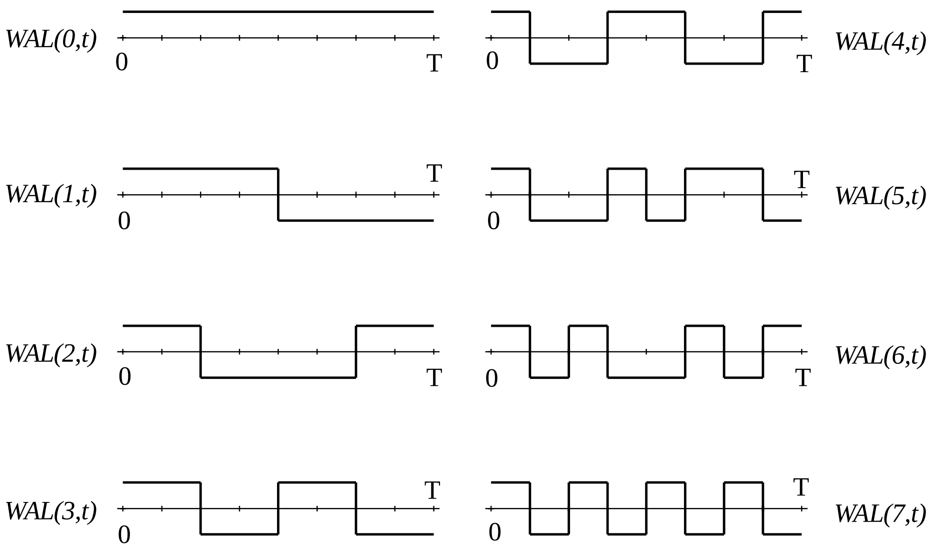
$$\tau_j \leq \left(\frac{TOL \times n!}{\left|v_j^{(n)}\right|}\right)^{\frac{1}{n}}$$

3.

$$\frac{\partial \Delta e_*}{\partial \tau} \geq 0 \Rightarrow e_*^{(1)} + e_*^{(2)}\tau + e_*^{(3)}\frac{\tau^2}{2} \geq 0$$

