Analysis of plates using an indirect Trefftz collocation method

Vitor A. Leitão and Carlos T. Fernandes
Departamento de Engenharia Civil
Instituto Superior Técnico
Av Rovisco Pais, 1049-001 Lisboa, Portugal
e-mail: vitor@civil.ist.utl.pt
ph: ++351 1 8418234
September 10, 2003

Abstract
In this work a multi-region indirect Trefftz collocation method for the analysis of bending of thin homogeneous plates is used to study plates subjected to non-trivial loading conditions. Techniques are discussed for the accurate implementation, in the context of indirect Trefftz methods, of linearly varying line and polygonal-shaped loads. Numerical examples are presented to illustrate the performance of the Trefftz formulation for thin plate bending problems for non-trivial load cases. Comparisons are made with other results available in the literature.

1 Introduction

The classical Trefftz method [1] consists in the solution of a partial differential equation by the superposition of a number of functions, which are themselves solutions of the homogeneous governing equation, appropriately scaled by a number of unknown parameters. These unknowns are then obtained from the approximate satisfaction of the boundary conditions by means of collocation or in an weighted residual sense.

The main characteristic of the Trefftz methods is, thus, the use of trial functions that satisfy, in a certain region (locally), all the governing differential equations of the problem. Reviews on the subject may be found in volume 24 of the journal Advances in Engineering Software, 1995, which was devoted to the Trefftz method.

A brief review of the main aspects of several Trefftz-based formulations and applications of the Trefftz approach to plate bending was presented by the authors in [2].

The purpose of this paper is to describe and to illustrate the application of a multi-region indirect Trefftz formulation for the analysis of plates subjected to non-trivial loading cases such as linearly varying line and polygonal-shaped loads. Details of the multi-region indirect Trefftz formulation are described in Leitão and Fernandes [2].

2 Bending of thin plates

The differential equation of the deflection surface of an homogeneous, isotropic, arbitrary thin plate under bending is the well known Lagrange equation:

\[ \nabla^4 w = \frac{p}{D} \]  

(1)
where:

- $w$ is the deflection of the middle surface of the plate;
- $\nabla^4$ is the biharmonic operator;
- $D = \frac{E t^3}{12 (1 - \nu^2)}$ is the flexural rigidity of the plate.

By neglecting the effect of deformation due to the shear stresses it is possible to identify the dependency between an additional shear force, $Q_n$, and $\frac{\partial M_{nt}}{\partial s}$ (the derivative of the twisting moment per unit length of a section of a plate normal to direction $n$). Thus, for each point on the boundary, two, of the following four, conditions are imposed:

$$w = w$$  \hspace{1cm} (2)
$$\frac{\partial w}{\partial n} = \frac{\partial w^*}{\partial n}$$  \hspace{1cm} (3)
$$M_n = M_n$$  \hspace{1cm} (4)
$$Q_n + \frac{\partial M_{nt}}{\partial s} = V_n$$  \hspace{1cm} (5)

where:

- $M_n$ is the bending moment per unit length of a section of a plate perpendicular to $n$ direction;
- $Q_n$ is the shearing force parallel to $z$ axis per unit length of section of a plate perpendicular to $n$ direction;
- $V_n = Q_n + \frac{\partial M_{nt}}{\partial s}$ is the effective shear force parallel to the $z$ axis of a section of a plate perpendicular to $n$ direction.

It should be noted that at each point on the boundary only the displacement (rotation) or the effective shear force (bending moment) may be imposed but not both simultaneously.

In case multi-regions are used it is necessary to enforce the continuity (of the deflections) and the equilibrium between the regions. Therefore the following boundary conditions have to be satisfied at the collocation points (CP’s) on the interfaces:

$$w_j = w_{j+1}$$  \hspace{1cm} (6)
$$\frac{\partial w_j}{\partial n} = -\frac{\partial w_{j+1}}{\partial n}$$  \hspace{1cm} (7)
$$M^j_n = M^{j+1}_n$$  \hspace{1cm} (8)
$$V^j_n = -V^{j+1}_n$$  \hspace{1cm} (9)

where $\Omega_j$ and $\Omega_{j+1}$ are two regions for which $\Gamma_j \cap \Gamma_{j+1} = \Gamma_I$.

