Solution for the Closed-Loop Problem in Pressurized Multipipe Systems
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**Abstract:** The fundamental background of the solution for the steady-state flow in pressurized water closed-loop pipe systems, without any reservoirs in-between, is presented in this paper. The use of the steady-state mass balance and energy equations to calculate discharges and heads in this type of hydraulic system leads to an undetermined problem. The way to solve this indeterminacy is to consider an additional continuity equation associated with the difference between initial and final conditions, taking into account fluid compressibility and pipe-wall deformability. A complete formulation is derived considering pressure and temperature changes in the hydraulic system. Simplified formulae are presented for isothermal flows in simple systems and multiple closed-loops with pipes in series and in parallel. This problem can also be solved by a pseudotransient analysis technique applied to steady-state conditions. Proposed solutions for this problem are applied to steady-state flows and tested for different system configurations.

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**Introduction**

The current paper focuses on the theoretical fundamentals for the solution of a practical problem that is frequently posed in the design of industrial systems, namely cooling and heating closed-loop pressurized pipe systems. These systems are typically described by a pipe loop with a pumping system that maintains the flow and overcomes the hydraulic resistance, without any connection to an external or in-between reservoir. A closed-loop system can be a single or complex pipe system arrangement with different pipe diameters and materials and other types of components (valves and pumps). The design of these systems is common in engineering and is based on flow network modeling and analysis (Kelkar and Patankar 2003); however, steady-state continuity and energy equations are not sufficient for a complete analysis of this type of system when there is no reservoir or any other in-between device for pressure control or leakage compensation.

This situation is similar to the statically undetermined solid-structure (e.g., a clamped beam): the reactions and bending moments in the supports cannot be calculated only from the static equilibrium conditions, but also depend on the deformation of the structure, and the static analysis has to be complemented with an elastic analysis. Similarly, in pipe closed-loops, an additional thermoelastic mass-balance equation has to be considered in order to eliminate the mathematical indeterminacy.

The following conceptual solution for this problem is presented and applied to steady-state flow analysis: an additional mass balance equation that takes into account pipe circumferential deformation and fluid compressibility due to the initial and final pressure and temperature distributions. Different approaches are presented to numerically solve this problem for isothermal flows: simplified formulae are presented for simple systems; a novel generalized formula is presented for complex pipe system arrangements; and the complete unsteady-state flow equations applied to the time-marching process toward the final steady-state solution (pseudotransient method) can also be used. Numerical examples with different system configurations are selected to establish and test proposed solutions.

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**Theoretical Concepts**

**Steady Flow in Pipe Networks**

Steady-state analysis is generally used to describe pressures and flow-rates in pressurized multipipe systems for a set of boundary conditions and demands. Equations can be formulated in terms of flows in pipes, heads at the nodes, or corrective discharges in the loops. Different methods can be used for solving equations (e.g., linear theory and the Newton-Raphson and Hardy-Cross methods).

Steady-state analysis is typically based on the general conservation principles: mass, momentum, and energy. The latter two are equivalent in what concerns the mechanical energy component. In practice, the thermal energy component as well as the change of density can be neglected in weakly compressible fluids (i.e., liquids), and the set of equations is defined by the flow balance at the nodes and the mechanical work-energy conservation in the loops.

The conservation of mass or the continuity principle states that
the net volumetric pipe discharge into a junction must equal the volumetric demand from the junction (junction continuity equation)

\[ \sum_k Q_k - Q_i' = 0 \]  

(1)

where \( Q_k \) = flow rate converging to node \( i \) through pipe \( k \), and \( Q_i' \) = demand assigned to node \( i \).

The work-energy conservation law requires that the algebraic sum of the head variation between the nodes is zero in each pipe-loop (mechanical energy loop equation)

\[ \sum_k R_k Q_k |Q_k|^{n-1} = \Delta WS + \sum_m \Delta H_m \]  

(2)

where \( R_k \) = pipe \( k \) hydraulic resistance; \( \Delta WS \) = difference of water level in pseudoloops; and \( \Delta H_m \) = head change in nonpipe element \( m \). Pumps and valve head changes can be described by

\[ \Delta H_{F,V} = A + BQ - CQ |Q| \]  

(3)

where \( A, B, C \) = coefficients that depend on the nonpipe elements (for valves, \( A = B = 0 \)). The simultaneous resolution of Eqs. (1)–(3) describes steady-state isothermal flows in pipe systems. After flow-rates are determined, the head losses in pipes are known, and the heads at nodes are calculated by starting at a known-head node and applying the head loss equation in pipes.

In pipe systems with temperature changes due to external heating or cooling sources, additional equations are necessary that are associated with (1) heat transfer in pipes (or temperature distributions) and nonpipe elements; and (2) thermal energy balance at nodes; also, (3) friction losses can be described by temperature-dependent functions (excluding for fully rough flows).

**Indeterminacy of Closed-Loops**

A pipe network can be described as a set of oriented link-elements (pipes, pumps, and valves) connected by node-elements (junctions and reservoirs). The number of equations is directly related to a fundamental relation between the number of link-elements \( E \), junction nodes \( N \), fixed-head nodes (reservoirs) \( R \), and independent (real) closed loops \( L \), according to Euler’s polyhedron formula (Walski 1985; Larock and others 1999)

\[ R + N - E + L = 1 \]  

(4)

A network that does not verify this geometric relation cannot be resolved due to ill-connectivity between elements. Even for well-connected systems, networks must have at least one fixed-head node to be solved by Eqs. (1)–(3); otherwise, the problem is undetermined.