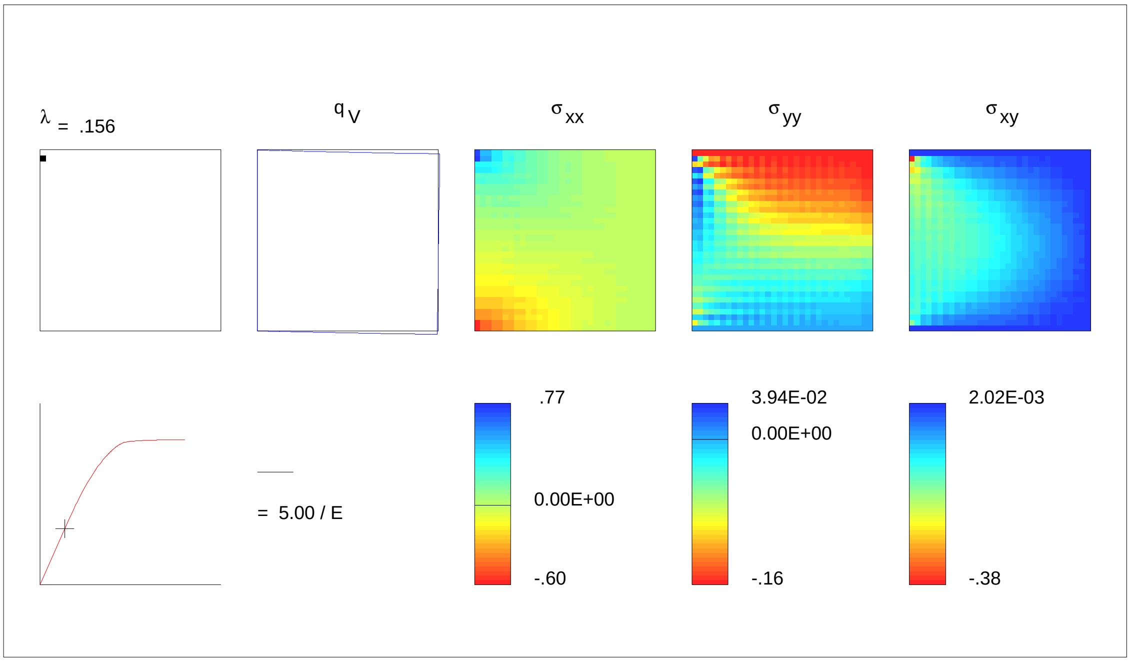
Examples of Application



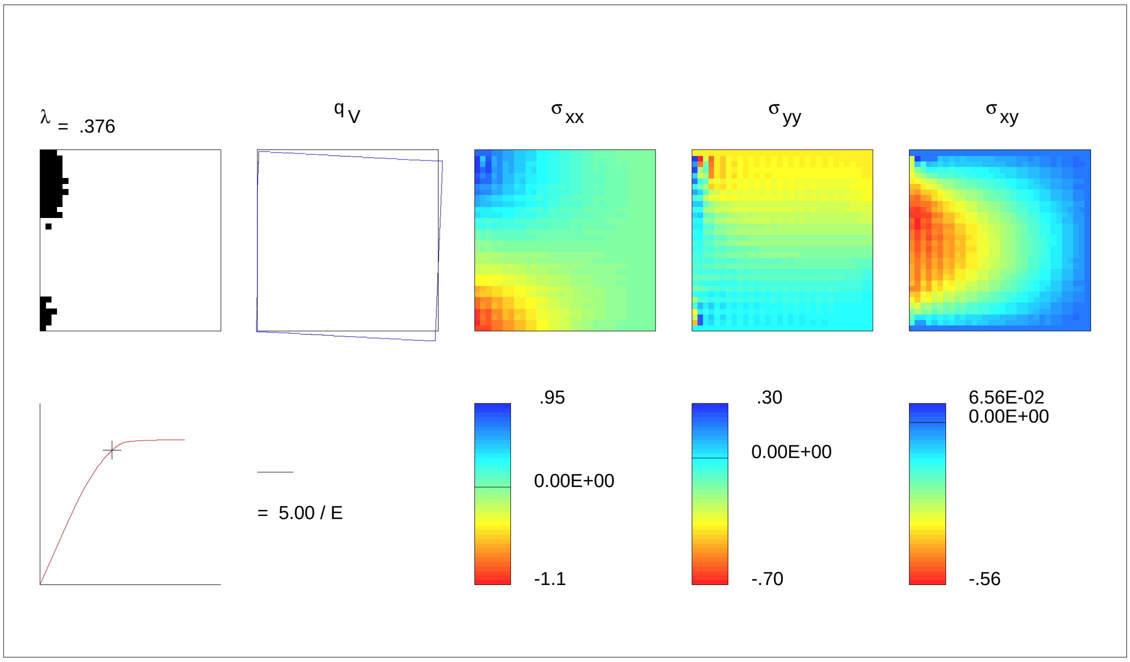


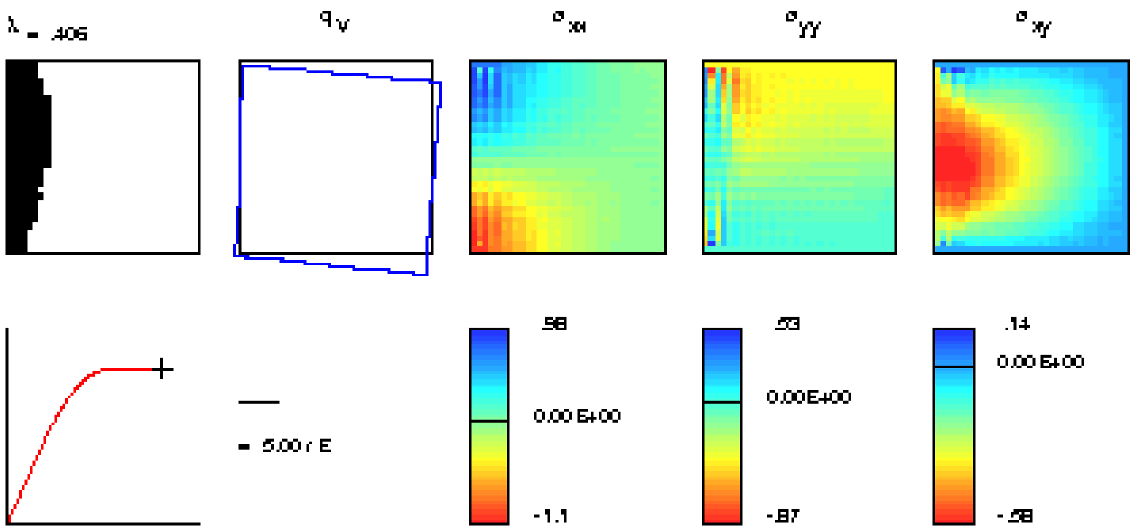
Examples of application

32×32 critical cell mesh



