Analysis of skewed plates by a Trefftz collocation method

V. M. A. Leitão and C. M. T. Fernandes

DECivil, Instituto Superior Técnico, Lisboa, Portugal

vitor@civil.ist.utl.pt, ctf@civil.ist.utl.pt

Abstract:

A boundary technique, based on the Trefftz method, is applied to the analysis of skewed thin plates. The formulation uses a superposition of the approximating functions of the displacements (and its derivatives) followed by the approximate satisfaction of the boundary conditions (bc) at suitably chosen boundary points. The least squares criterion is used to obtain the unknowns of the problem, that is, the coefficients or scaling factors of the approximating functions used. Numerical examples are presented.

1 INTRODUCTION

The analysis of the structural behaviour of skewed plates poses some difficulties due to the singularities which may develop at the corners especially for large degrees of skewness and/or for specific boundary conditions, Timoshenko and Woinowsky-Krieger (1959). This singular behaviour, as described by Williams (1951), explains the occurrence, at the obtuse corners of skewed plates, of a great number of cracks and/or excessive deformation typical of skewed plates. Despite the numerical difficulties in the analysis and the problems described above, the use of skew plates in structures is increasing.

Besides the use of simplified methods and/or tables, the design of such plates requires, generally, the use of numerical techniques such as the finite differences method (FDM), the finite element (FEM) and the boundary element methods (BEM). An alternative numerical technique based on a Trefftz method is now presented. The formulation uses, as approximating functions, a suitably truncated series of functions. These functions are obtained by satisfying both the governing equation and the bc at the sides containing the obtuse corners. The remaining bc are approximated in the least squares sense. The particular Trefftz technique used here, the multi-region indirect Trefftz collocation method as described in Leitão and Fernandes (1998), is a multi-region one (which means that the necessary continuity conditions are appropriately enforced at the interfaces in the case of subregions); it is indirect (in the sense that the unknowns of the problem are only the scaling factors of the solutions used); and it is based on the collocation approach meaning that the system of equations is assembled by collecting all the required terms from the collocation of the kinematic or static conditions at a number of boundary points. Numerical tests are presented to illustrate the performance of the Trefftz formulation for the analysis of bending of skewed thin plates subjected to uniform loading and for various types of boundary conditions. The comparison of the results obtained so far with others available in the literature is good.

2 THIN PLATE PROBLEM

Consider a plate and a polar co-ordinate system (r,θ) centered at an obtuse corner. The Lagrange equation, that is, the governing equation for the bending of thin, elastic, isotropic and homogeneous plates is $(\nabla^2)^2 w = p/D$, where $\nabla^2 = \partial^2 w/\partial r^2+(1/r)\partial w/\partial r+(1/r^2)\partial^2 w/\partial\theta^2$ and $D = Et^3/(12(1-\nu^2))$ is the flexural stiffness of the plate. Along the edge of outer normal $\vec{n}$ *two* bc are imposed: the deflection (w) or the effective normal shear force $(V_n=Q_n+\partial M_{nt}/\partial t)$ and the normal rotation $(\partial w/\partial n=\cos\phi\partial w/\partial r+(\sin\phi/r)\partial w/\partial\theta)$ or the normal bending moment $(M_n=M_r\cos^2\phi+M_\theta\sin^2\phi+2M_{r\theta}\sin\phi\cos\phi)$.

Two of the four conditions, $w=\bar{w}$, $\partial w/\partial n=\partial\bar{w}/\partial n$, $M_n=\bar{M}_n$ and $V_n=\bar{V}_n$, must be imposed at the boundary, $\Gamma = \Gamma_u+\Gamma_f$.