Fig. 1 presents two examples of well-connected systems [i.e., those that verify Eq. (4)], and therefore flow distribution in all pipes can be calculated: (1) a nonautonomous, externally unconnected network; and (2) an autonomous closed-loop system with a pump. In both, the number of fixed-head nodes is null (\( R = 0 \)). The difference between them is that system (a) does not have a solution with respect to head distributions unless one node head is known, whereas system (b) has a solution providing that an additional mass balance equation is added to classic equations. System (b) is the simplest form of the undetermined autonomous closed-loop system.

**Basic Autonomous Ring System Problem**

Any closed-loop system can be simplified to an equivalent two-component system—the basic autonomous ring system (BARS)—which is composed of an equivalent single pipe and an equivalent single pump (Fig. 2). The former will describe the head loss, heat transfer, and flow storage in the system, and the latter will assure the head source. This equivalent pipe system has two basic nodes: an inflow node 1 and an outflow node 2, located downstream and upstream of the pump, respectively. The simplified BARS concept is important for obtaining analytical solutions and running sensitivity analyses.

By applying the mechanical energy loop equation [Eq. (2)] to the BARS, the total flow rate \( Q_i \) between nodes 1 and 2 can be straightforwardly calculated in isothermal flows. Head distribution will obey the following set of equations obtained by applying Eqs. (2) and (3):

\[ \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = \begin{bmatrix} \Delta H_{F,V}(Q_i) \\ \Delta H_{F,V}(Q_i) \end{bmatrix} \]  

(5)

This system \((Ax=b)\) is undetermined when using the classic steady-state equations only. This can be mathematically proven by calculating the determinant of matrix \( A \) and verifying that it is null (i.e., matrix \( A \) eigenvalues are linearly dependent). In fact, it is possible to obtain the head difference between the two nodes, but not each head \( H_i \) nor the head distribution along the pipe.

In closed-loop cooling pipe systems, large temperature variations, due to heat sources and sinks, can occur along the pipe, and the isothermal hypothesis may not be theoretically valid.

**Conceptual Solution**

**Thermoelastic Constraint Condition**

The solution for the BARS problem by means of steady-state analysis is possible as long as an additional independent condition related to the fluid mass balance along the pipe system between
initial and final conditions is considered. Basic hypotheses considered are the following (Almeida and Koelle 1992): (1) flow is one-dimensional (1D) with a pseudouniform velocity profile at each cross-section; (2) fluid is one-phase, homogeneous, and weakly compressible (i.e., liquid); (3) pipe-elements are straight and uniform, with constant (circular) cross-section, and constrained from axial movement, and pipe walls are linear-elastic; (4) nonpipe elements are rigid, with negligible fluid volume inside; and (5) the velocity head component is relatively small and can be neglected.

By applying the mass conservation principle to a deformable control volume inside the pipe wall, along the total closed-loop pipe length the L, it yields (White 1999)

\[
dM = 0 \Leftrightarrow \int_{V_i} \frac{\partial p}{\partial t} dV + \int_{S_L} \rho \frac{\partial V}{\partial t} n dS = 0
\]

where \(\rho\)=fluid mass density; \(V\)=absolute fluid velocity across the control surface; \(x\)=distance along the pipe axis; and \(t\)=time. By considering the control pipe surface equal to the lateral \(S_L\) and the cross-sectional (transversal) \(S_T\) surfaces

\[
\int_{S_L} \frac{\partial p}{\partial t} dS + \int_{S_T} \rho \frac{\partial V}{\partial t} \overline{n} dS = 0
\]

where \(S\)=pipe cross-sectional area; and \(V_L, V_T\)=fluid velocity through the lateral or transversal control surfaces, respectively. Considering the mass density constant at each cross section, lateral velocity \(V_L = \frac{\partial R}{\partial t} (R=\text{pipe radius function of } x \text{ and } t)\), and lateral area \(S_L = 2\pi RL\), and by accepting the continuity condition in the last integral, it yields

\[
\int_{S_L} \frac{\partial p}{\partial t} dx + \int_{S_L} \rho 2\pi R \frac{\partial R}{\partial t} dx + \int_{S_T} \frac{\partial (\rho V_T)}{\partial x} dS dx = 0
\]

Considering that \(\frac{\partial S}{\partial t} = 2\pi R \frac{\partial R}{\partial t}, V_T \text{ and } \rho \text{ are constant at each cross section (i.e., they are just functions of } x \text{ and both cross-sections are perpendicular to the pipe axis}

\[
\int_{S_L} \frac{\partial p}{\partial t} dx + \int_{L} \frac{\partial S}{\partial t} dx + \int_{S_T} \frac{\partial (\rho V_T)}{\partial x} dS dx = 0
\]

Resolving the last integral and knowing that \(Q = SV_T\)

\[
\int_{L} \left( \frac{dp}{dt} + \frac{dS}{dt} \right) dx + \int_{S_L} \rho S \frac{\partial V_T}{\partial x} dx = 0
\]

Introducing Eqs. (14) and (15) into Eq. (13) and replacing pressure by piezometric head \(H\), the following global thermoelastic constraint condition is obtained:

\[
\int_{L} \frac{S}{\alpha^2} dH dx + \int_{L} \rho S(2\alpha_L - \beta_V) dT dt dx = 0
\]

For a single-phase substance (e.g., water, air, or solid), any two basic properties (typically, pressure \(p\) and temperature \(T\)) are sufficient to describe all the other properties (e.g., mass density, viscosity, or cross-sectional area) (White 1999). In fluids, this relationship is known as the thermodynamic condition or state equation. Considering \(S = f(p, T), \rho = f(p, T, p(x, t), T(x, t))\), it yields

\[
\int_{L} \rho S \left[ \frac{1}{S} \frac{\partial S}{\partial T} + \frac{1}{\rho} \frac{\partial \rho}{\partial T} \right] dT = 0
\]