Once the displacement field is known it is straightforward to evaluate the shear forces, the bending and the twisting moments.

### 3 Approximating functions of the deflection field

The linearity of the differential equations (of the plate bending problem) allows for the deflection field to be expressed as the sum of the solution of the homogeneous part ($w_h$) and a particular solution of the nonhomogeneous part ($w_p$). The complex form representation is a convenient way of describing the infinite set of linearly independent trial functions which form the complete basis of solutions of the governing differential equations of the plate bending problem.

The homogeneous solution used here is part of a general solution for thick plates given by Piltner [3] (neglecting the terms involving $h^2$, $z^2$, and $z^3$) and takes the form:

$$w_h = N_c c,$$  \hspace{1cm} (10)
where:

\[ N_w = \frac{1 + \nu}{E} \begin{bmatrix} N_0 & N_1 & \cdots & N_k & \cdots & N_l \end{bmatrix} \]

and

\[ N_0 = \begin{bmatrix} x & y & 1 \end{bmatrix}, \]

\[ N_1 = \begin{bmatrix} r^2 \end{bmatrix}, \]

\[ N_k = \begin{bmatrix} r^2 \Re (\zeta)^k & -r^2 \Im (\zeta)^k & \Re (\zeta)^k & -\Im (\zeta)^k \end{bmatrix}, \]

with \( \zeta = x + iy \) and \( r^2 = x^2 + y^2 \).

The number of terms in \( c \) is \( n = 4l \). The three first terms are the rigid body motions (two rotations and one translation). Note that a different number of the expansion terms, for each region, can be used.

**Particular solutions**

For the trivial types of loading such as uniformly distributed and point loads the solutions are well known.

Considering \( p \) as the value of the distributed load the particular solution is:

\[ w_p = \frac{pr^4}{64D} \]

and for a point load, \( P \), located at \((x_P, y_P)\), the particular solution may be taken as

\[ w_p = \frac{Pr^2 \ln r_P^2}{16\pi D} \]

where \( r_P^2 = (x - x_P)^2 + (y - y_P)^2 \).

For other types of loading, such as linearly varying line and polygonal-shaped loads, the particular solutions are more difficult to obtain.

The technique normally used is the integration in the loaded region of the fundamental solution for a unit point load in an infinite plate; this integration may be carried out numerically or analytically. Costa and Brebbia [4], Hartmann [5], Burgess and Mahajerin [6] and Abdel-Akher and Hartley [7] are some of the authors that have used this technique in the fields of BEM and BIE. An application of the fundamental solution integration in the area of Trefftz methods has been done by Venkatesh and Jirousek [9].

None of the previous results, for one reason or another, could be used, as they were, for the indirect Trefftz formulation being described here. It was, therefore, necessary to adapt the integration schemes in a suitable manner. Details of the implementation may be found in [10] and [11].

**4 Enforcing the boundary conditions**

The deflection field may be, as seen in the previous section, expressed as the sum of the particular and the complementary solutions:

\[ w = w_p + N_w c. \quad (11) \]

The unknowns of the problem, that is, the \( c \) coefficients, are found after imposing the boundary conditions at selected points on the boundary. At each collocation point located on the boundary, two equations are formed.

After carrying out the necessary derivatives, the following equations are obtained:

\[ N_w c = \overline{w} - w_p, \quad (12) \]

\[ N_{\overline{w}} c = \frac{\partial \overline{w}}{\partial n} - \frac{\partial w_p}{\partial n}, \quad (13) \]

\[ N_M c = \overline{M} - M_{np}, \quad (14) \]

\[ N_V c = \overline{V} - V_{np}. \quad (15) \]
Let $n_w, n_{\frac{\partial w}{\partial n}}, n_M, n_V$ denote the number of CP where the quantities $w, \frac{\partial w}{\partial n}, M, V$ are known.