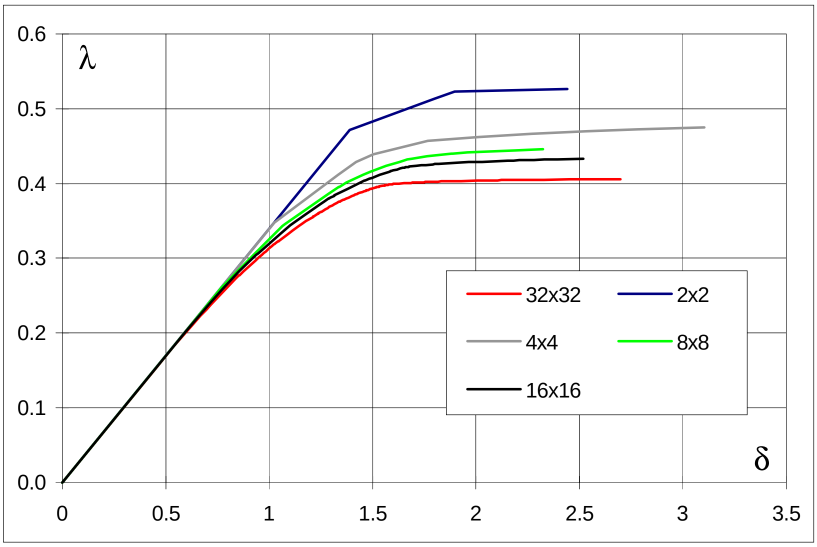
Examples of application





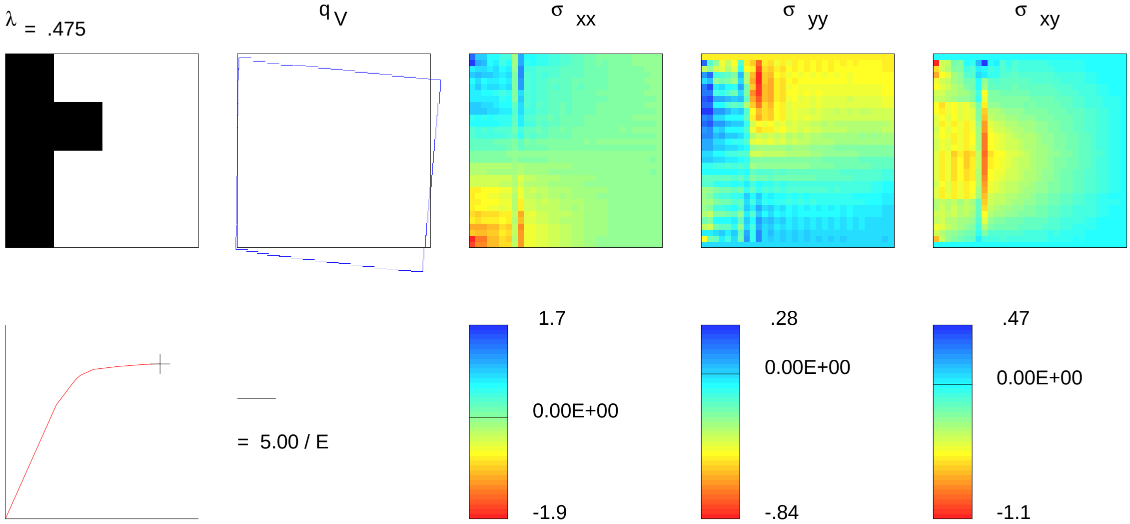
Examples of application

critical cells	*loading steps*	λ_c
2×2	4	0.527
4×4	10	0.475
8×8	20	0.446
16×16	68	0.433
32×32	180	0.406