Pipe circumferential deformation (first term of the first integral) and fluid compressibility (second term of the first integral) properties are a function of the fluid internal pressure \(p\); thus, these can be merged in the elastic wave celerity, \(a\), as follows (Chaudhry 1987):

\[
\frac{1}{a^2} = \rho \left( \frac{D}{eE} + \frac{1}{K} \right) = \rho \left( \frac{1}{S} \frac{\partial S}{\partial T} + \frac{1}{\rho} \frac{\partial \rho}{\partial T} \right)
\]

where \(K\)=fluid isothermal compressibility \(K = 1/\rho (\partial p/\partial T)_p, e = \text{pipe-wall thickness}, \) and \(E\)=pipe modulus of elasticity.

Pipe circumferential thermal expansion (first term of the second integral) and the fluid volumetric thermal expansion (second term of the second integral) are a function of temperature \(T\); the first can be described in terms of the linear expansion coefficient of the pipe material \(\alpha_L = 1/\rho (\partial \rho/\partial T)_p\), whereas the second is described by the coefficient of volumetric expansion of the fluid \(\beta_V = -1/\rho (\partial p/\partial T)_p\), as follows:

\[
2\alpha_L - \beta_V = \left[ \frac{1}{S} \frac{\partial S}{\partial T} + \frac{1}{\rho} \frac{\partial \rho}{\partial T} \right]
\]

Introducing Eqs. (14) and (15) into Eq. (13) and replacing pressure by piezometric head \(H\), the following global thermoelastic constraint condition is obtained:

\[
\int_{L} \frac{S}{\alpha^2} dH dx + \int_{L} \rho S(2\alpha_L - \beta_V) dT dt dx = 0
\]

\[
\int_{L} \rho S(2\alpha_L - \beta_V) dT dt dx = 0
\]
temperature- and pressure-dependent functions. For water in the liquid state at atmospheric pressure (ca. 1 MPa), the most variable parameters with temperature are \( \mu \) and \( \beta \), and density and isothermal compressibility are nearly constant.

Assuming that water volumetric expansion can be described by a power function of the type \( \beta_v(T) = AT^B \) (\( A, B \) constants), the pipe linear expansion coefficient \( \alpha_L \) is temperature independent, and temperature distribution varies linearly between initial \( T_0(x) \) and final \( T_f(x) \) steady-state conditions, time integration of Eq. (16) yields

\[
\int L \frac{\gamma S}{\alpha^2} [H_f(x) - H_0(x)] dx + \int L \left( \rho S \alpha_L [T_f(x) - T_0(x)] - \rho S A \frac{B + 1}{B} [T_f^{B+1}(x) - T_0^{B+1}(x)] \right) dx = 0
\]

(17)

**Solutions for Isothermal Flows**

Closed-loops with isothermal flows allow the development of simple analytical solutions. For this purpose, it is assumed that temperature changes due to fluid friction are negligible and there are no local head losses (i.e., real or equivalent head distribution varies linearly along each pipe axis). For this case, several simplified formulas are developed and general computational techniques are presented.

**Head Constraint Equation for Isothermal Flows**

In isothermal flows, mass distribution described by Eq. (16) will only depend on the elastic effects due to the internal pressure variation with time (elastic constraint condition)

\[
\int L \frac{dH}{dt} dx = 0
\]

(21)

where \( \alpha = \) an elastic coefficient defined by \( \alpha = S/a^2 \) assuming constant fluid density. For viscoelastic pipes, both Eqs. (16) and (21) are valid as long as the wave celerity is calculated with the static modulus of elasticity (long-term pipe behavior), as initial and final conditions correspond to steady-state flows (asymptote of creep function) and are not affected by the short time-dependency of creep function (Covas et al. 2004; 2005).

This elastic constraint condition is valid at each time since the initial \( (Q_0, H_0) \) until the final steady-state conditions \( (Q_f, H_f) \). According to the basic hypothesis, the elastic coefficient \( \alpha \) is a time-independent parameter, and no matter the transient head evolution in time along the pipe, the time integration of the elastic constraint equation between the initial and the final conditions will give the final head form

\[
\int L \alpha [H_f(x) - H_0(x)] dx = 0
\]

(22)

This head constraint equation resolves the indeterminacy of the closed-loop problem for this simplified case. In the equivalent BARS and knowing that the head distribution along each pipe is a continuous function, there is a pipe section \( x_s \) where \( H_f = H_0 \) (center of heads).
Simplified Formulas for Simple Systems

Uniform BARS Pipe
For the uniform elastic pipe, the nonzero parameter $\alpha$ can be removed and Eq. (22) will simplify to $\int L \Delta H(x) dx = 0$, where $\Delta H = H_i(x) - H_0(x)$. This corresponds to an antisymmetrical head distribution with Area 1 = Area 2, in which $\Delta H = 0$ for $x = x_i = 0.5L$ (center of heads at the midlength of the pipe) and $H_2 - H_0 = H_0 - H_1$ (Fig. 3).

The antisymmetrical head distribution is the basic and simplest solution for the closed-loop problem and means that the pump head $\Delta H^*$ will generate symmetrical heads above and below $H_0$.

For the ideal case of an incompressible fluid and rigid pipe, $\alpha$ is zero and the integral on Eq. (22) is null, and the elastic constraint cannot resolve the system indeterminacy; however, in real-life systems, the full rigid hypothesis is not reasonable.