If $m > n$ an overdetermined system of equations is obtained. A criterion is necessary in order for a solution to be obtained. In this approach the coefficients defining the displacement field are found by the least squares method, that is, the minimization of the sum of the square of the residuals at each CP:

$$\text{Minimize } \left\{ W^2_w \sum_{i=1}^{n_w} (w_i - w_i^*)^2 + W^2_{\frac{\partial w}{\partial n}} \sum_{i=1}^{n_{\frac{\partial w}{\partial n}}} \left( \frac{\partial w_i}{\partial n_i} - \frac{\partial w_i^*}{\partial n_i} \right)^2 + W^2_M \sum_{i=1}^{n_M} (M_i - M_i^*)^2 + W^2_V \sum_{i=1}^{n_V} (V_i - V_i^*)^2 \right\}$$

(16)

The weights $W_w, W_{\frac{\partial w}{\partial n}}, W_M, W_V$ are used to tune the relative strength of the different boundary conditions and, also, to restore the homogeneity of the physical dimensions. The minimization of (16) in order to $c$ leads to the following system of equations:

$$\left( D^TD \right) c = D^T \left( \bar{d} - d_p \right)$$

(17)

where matrix $D$ contains all the terms of the series $N_w, N_{\frac{\partial w}{\partial n}}, N_M, N_V$; $\bar{d}$ contains the bc’s, and $d_p$ represents the particular solutions values at the CP’s.

Each equation of system (17), appropriately multiplied by its associated weight, represents a boundary condition being enforced at a particular CP where each column of $D$ represents the influence of a given function (of the set of trial functions being used) at that particular CP. Note that $D$ is not a square matrix.

Once the displacement field is known it is straightforward to evaluate the shear forces, the bending and the twisting moments.

5 Numerical tests

The numerical tests here reported concern only the cases of plates loaded in a non-trivial way. The cases of two uniformly loaded rectangular zones and of a linearly varying line load were analysed. For more trivial types of loading the reader is referred to Ref. [2].

Rectangular plate subjected to two uniform rectangular loads

Consider the rectangular plate represented in Figure 1, where $b = 2a$, subjected to two uniform rectangular loads. The resulting load is $\bar{P}$ and the value of the uniform load in each zone is $p = \frac{\bar{P}}{b^2}$.

The results obtained by the present method are compared to the results obtained by using the Navier solution, that is, a double series solution which, for the case a unique rectangular load, is given by Timoshenko [8]; it is worth mentioning that Venkatesh and Jirousek [9] have also analysed and obtained good results for this case using a mesh of 12 hybrid Trefftz elements. The coefficients of the series take the values:

$$a_m = \frac{16}{a b u v} \sin \frac{m \pi \xi}{b} \sin \frac{n \pi \eta}{a} \sin \frac{m \pi u}{2b} \sin \frac{n \pi v}{2a}$$

(18)

where $m = 1, 2, 3, \ldots$ and $n = 1, 2, 3, \ldots$.

Constant values $u, v, \xi$ and $\eta$ are defined in Figure 2 and $\bar{P}$ is the total load, $\bar{P} = u v \bar{p}$.

The solution for a rectangular plate subjected to two uniform rectangular loads is then obtained by superposing the Navier series for both rectangular loads.

In Table 1 the results, the deflection at the center and the components of the moment tensor at point $E$, obtained by the present method (with 20 collocation points per side and a number of 20 x 4 terms of the infinite series which constitutes the general solution of the homogeneous plate bending problem thus leading to a total of 80 unknowns) and
Figure 1: Rectangular plate subjected to two uniform rectangular loads.

Figure 2: Rectangular plate subjected to one uniform rectangular loads.
Table 1: Values of $w^{adim \ A}$, $M^{adim \ E}_{xy}$, $M^{adim \ E}_{xx}$ and $M^{adim \ E}_{yy}$ for a rectangular plate subjected to two uniform rectangular loads

<table>
<thead>
<tr>
<th></th>
<th>$w^{adim \ A}$</th>
<th>$M^{adim \ E}_{xy}$</th>
<th>$M^{adim \ E}_{xx}$</th>
<th>$M^{adim \ E}_{yy}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.139705</td>
<td>0.73549</td>
<td>0.49685</td>
<td>0.78522</td>
</tr>
<tr>
<td>14</td>
<td>0.134859</td>
<td>0.75186</td>
<td>0.49590</td>
<td>0.80496</td>
</tr>
<tr>
<td>18</td>
<td>0.134665</td>
<td>0.75783</td>
<td>0.49570</td>
<td>0.80591</td>
</tr>
<tr>
<td>22</td>
<td>0.134657</td>
<td>0.75715</td>
<td>0.49585</td>
<td>0.80604</td>
</tr>
<tr>
<td>Series</td>
<td>0.134659</td>
<td>0.75727</td>
<td>0.49593</td>
<td>0.80601</td>
</tr>
</tbody>
</table>