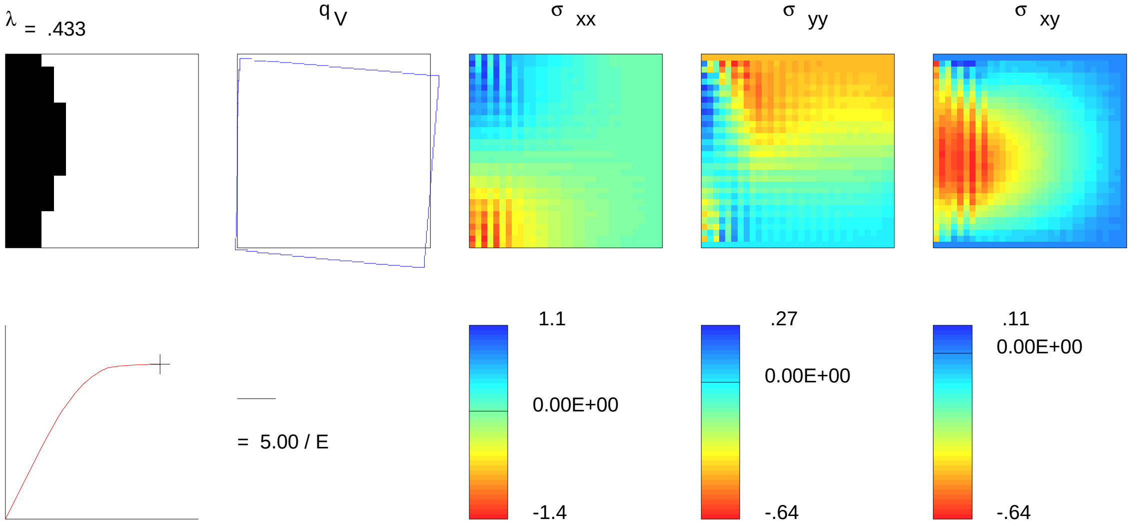


Examples of application

4 × 4 critical cell mesh

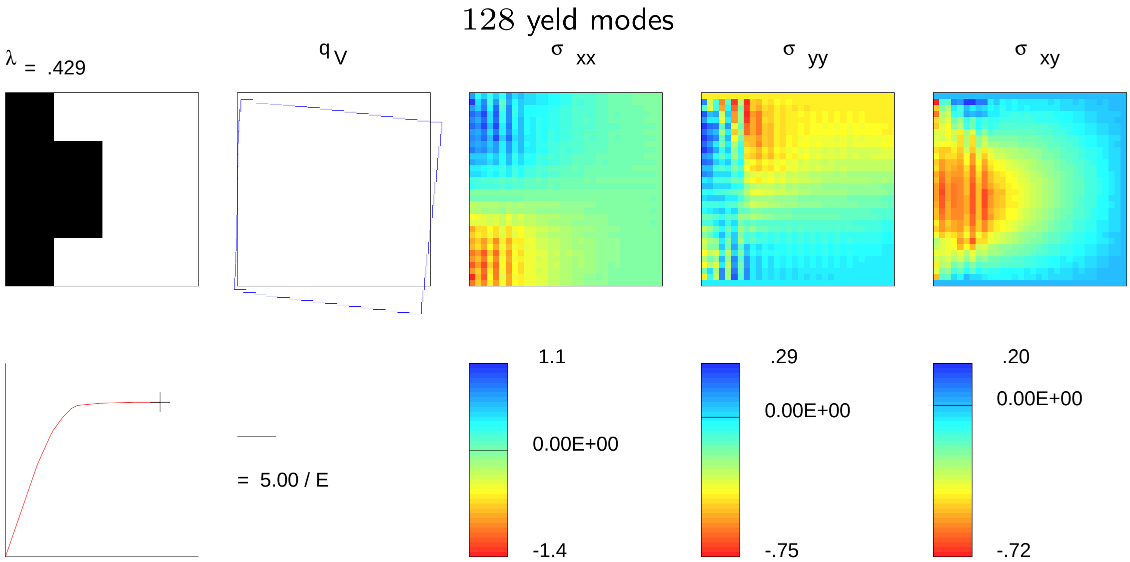


16 × 16 critical cell mesh

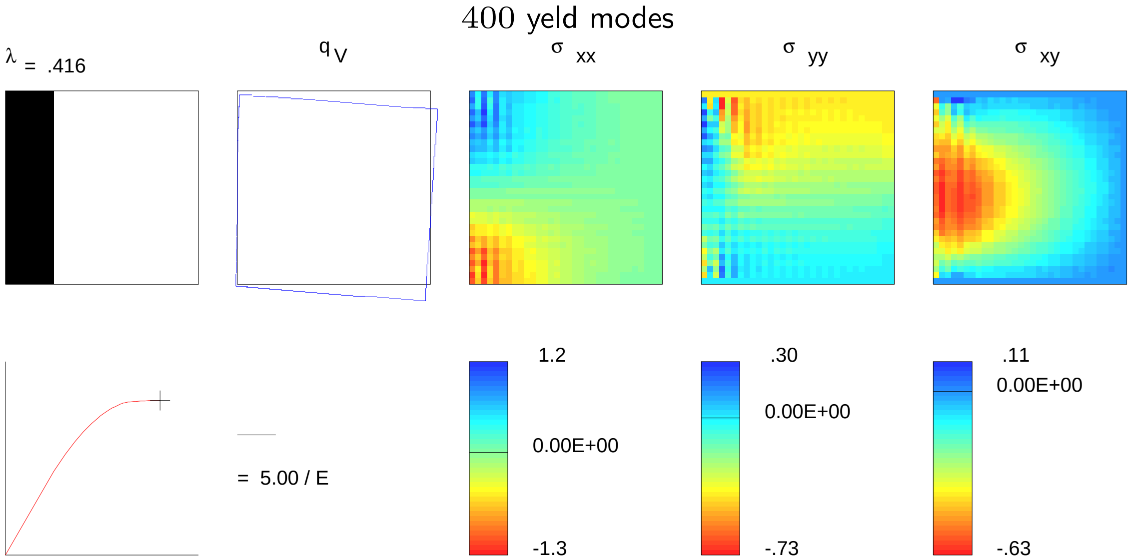


Examples of application