Nonuniform BARS Pipe
For the nonuniform BARS, the pipe system can be decomposed in $i$ uniform pipe reaches with $\alpha_i$ and $L_i$. A simplified case is the BARS, with two pipes in series with lengths $L_1$ and $L_2$, cross-sectional areas $S_1$ and $S_2$, wave speeds $a_1$ and $a_2$, and head gradients $\phi_1$ and $\phi_2$, as depicted in Fig. 4.

For this case, the center of heads is obtained by the spatial integration of Eq. (21)

$$\int L \frac{\alpha}{\partial H}{dx} = 0: \alpha_1 \text{Area } 1 + \alpha_2 \text{Area } 2 = \alpha_3 \text{Area } 3 \quad \text{(see Fig. 4)}$$

After mathematical manipulations, the pipe section $x_s$, where $H_j = H_0$, (center of heads) is determined by (Fig. 4)

$$\frac{x_s}{L} = \frac{L_1}{L} \left( 1 + \frac{1}{\psi} \frac{\beta_2}{\beta_1} - \frac{\phi_1}{\phi_2} \right)$$

where

$$\beta = \frac{\phi_2}{a_1^2}$$

Table 1. Simplified Ring Systems (BARS) with $\Phi=1$: Sensitivity Analysis

<table>
<thead>
<tr>
<th>$\psi$</th>
<th>$\beta$</th>
<th>$x_s/L$</th>
<th>$\lambda$</th>
<th>Observation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1/2 $L$</td>
<td>1</td>
<td>Uniform ring</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>3/4 $L$</td>
<td>3</td>
<td>Half rigid ring (reach 1)</td>
</tr>
<tr>
<td>$\infty$</td>
<td>1/4 $L$</td>
<td>1/3 $L$</td>
<td>1</td>
<td>Half rigid ring (reach 2)</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>3/4 $L$</td>
<td>2</td>
<td>1/3 rigid ring (reach 1)</td>
</tr>
<tr>
<td>1/2</td>
<td>$\infty$</td>
<td>1/3 $L$</td>
<td>1</td>
<td>1/3 rigid ring (reach 2)</td>
</tr>
</tbody>
</table>

$$x_s = \frac{L_1}{L} \left[ 1 + \frac{1}{\psi} \frac{\beta_2}{\beta_1} - \frac{\phi_1}{\phi_2} \right]$$

$\frac{x_s}{L}$

$$\int L \frac{\alpha}{\partial H}{dx} = 0: \alpha_1 \text{Area } 1 + \alpha_2 \text{Area } 2 = \alpha_3 \text{Area } 3 \quad \text{(see Fig. 4)}$$

$$\frac{x_s}{L} = \frac{L_1}{L} \left( 1 + \frac{1}{\psi} \frac{\beta_2}{\beta_1} - \frac{\phi_1}{\phi_2} \right)$$

where

$$\beta = \frac{\phi_2}{a_1^2}$$

Under the hypothesis of uniform head gradient ($\Phi=1$), the heads at nodes 1 and 2 (pump nodes) obey the following head ratio $\lambda$ as a function of $x_s/L$:

$$\lambda = \frac{H_1 - H_0}{H_0 - H_2} = \frac{x_s}{L} \left( 1 - \frac{x_s}{L} \right)^{-1}$$

and the final $H_i$ linear distribution is fixed as a function of $\lambda$. In fact, replacing the first equation of the system of Eq. (5) with Eq. (29), it yields

$$\begin{bmatrix} +1 + \lambda \\ +1 - 1 \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = \begin{bmatrix} (\lambda + 1)H_0 \\ \Delta H_1(Q_i) \end{bmatrix}$$

As $\lambda > 0$ due to $x_s < L$, this system of equations, in opposition to Eq. (5), is determined; thus, final head values at nodes 1 and 2 can be calculated as well as head distribution along the pipe. This simplified hypothesis allows analysis of the influence of the pipe elastic properties on the head distribution should the head gradient be constant.

A brief sensitivity analysis regarding Eqs. (23) and (27) is presented in Table 1. Knowing $\lambda$, $H_1$, $H_2$, and the pipe profile, it is possible to obtain the maximum and minimum head and pressure values. For a horizontal pipe, the maximum pressure corresponds to node 1 and the minimum pressure to node 2.
This sensitivity analysis shows the following effects of pipe-elastic properties on the final head distribution along the system: (1) the stiffer is the upstream part of the pipe system ($\beta<1$), the further away from node 1 in section $x_j$, resulting in $\alpha>1$ or $H_1-H_0>H_0-H_2$; (2) the location of the ring center of heads ($x_j$) will impose a greater or lesser final pressure at one or other side of the pump (nodes 1 and 2); and (3) the location of the ring center of heads and the initial head $H_0$ are the determining parameters for the final maximum and minimum pressures.

**General Solution for Multipipe Systems**

In engineering practice, the closed-loop can be a complex pipe system and the transformation into the equivalent BARS configuration may not be practical for design purposes. For isothermal flows, two computational techniques can be applied to resolve this undetermined problem: (1) the conditioned steady-state analysis; or (2) the pseudotransient elastic analysis. The first technique is based on the general steady-state continuity and energy equations coupled with the elastic constraint equation. The second is the use of a computer model developed for water hammer (or pressure transients) analysis applied to the time-marching process toward the final steady-state solution. In this case, the deformation of pipe and fluid are implicitly considered in the differential equations, and the solution can be obtained.