Figure 3: Rectangular plate subjected to linearly varying line load.

by the use of the Navier solution are shown. It should be mentioned that the following scales are used for, respectively, the deflection $w^{adim} = \frac{w}{D}$ and the components of the moment tensor $M^{adim \ ij} = \frac{M_{ij}}{E \pi^2}$.

Rectangular plate subjected to linearly varying line load

Consider the plate represented in Figure 3 which is subjected a linearly varying load along segment $FH$. The total load is $\overline{PF}$ and the values of the load at $F$ and $H$ are, respectively, $p_F$ and $p_H$.

Even though the use of a single-region would lead to sufficiently accurate results it was decided to study the effect of mesh distortion when two sub-regions are considered. Figure 4 shows the mesh (where 10 collocation points per side and a total of $15 \times 4$ terms of the series were considered) and the distortion parameter $\gamma$ used.

For comparison terms it was necessary to determine a double series solution by integrating the point load along the line. After expressing the load in the global system of coordinates by $p(x) = p_0 + p_1 x$, the coefficients of the series take the values:

$$a_{m \, n} = \frac{4}{ab} \int_{\xi - \frac{a}{2}}^{\xi + \frac{a}{2}} (p_0 + p_1 x) \sin \frac{m \pi x}{b} \sin \frac{n \pi y}{a} \, dx. \quad (19)$$

where $v = 0$ and constant values $u$, $\xi$ and $\eta$ are shown in Figure 3. This expression may be decomposed into two parts, in the form $a_{m \, n} = a_{m \, n}^{p_0} + a_{m \, n}^{p_1}$ where

$$a_{m \, n}^{p_0} = \frac{8 p_0}{\pi m b} \sin \frac{n \pi \eta}{a} \sin \frac{m \pi \xi}{b} \sin \frac{m \pi u}{2b}$$

and

$$a_{m \, n}^{p_1} = \frac{4 p_1}{\pi^2 m^2 a} \sin \frac{m \pi \eta}{a} \left[ 2b \cos \frac{m \pi \xi}{b} \sin \frac{m \pi u}{2b} + 2\xi m \pi \sin \frac{m \pi \xi}{b} \sin \frac{m \pi u}{2b} - 2 m \pi \cos \frac{m \pi \xi}{b} \cos \frac{m \pi u}{2b} \right]$$
Figure 4: Rectangular plate subjected to linearly varying line load. Mesh distortion parameter, $\gamma$.

Table 2: Values of $u^{\text{adim } A}$, $M^{\text{adim } A}_{xx}$, $M^{\text{adim } A}_{yy}$, $Q^{\text{adim } A}_x$ for region 1 of the rectangular plate subjected to linearly varying line load.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$10u^{\text{adim } A}$</th>
<th>$M^{\text{adim } A}_{xx}$</th>
<th>$M^{\text{adim } A}_{yy}$</th>
<th>$Q^{\text{adim } A}_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.136023</td>
<td>0.106680</td>
<td>0.203068</td>
<td>0.364910</td>
</tr>
<tr>
<td>0.10</td>
<td>0.136015</td>
<td>0.106626</td>
<td>0.203047</td>
<td>0.364553</td>
</tr>
<tr>
<td>0.15</td>
<td>0.136015</td>
<td>0.106626</td>
<td>0.203046</td>
<td>0.364560</td>
</tr>
<tr>
<td>0.20</td>
<td>0.136016</td>
<td>0.106630</td>
<td>0.203048</td>
<td>0.364594</td>
</tr>
<tr>
<td>0.25</td>
<td>0.136016</td>
<td>0.106632</td>
<td>0.203050</td>
<td>0.364632</td>
</tr>
<tr>
<td>0.30</td>
<td>0.136015</td>
<td>0.106630</td>
<td>0.203048</td>
<td>0.364602</td>
</tr>
<tr>
<td>0.35</td>
<td>0.136016</td>
<td>0.106629</td>
<td>0.203048</td>
<td>0.364585</td>
</tr>
<tr>
<td>0.40</td>
<td>0.136016</td>
<td>0.106631</td>
<td>0.203049</td>
<td>0.364613</td>
</tr>
<tr>
<td>0.45</td>
<td>0.136015</td>
<td>0.106629</td>
<td>0.203048</td>
<td>0.364584</td>
</tr>
<tr>
<td>0.50</td>
<td>0.136015</td>
<td>0.106630</td>
<td>0.203048</td>
<td>0.364606</td>
</tr>
<tr>
<td>Series</td>
<td>0.136015</td>
<td>0.106630</td>
<td>0.203048</td>
<td>0.364603</td>
</tr>
</tbody>
</table>