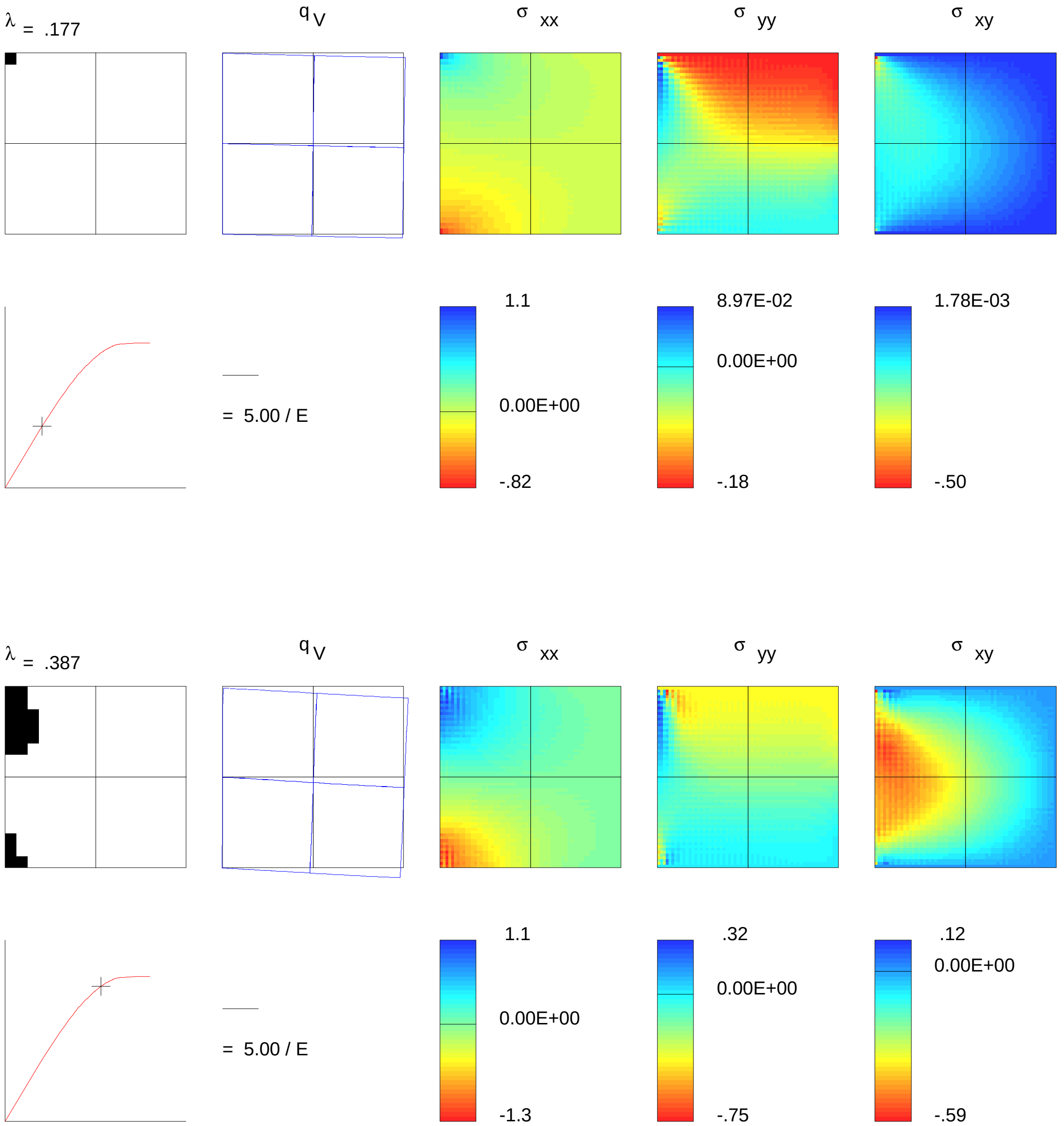
Degree	*yield modes*	*loading steps*	λ_c
constant	16	10	0.476
linear	64	15	0.437
parabolic (8)	128	20	0.429
parabolic (9)	144	27	0.428
cubic (12)	192	41	0.423
quartic	400	39	0.416