**Conditioned Steady-State Analysis**

The integral in Eq. (22) can be transformed into a general algebraic equation and applied to a set of uniform pipes. Assuming the equivalent continuous head loss along each uniform pipe, the generalized elastic constraint equation can be written as follows:

$$
\sum_{p=1}^{n} \sum_{i=1}^{N} \beta_{p,i} h_{p,i} + \sum_{j=1}^{M} \left( \sum_{i=1}^{N} \beta_{i,j}^* \right) h_j = 0
$$

(31)

where $E_{np}$ is the number of nonpipe link-elements (i.e., pumps); $N$ is the number of junction nodes; $M$ is the number of pipes connected to node $j$ (excluding pump nodes); $h_j$ is the relative head at node $j$ defined by $h_j = H_j - H_0$ (excluding pump nodes); $h_{p,i}$ is the relative head at a pump node (pump $p$ at upstream node 1 or downstream node 2); and $eta_{i,j}^*$ is the pipe characteristic coefficient of each pipe $i$ connected to node $j$ (junction) or node $i$ (pump node)

$$
\beta_{i,j}^* = \left( \frac{L_i S_i}{a_{i,j}} \right) = \beta_{i,j}^*
$$

(32)

Eq. (31) is equivalent to Eq. (18), but with a simpler form that can be directly added to the system of head equations and will resolve the problem indeterminacy for any combination of pipes in series and in parallel. Node heads $H_i$ can be obtained by solving the set of head equations with one of them replaced by the elastic constraint Eq. (31). No matter how complex the system is, this constraint equation is sufficient to make the problem solvable.

Considering the pipe system depicted in Fig. 5, with one pump ($E_{np}=1$), nine pipes and nine nodes ($N=9$), when applying Eq. (31), it yields

$$
\begin{align*}
\beta_{p,1}^* h_{p,1} + (\beta_{1,1}^* + \beta_{2,1}^*) h_2 + (\beta_{1,2}^* + \beta_{3,2}^* + \beta_{1,3}^*) h_{3} + (\beta_{1,4}^* + \beta_{2,4}^*) h_4 \\
+ (\beta_{3,5}^* + \beta_{2,5}^*) h_5 + (\beta_{3,6}^* + \beta_{4,6}^* + \beta_{5,6}^*) h_{6} + (\beta_{4,7}^* + \beta_{2,7}^*) h_7 \\
+ (\beta_{5,8}^* + \beta_{2,8}^*) h_8 + \beta_{5,9}^* h_{9} = 0
\end{align*}
$$

(33)

Knowing pipe characteristic coefficient for each pipe $\beta_{p,i}^*$, the linear equation can be added to the set of ($N-1$) node head equations, and final head distribution can be obtained. Due to the typical $a_i$ and $S_i$ values, pipe parameters $\beta_{i,j}^*$ can have a very small order of magnitude, and for complex pipe systems with several nodes and secondary loops, this technique needs an efficient numerical procedure in order to solve the set of equations. However, Eq. (31) coupled with a general steady-state analysis code allows the final solution for any closed-loop, multipipe system (for isothermal flows).

**Pseudotransient Elastic Analysis**

A pressure-transient solver, based on the elastic continuity and the momentum equation, is also a very good solver platform for the closed-loop problem in isothermal flows. This model controls fluid compressibility and pipe circumferential deformation through the continuity equation and the wave speed parameter. It is basically oriented for the unsteady-state flow analysis, but it has also been used for steady-state analysis (time-marching technique or pseudotransient) as a means of obtaining the final solution based on any initial approximate condition (Shimada and Okushima 1984). The flow establishment process from the initial hydrostatic ($Q_0=0$ and $H=H_0$) condition until the final solution can be obtained by a numerical method (e.g., method of characteristics). Current computer models incorporate different types of components in complex pipe networks.

**Examples and Sensitivity Analysis**

This section presents three examples with the application of proposed solutions to solve the BARS problem: the first two refer to isothermal flows, whereas the third illustrates a flow with an increasing-decreasing linear temperature variation for different pipe materials.

**Example 1: Nonuniform BARS with Two Pipes in Series (Isothermal Flow)**

Consider a single closed-loop system composed of two smooth-wall ($K_s=100$ m$^{0.5}$/s$^{1/2}$) horizontal pipes in series (pipes 1 and 2) with $L_1 = L_2 = 100$ m, $D_1 = D_2 = 100$ mm, $S_1 = S_2 = 0.00785$ m$^2$, $a_1 = 1,000$ m/s, and $a_2 = 800$ m/s, and a pump with the head curve $H = -40,000 Q^2 + 36$ with $H(m)$ and $Q(m^2/s)$, as presented in Fig. 4. The initial steady state is the hydrostatic (no-flow) condition with $H_0=30$ m (the datum level is the pipe axis). At a certain time, the pump starts up and a new steady state is attained.
The variation of the center of heads $x_i/L$ with the ratio $\sqrt{\beta}=a_i/a_1$, for $\gamma=1$ and $\Phi=1$, given by Eq. (27), is presented in Fig. 6(a). This function has two asymptotes ($x_i/L=0.25$ and $x_i/L=0.75$) corresponding to extreme conditions in which one of the pipes (half of the ring) is considered as rigid ($a_1$), and the other is elastic with wave speed $a_2$ significantly lower than $a_1$ ($a_1/a_2=\infty$). In these cases, if only half of the ring is elastic, only that part will contribute to the head distribution according to Eq. (21), and the center of heads is at the middle of the pipe. This conclusion is consistent with results in Table 1. Fig. 6(b) presents the solution of the BARS with two pipes in series with the head distribution along the pipeline for the average and extreme conditions referred to above ($a_1/a_2=0, 1.0, \infty$). The full rigid hypothesis is a theoretical and an extreme case, as in practice the fluid is compressible and maximum wave speed is $a=\sqrt{K/p}$ (for water at $T=20^\circ$C, $a=1,400$ m/s).