where $m = 1, 2, 3, \ldots$ and $n = 1, 2, 3, \ldots$. The line load is defined by $p_F = 0$ and $p_H = \frac{P}{2a}$. In Tables 2 and 3 results are shown at center point $(A)$. As this point is on the interface, that is, belongs to the two regions, each table represents the results obtained by considering the expansion for each of the regions. It may be observed that the effect of mesh distortion is negligible.

6 CONCLUSIONS

The multi-region Trefftz collocation method for the analysis of plate bending described in this work has proved to be effective for all the non-trivial loading cases it was applied to. Excellent approximations can be obtained using the assumed deflection field a priori satisfying the differential equations of the problem, the T-functions. The collocation procedure used is very flexible, easy to implement and efficient in terms of CPU time consumption.
### Table 3

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$10^3 w^{adim , A}$</th>
<th>$M_x^{adim , A}$</th>
<th>$M_y^{adim , A}$</th>
<th>$Q_x^{adim , A}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.05</td>
<td>0.136021</td>
<td>0.106665</td>
<td>0.203053</td>
<td>0.364395</td>
</tr>
<tr>
<td>0.10</td>
<td>0.136014</td>
<td>0.106621</td>
<td>0.203045</td>
<td>0.364678</td>
</tr>
<tr>
<td>0.15</td>
<td>0.136015</td>
<td>0.106624</td>
<td>0.203044</td>
<td>0.364667</td>
</tr>
<tr>
<td>0.20</td>
<td>0.136016</td>
<td>0.106628</td>
<td>0.203046</td>
<td>0.364632</td>
</tr>
<tr>
<td>0.25</td>
<td>0.136015</td>
<td>0.106629</td>
<td>0.203047</td>
<td>0.364606</td>
</tr>
<tr>
<td>0.30</td>
<td>0.136015</td>
<td>0.106631</td>
<td>0.203049</td>
<td>0.364591</td>
</tr>
<tr>
<td>0.35</td>
<td>0.136016</td>
<td>0.106632</td>
<td>0.203049</td>
<td>0.364591</td>
</tr>
<tr>
<td>0.40</td>
<td>0.136016</td>
<td>0.106630</td>
<td>0.203048</td>
<td>0.364601</td>
</tr>
<tr>
<td>0.45</td>
<td>0.136015</td>
<td>0.106632</td>
<td>0.203049</td>
<td>0.364646</td>
</tr>
<tr>
<td>0.50</td>
<td>0.136015</td>
<td>0.106630</td>
<td>0.203048</td>
<td>0.364616</td>
</tr>
<tr>
<td>Series</td>
<td>0.136015</td>
<td>0.106630</td>
<td>0.203048</td>
<td>0.364603</td>
</tr>
</tbody>
</table>

Table 3: Values of $w^{adim \, A}$, $M_x^{adim \, A}$, $M_y^{adim \, A}$, $Q_x^{adim \, A}$ for region 2 of the rectangular plate subjected to linearly varying line load.

### ACKNOWLEDGEMENTS

This work has been partially supported by ICIST (of which the first author is a member), by the program PRAXIS XXI as part of research project (2/2.1/CEG/33/94) and by JNICT.

### References