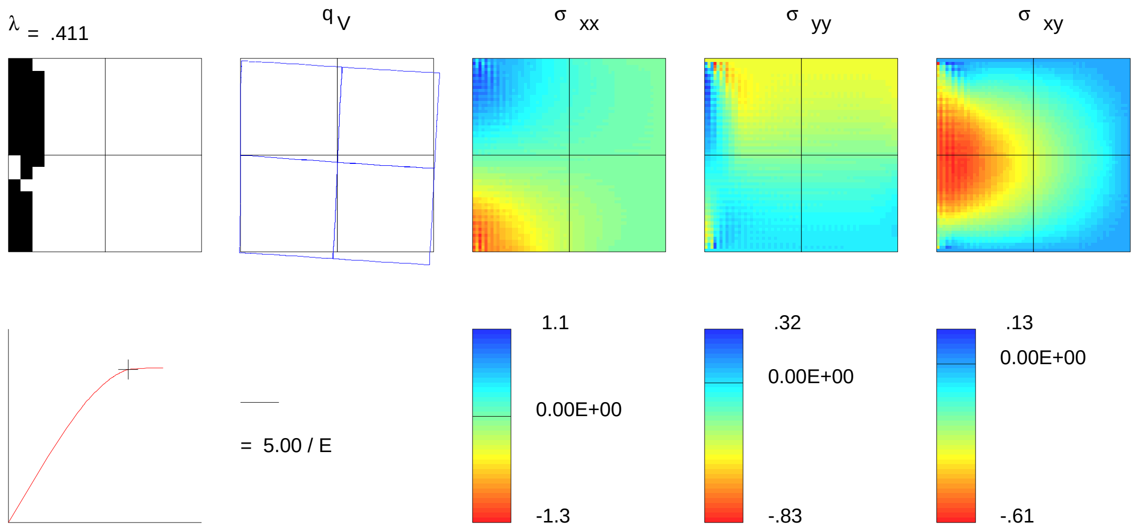
Examples of application

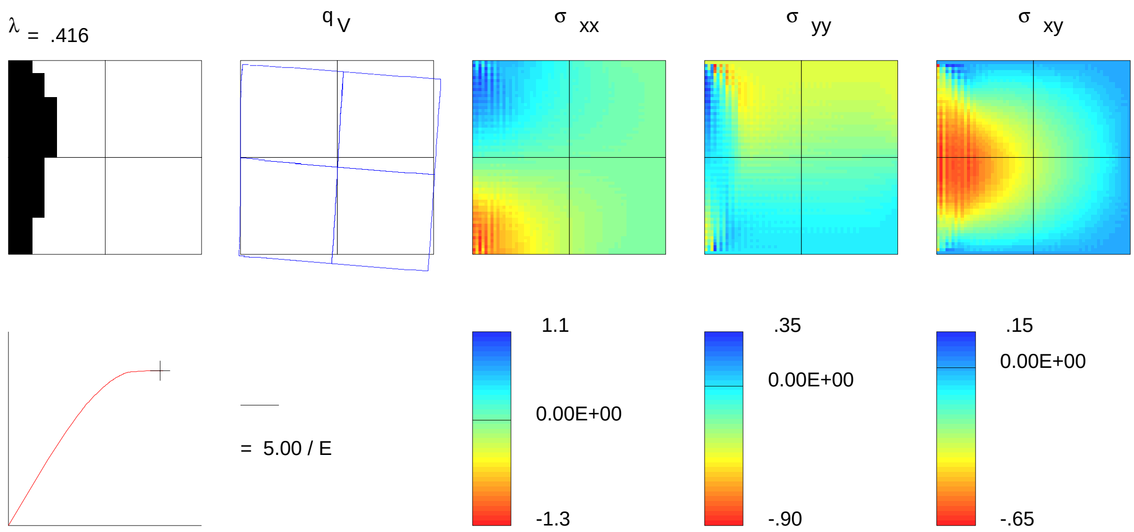


Examples of application

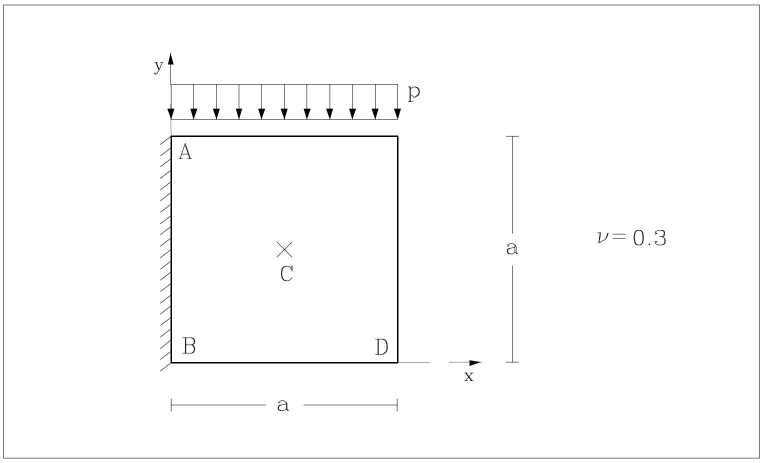


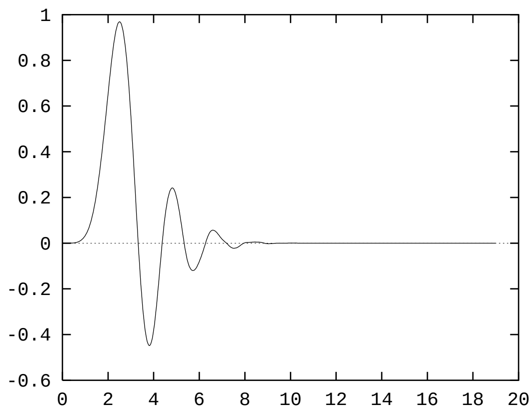
Examples of application

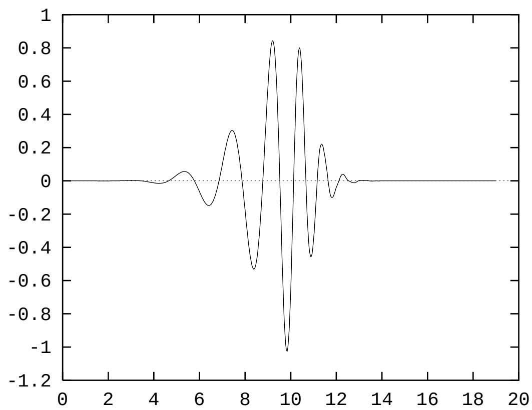




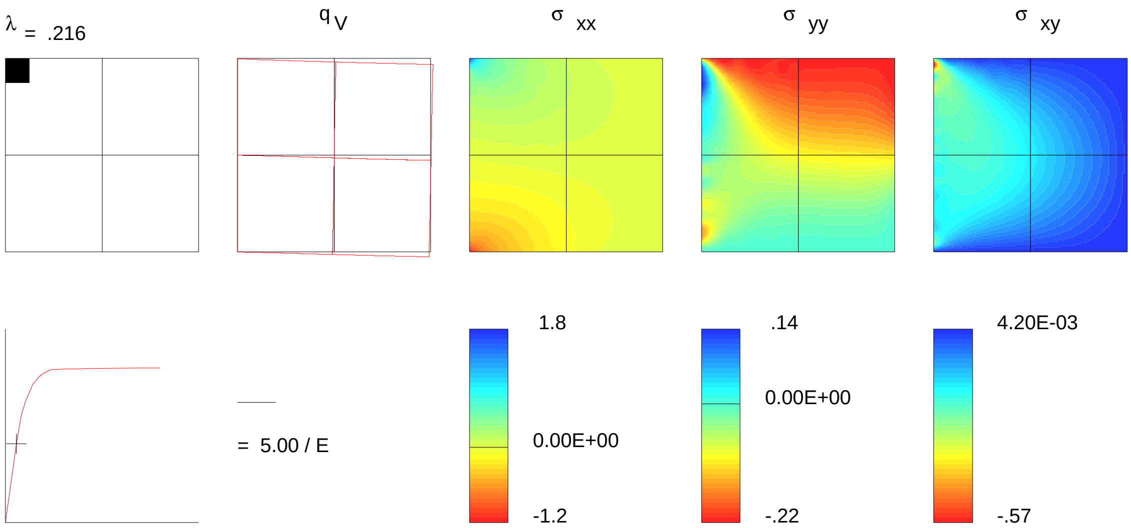
Examples of application

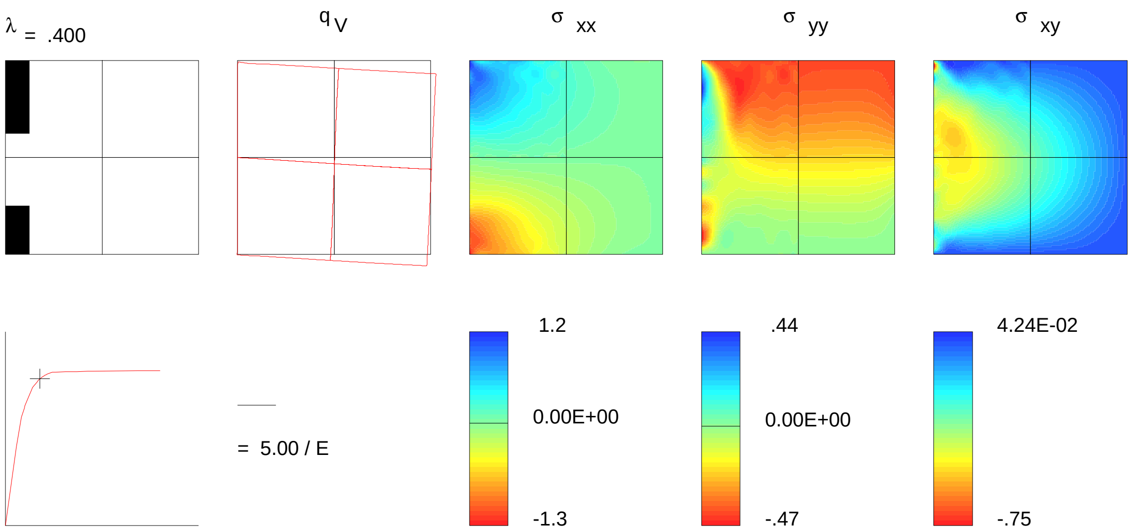