**Conditioned Steady-State Analysis**

The solution by conditioned steady-state analysis requires the simultaneous resolution of custom steady-state equations and the generalized elastic constraint (Eq. (31)). Should the problem be formulated in terms of $H$-equations for which the primary set of unknown variables is the head throughout the pipe system, one $H$-equation is written at each junction node, constituting a system of $N$ equations. The discharge is generally defined as a function of the head loss along each pipe $Q_i=\rho/(H_f-H_i)^{1/n}$. For the current case, there are three $H$-equations ($N=3$). Writing the continuity equations at nodes 1, 3, and 2, it yields

$$Q_1 = Q_2 = Q_3 = 0 \Rightarrow F_1 \equiv Q_1 - C_1 (H_3 - H_1)^{1/n1} = 0$$

(35a)

$$Q_1 - Q_3 = 0 \Rightarrow F_2 = C_1 (H_3 - H_1)^{1/n1} - C_3 (H_3 - H_2)^{1/n1} = 0$$

(35b)

$$Q_2 - Q_3 = 0 \Rightarrow F_3 = C_2 (H_2 - H_3)^{1/n1} - Q_3 = 0$$

(35c)

where

$$Q_0 = \frac{-B + \sqrt{B^2 - 4A[C - (H_2 - H_1)]}}{2A}$$

(36)

Eq. (35c) is linearly dependent on the other two (35a), (35b) and is therefore redundant, and the additional constraint equation is needed for the solution of the problem in Eq. (31)

$$F_3 = \beta_{13} (H_1 - H_0) + \beta_{12} (H_2 - H_0) + (\beta_{13} + \beta_{23})(H_3 - H_0) = 0$$

(37)

The set of Eqs. (35a), (35b), and (37)—in matrix notation $\mathbf{F}(\mathbf{X})=0$—is now determined and can be solved by a numerical method. By the Newton-Raphson method, the solution can be iteratively obtained by $\mathbf{X}_{n+1} = \mathbf{X}_n + \mathbf{J}^{-1}(\mathbf{X}_n)\mathbf{F}(\mathbf{X}_n)$ where $\mathbf{X}=$ column vector of unknowns (heads), $\mathbf{F}(\cdot)$ is the column vector of equations, $\mathbf{J}^{-1}(\cdot)=\text{inverse of the matrix } \mathbf{J}$, and $\mathbf{J}(\cdot)=\text{Jacobian}$, which is the matrix of the derivatives defined as $J_{ij} = \partial F_j/\partial x_i$. Substituting $\mathbf{X}$, $\mathbf{F}(\cdot)$, and $\mathbf{J}^{-1}(\cdot)$ by the corresponding system of equations, it yields

$$\mathbf{X} = [H_1 H_3 H_2]^T$$
Starting with \( X(0) = [40.00, 35.01, 30.02, 0.00]^T \) and setting the convergence criteria in the relative head change equal to 0.0001, the final solution is found. Fig. 7(a) shows the initial and final hydraulic grade line (HGL) along the pipeline.

**Pseudotransient Method**

The pseudotransient method is a very efficient and simple technique as long as a pressure transient solver is available. The topology and physical characteristics of the system defined as input data and the no-flow hydrostatic head \( Q = 0 \) and \( H_0 = 30 \text{ m} \) along the pipe ring are set as the initial conditions \( t = 0 \) s. The solver is run for a certain period of time \( \Delta T \) until the new final steady state is attained [Fig. 7(b)]. In the current case, the in-house software developed at Instituto Superior Técnico was used. The initial no-flow hydrostatic condition was maintained during 5 s, and at \( t = 5 \) s the pump was started and the program was run for \( \Delta T = 20 \) s. The results of the simulations in terms of heads at nodes are presented in Fig. 7(b), and these are consistent with results and accuracy obtained for the conditioned steady-state analysis [Fig. 7(a)].

**Example 2: Nonuniform BARS with Four Pipes in Series (Isothermal Flow)**

Consider a more complex system with four pipes in series with different physical characteristics presented in Table 2 and five nodes. The pump (defined between nodes 1 and 5) has the same characteristic head-curve as that of Example 1. The initial steady state is the hydrostatic condition with \( H_0 = 30 \text{ m} \) (the datum level is the pipe axis).

The flow rate can be straightforwardly calculated by Eq. (34), in which \( \Sigma R_k \) corresponds to the summation of \( R_k = L_k / (K_{0k} S_k R_{i0k}^2) \) of the four pipes in series: \( Q = 0.0228 \text{ m}^3/\text{s} \). Heads along the pipe system \( H_j \) with \( i = 1 \ldots 5 \), as well as upstream and downstream of the pump \( H_1 \) and \( H_5 \), will be determined only by conditioned steady-state analysis (simplified formulas cannot be applied, and the pseudotransient method can be used but will not add any further information to this example).

**Conditioned Steady-State Analysis**

For the current case, there are five \( H \)-equations. Writing the continuity equations at nodes 1 to 4 (node 5 is excluded as the equation is linearly dependent), it yields

\[
\begin{align*}
\text{Node 1:} & \quad Q_1 - Q_{12} = 0 \Rightarrow F_1 = Q_1 - C_{12}(H_2 - H_1)^{1/n_{12}} = 0 \\
\text{Node 2:} & \quad Q_{12} - Q_{23} = 0 \Rightarrow F_2 = C_{12}(H_2 - H_1)^{1/n_{12}} - C_{23}(H_3 - H_2)^{1/n_{23}} = 0 \\
\text{Node 3:} & \quad Q_{23} - Q_{34} = 0 \Rightarrow F_3 = C_{23}(H_3 - H_2)^{1/n_{23}} - C_{34}(H_4 - H_3)^{1/n_{34}} = 0 \\
\text{Node 4:} & \quad Q_{34} - Q_{45} = 0 \Rightarrow F_4 = C_{34}(H_4 - H_3)^{1/n_{34}} - C_{45}(H_5 - H_4)^{1/n_{45}} = 0
\end{align*}
\]

![Fig. 7. Nonuniform BARS with two pipes in series: (a) initial and final head distributions along the pipe; (b) results of the pseudotransient simulation (Example 1)](image-url)
Flow rate $Q_p$ is given by Eq. (36), replacing $H_2$ by $H_3$. The fifth equation is the generalized head constraint Eq. (31)

$$F_3 = (B_{12}^* - H_0) + (B_{12} + B_{23}^*) (H_2 - H_0) + (B_{34}^* + B_{34}) (H_3 - H_0)$$

The set of equations has been solved by the Newton-Raphson method, in which the unknowns are heads at the five nodes. Fig. 8 shows the HGL along the ring, and these results deserve the following comments:

1. The slope of the HGL varies from pipe to pipe and increases with the increase of $K_S$ and the decrease of the diameter (Table 2): pipes 2–3 and 4–5 have the highest slopes; and

2. The center of the heads is located at $x=58.28$ m, that is, closer to the downstream end of the pump (between nodes 2 and 3); this is because pipe 1–2 has the highest contribution for the integral in Eq. (22), i.e., $a_{1.2} \gg a_{i.i+1}$ (i.e. 1) (Table 2), and its part is almost the same as the sum of the other three.

**Example 3: Uniform Cooling System with Linear Temperature Variations**

Consider a uniform BARS with $L=200$ m, $D=100$ mm, $S=0.00785$ m$^2$, and a pump with the head-curve presented in Example 1. Initial steady state is described by a constant temperature $T_0=20^\circ$C (293.15 K) and head $H_0=30$ m, and final conditions by an increasing temperature between $T_1=20^\circ$C (293.15 K) and $T_2=30^\circ$C (303.15 K), in pipe 1, and decreasing temperature between $T_2$ and $T_1$, in pipe 2 [Fig. 9(b)]. Friction losses and the hydraulic grade line will depend on temperature distribution, as viscosity is temperature dependent [Fig. 9(a)]. Assuming $\rho=1.000$ kg/m$^3$, $K=50 \times 10^{-11}$ Pa$^{-1}$, and the temperature-dependent water properties are described by $\beta_1(K^{-1})=2 \times 10^{-17}T^{5.316}$ and $\mu(\text{kg/m/s})=7 \times 10^{10}T^{-5.2081}$.

![Fig. 8. Nonuniform BARS with four pipes in series: initial and final head distributions (Example 2)](image)

![Fig. 9. Uniform BARS with nonuniform linear temperature variation: (a) head; (b) temperature distributions (Example 3)](image)

![Fig. 10. Examples of water thermal expansion and dynamic viscosity variations with temperature [adapted from ThermExcel (2004)]](image)
4. Uniform pipes (i.e., with the same elastic coefficient $\alpha=S/\rho A^2$) without any temperature variation have the same head distribution, independently of the pipe material; and

5. The nonlinearity of the head distribution is not visible in Fig. 11 as temperature variation is relatively small (only $10^\circ C$) and viscosity is almost constant.

Fig. 12 shows (a) the HGL and (b) the head losses $\Delta H$ and the hydraulic gradient $J$ obtained for a PE pipe with $T_1=0^\circ C$ and $T_2=100^\circ C$ [Fig. 9(b)]. In this case [Fig. 12(a)], the HGL is higher than in Fig. 11 because the temperature variation is much higher (ca. $80^\circ C$). The nonlinearity of the HGL due to viscosity dependence on temperature is clear in Fig. 12(b), and the hydraulic gradient $J$ curve is similar (although not linear) to the temperature variation with a maximum associated with the maximum temperature $T_2$ (at midlength of the pipe as $L_1=L_2$).

**Concluding Remarks**

The analysis of the closed-loop was motivated by a practical problem that is frequently posed in the design of industrial systems and leads to a theoretically undetermined head distribution when using steady-state continuity and energy equations only. The indeterminacy of the closed-loop problem can be resolved by the thermoelastic constraint Eq. (16)—an additional mass balance equation that takes into account pipe circumferential deformation and fluid compressibility due to the initial and final pressure and temperature distributions. Analytical algebraic solutions [Eqs. (17)-(20)] have been developed for linear temperature variations in time (between initial final conditions) and along the pipe axis.

For isothermal flows, three different equivalent methods can be applied to numerically solve the problem: (I) simplified formulae for the calculation of the location of the center of heads $x_i$ [Eqs. (23) and (27)] and the ratio of the head values at pump nodes $\lambda$ [Eq. (29)]; (II) conditioned steady-state analysis with the general head constraint equation valid for any type of closed-loop system [Eq. (31)]; and (III) pseudotransient analysis to solve final steady state given an initial hydrostatic condition. These solutions assume that the temperature variation due to fluid viscous shear stress is negligible (this is true in water pipe systems).

Method (I) can only be applied to systems with one or two pipes in series, whereas (II) and (III) can be applied to any multipipe system. Although (III) is the most efficient and easy to use, given the readily available transient codes, (I) allows a better understanding of the problem and the effect of system elastic/physical properties on the final solution, and (II) involves an additional equation [Eq. (31)], which can be incorporated as an up-grade to steady-state codes to solve the closed-loop problem. Finally, the blind-use of (III), without the understanding of the essence of the problem, is like using a black box code that outputs meaningless results and can mislead one from the final solution.