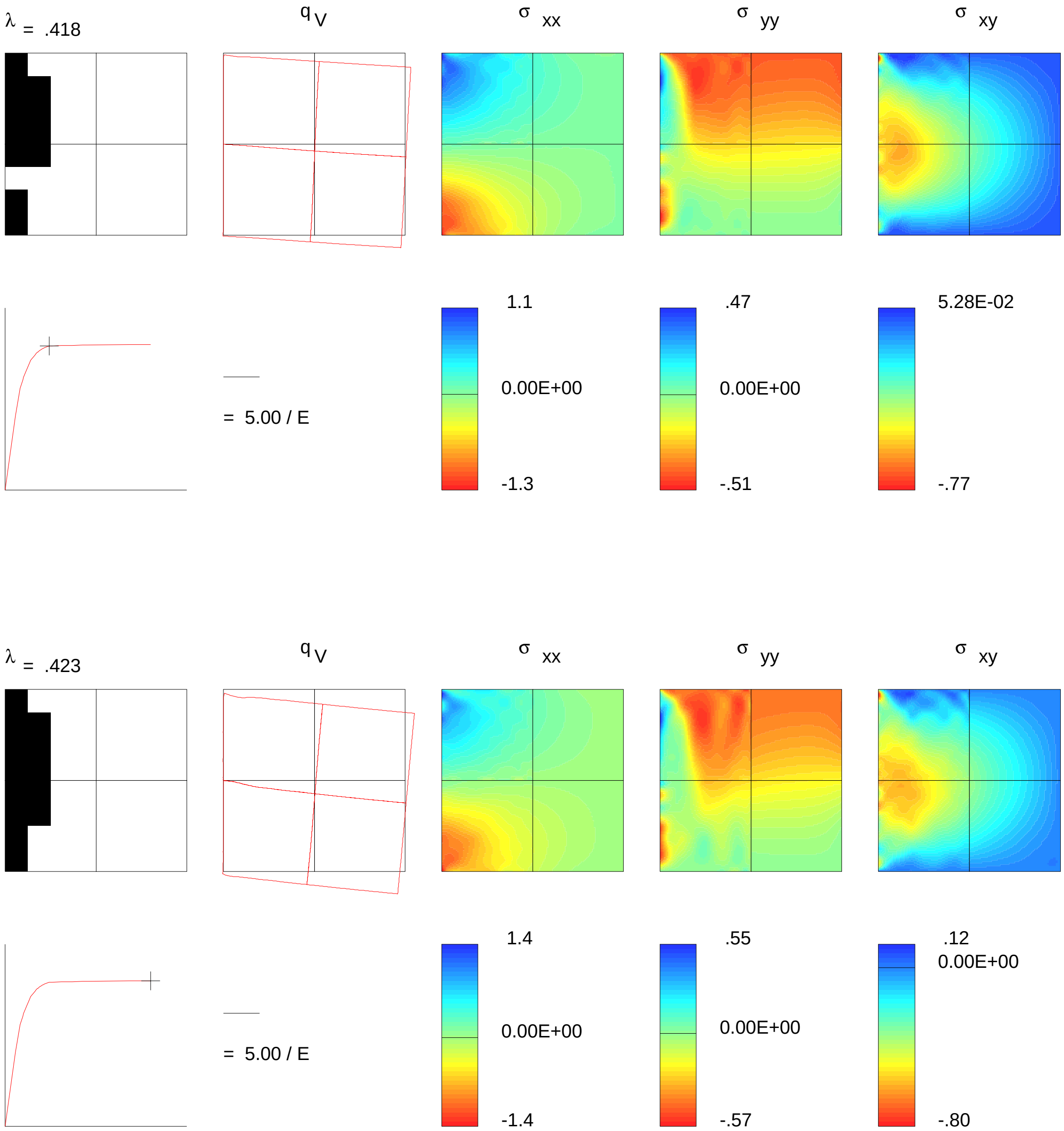


Examples of application

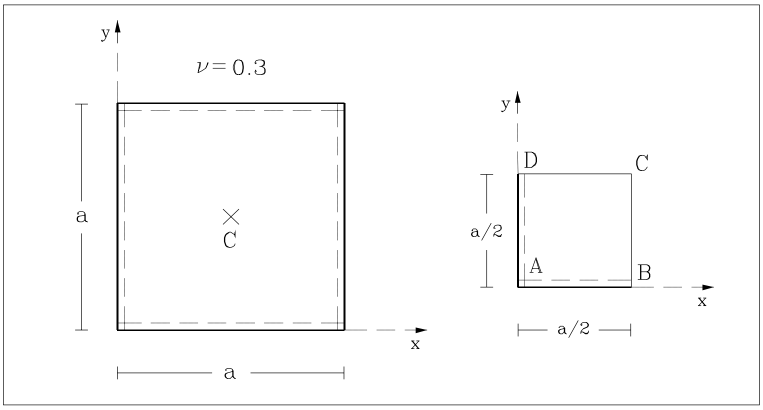




Examples of application



Examples of application



Examples of application

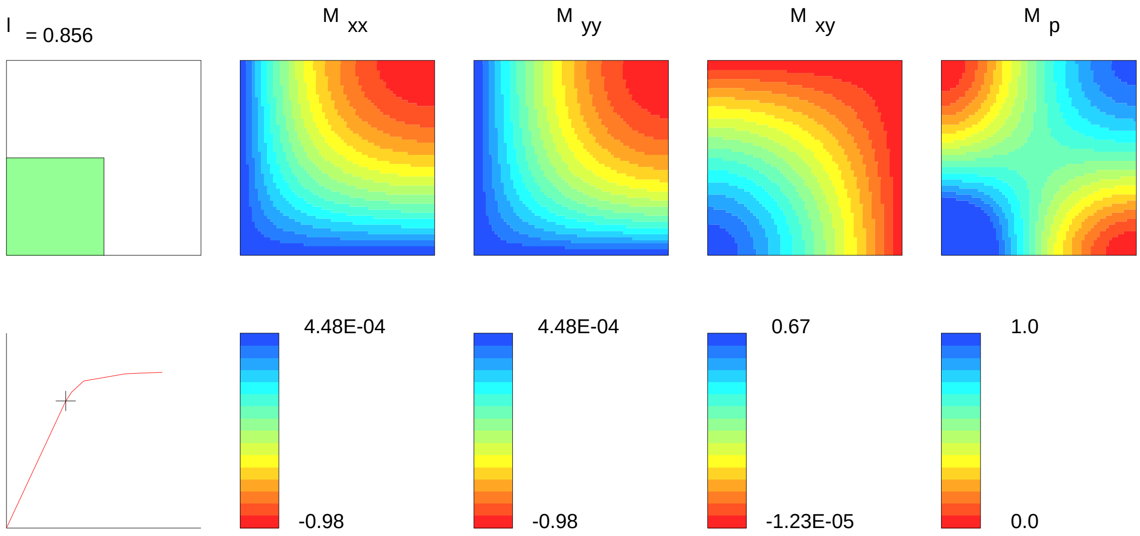
	Degree				
c. cells	0	1	2	3	4
1×1	0.529	0.481	0.461	0.448	0.439
2×2	0.492	0.437	0.431	0.418	0.418
4×4	0.441	0.422	0.415	0.408	0.408

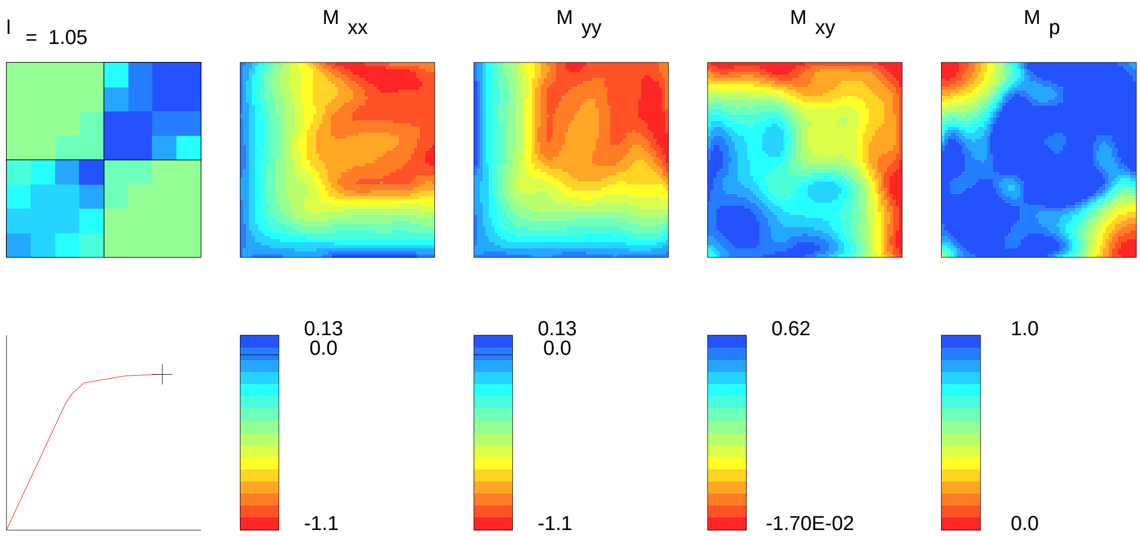
$$\lambda^* = \frac{\lambda\, a^2}{6\, m_0}$$

Model	*Degree*	n_{modes}	λ_e^*	λ_c^*
Present	0	1	0.9503	1.1208
	1	4	0.8560	1.0522
	2	8	0.8150	1.0495
	3	12	0.7929	1.0465
	4	25	0.7796	1.0436
Ang and Lopez			0.74	1.031
Belytschko and Velebit			0.810	1.068

$$1.036 < \lambda_c^* < 1.106$$

Examples of application





Examples of application

Plastic cells	λ_e^*	λ_c^*
2×2	0.8560	1.0522
4×4	0.7778	1.0443
8×8	0.7514	1.0329