In closed-loops with no reservoirs to maintain head and compensate for mass changes, final heads will depend on the spatial...
distribution of initial and final pressures and temperatures, as well as the thermoelastic properties of the pipe and fluid. For isothermal flows in uniform single loops, the head distribution is such that the increase of head downstream of the pump is compensated for by an equal head decrease upstream of the pump. A temperature distribution will change the solution: an increase from an initial to a final steady-state temperature will raise the whole head distribution along the pipe system when compared to the solution for isothermal flow. The higher the head distribution is, the higher will be the temperature increase and the stiffness of the pipe material. Flexible and highly deformable pipes, for small temperature changes, have head distributions similar or very close to isothermal flows, whereas rigid pipes, even for small temperature variations, may significantly increase maximum pressures.

The knowledge of the head final distribution along pipes is very important for the specification of pipe materials, pressure classes, and wall-thicknesses, as well as maximum allowable temperature changes, in order not to exceed maximum resistant pressures and consequently avoid pipe bursts. Low final pressures can also be an operational problem, causing cavitation, pipe wall instability, unwanted pollution, or air intrusion into the pipe.

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Notation

The following symbols are used in this paper:

\( A, B, C \) = coefficients or constants (\(-\));

\( a \) = elastic wave celerity (m/s);

\( b \) = exponent of temperature-dependent head losses \( J(x)=CT^b \) (\(-\));

\( D \) = pipe internal diameter (m);

\( E \) = number of link-elements (pipes, pumps, and valves) (\(-\));

\( E_{np} \) = number of nonpipe link-elements (\(-\));

\( e \) = pipe wall thickness (m);

\( H \) = piezometric head or simply head (m);

\( h_i \) = relative head at node \( i \) defined by \( h_i=H_i-H_0 \) excluding pump nodes (m);

\( h_{p,i} \) = relative head at pump node (pump \( p \) between nodes 1 and 2) (m);

\( J \) = hydraulic gradient (i.e., head loss per unit length) \( J=g\phi/\cos \theta \) (\(-\));

\( K \) = bulk modulus of elasticity of fluid \( K=\frac{1}{\rho}\frac{d}{d\phi} \) (Pa\(^{-1}\));

\( K_S \) = Gauckler-Manning-Stickler coefficient (m);

\( L \) = pipe length, total system length (m\(^{1/3}\) s\(^{-1}\));

\( M \) = total fluid mass (\(-\));

\( M_j \) = number of pipes connected to node \( j \), excluding pump nodes (\(-\));

\( m, m' \) = slope of temperature or head variation along pipe axis (\(-\));

\( N \) = number of junction nodes (\(-\));

\( n \) = exponent of discharge in head loss formula (\(-\));

\( P_L \) = circumferential perimeter of pipe cross-section (m);

\( p \) = pressure (Pa);

\( Q_c \) = demand or water loss assigned to node \( i \) (m\(^3\)/s);

\( Q_k \) = flow rate converging to (or diverging from) a node through pipe \( k \) (if flow converges to the node, this flow rate is positive) (m\(^3\)/s);

\( Q_s \) = final steady-state flow-rate in a single loop (m\(^3\)/s);

\( R \) = pipe radius (m), number of fixed-head nodes (\(-\));

\( R_k \) = pipe hydraulic resistance [when using Gauckler-Manning-Stickler formula for headloss calculation, \( R=L/(K_S R_k^{1/2}) \) (m\(^3\)/s\(^{-2}\));

\( S \) = pipe cross-sectional area (m\(^2\));

\( S_C \) = surface of control volume (m\(^2\));

\( S_L, S_T \) = lateral or transversal surfaces of the control volume, respectively (m\(^2\));

\( T \) = temperature (K);

\( t \) = time (s);

\( V \) = fluid velocity (m/s\(^{-1}\));

\( V_L, V_T \) = velocity through lateral and transversal control surfaces, respectively (m/s\(^{-1}\));

\( x \) = distance along pipe axis (m);

\( x_c \) = position of center of heads (m);

\( \alpha \) = elastic coefficient \( \alpha \) defined by \( \alpha=S/a^2 \) (s\(^2\));

\( \alpha_L \) = linear expansion coefficient of pipe material \( \alpha_L=\frac{1}{E} \) (K\(^{-1}\));

\( \beta_L \) = volumetric expansion coefficient of fluid \( \beta_V=1/\rho \) (K\(^{-1}\));

\( \beta^* \) = characteristic coefficient of pipe defined by \( \beta^*=LS/a^2=La \) (ms\(^3\));

\( \gamma \) = fluid specific weight (N/m\(^3\));

\( \Delta H' \) = head change through nonpipe elements (m);

\( \Delta W_S \) = difference of level in water stored in pseudoloops (m);

\( \theta \) = angle of pipe axis with horizontal axis (rad);

\( \mu \) = dynamic viscosity (kg/ms);

\( \nu \) = cinematic viscosity (m\(^2\)/s);

\( \rho, \rho_i \) = fluid mass density (kg/m\(^3\));

\( \Phi \) = dimensionless parameter defined by \( \Phi=\psi/\phi \) (\(-\));

\( \phi_i \) = angle of hydraulic grade line in pipe \( i \) with horizontal axis (rad); and

\( \psi \) = dimensionless parameter defined by \( \psi=L_2/L_1 \) (\(-\)).

Subscripts

\( f \) = final steady-state condition;

\( i, m, k \) = pipe or node \( i, m, \) or \( k \);

\( ij \) = pipe between nodes \( i \) and \( j \);

\( P, V \) = pump or valve;

\( 0 \) = initial steady-state condition; and

\( 1, 2 \) = pipe or node 1 or 2.

References