In certain cases, the use of subregions, $\Omega^1,\ldots,\Omega^j, \Omega^{j+1},\ldots,\Omega^n$, may be advantageous. To ensure continuity the following *four* bc must be satisfied at the interface, $\Gamma_I = \Omega^j\cap\Omega^{j+i}$, of the two subregions Ω^j e Ω^{j+I}, $w^j = w^{j+1}$, $\partial w^j/\partial n=-\partial w^{j+1}/\partial n$, $M_n^j=M_n^{j+1}$, and $V_n^j = -V_n^{j+1}$.

3 SOLUTION OF THE LAGRANGE EQUATION

The solution, for each subregion, is given by adding a particular solution, w_p, and a homogeneous one, w_c. This may take the form referred by Thein (1979):

$$w = r^{\lambda+1}\{C_1\sin(\lambda+1)\theta+C_2\cos(\lambda+1)\theta+C_3\sin(\lambda-1)\theta+C_4\cos(\lambda-1)\theta\} \quad (1)$$

where C_1, C_2, C_3, C_4 and λ are constants which depend on the *four* bc at the corner edges. By substituting (1) in the bc a four by four system of equations is obtained. Solutions other than the trivial one are found by making the determinant equal to 0 thus finding the root λ of the transcendent equation (this is achieved by the Muller's method for example). The six possible equations are represented in Table 1. The eigen vectors associated to each root will form the (infinite) set of eigen functions, see Thein (1979).

Table 1: Transcendent equations

	Boundary conditions		Transcendent equation
Case	$\theta = -\theta_0$	$\theta = \theta_0$	
1	Clamped	Clamped	$\sin^2 2\lambda\theta_0 = \lambda^2\sin^2 2\theta_0$
2	Free	Free	$(3+\nu)^2\sin^2 2\lambda\theta_0 = (1-\nu)^2\lambda^2\sin^2 2\theta_0$
3	Clamped	Free	$(3+\nu)(1-\nu)\sin^2 2\lambda\theta_0 = 4-(1-\nu)^2\lambda^2\sin^2 2\theta_0$
4	Clamped	SS	$\sin 4\lambda\theta_0 = \lambda\sin 4\theta_0$
5	Simply supported	Free	$(3+\nu)\sin 4\lambda\theta_0 = -(1-\nu)\lambda\sin 4\theta_0$
6	Simply supported	Simply supported	$\sin^2 2\lambda\theta_0 = \sin^2 2\theta_0$

The particular solution for a uniform load, $\bar{p}$, obtained by satisfying the bc, is:

$$w_p = (\bar{p}r^4)/4D\,\{A_1\cos4(\theta+\theta_0)+A_2\sin4(\theta+\theta_0)+A_3\cos2(\theta+\theta_0)+A_4\sin2(\theta+\theta_0)+1\}. \quad (2)$$

4 IMPOSING THE BOUNDARY CONDITIONS

The unknowns of the problem are determined by the collocation of the appropriate equations at selected points at the edges not containing the obtuse corner. To express, adequately, those equations it is useful to use symbolic manipulation packages such as Mathematica of Wolfram (1992). The number of collocation equations must be, at least, equal to that of unknowns. The *four* types of equations take the form, $\mathbf{N}_w\mathbf{c} = \bar{w} - w$, $\mathbf{N}_{\partial w/\partial n}\mathbf{c} = \partial\bar{w}/\partial n - \partial w/\partial n$, $\mathbf{N}_{Mn}\mathbf{c} = \bar{M}_n - M_n$ and $\mathbf{N}_{Vn}\mathbf{c} = \bar{V}_n - V_n$.

From those equations a global system is obtained, $\mathbf{Dc} = \bar{\mathbf{d}} - \mathbf{d}$, that is, usually, overdetermined. The Singular Value Decomposition (SVD) algorithm, Press *et al.* (1994), may be used for solving the system. After finding the solution, the **c** coefficients, it is straightforward to obtain the displacements, rotations and moments by superimposing the solutions, w_p e w_c .

5 NUMERICAL EXAMPLES

To assess the method, several skewed plates were analysed. Results are shown here for (I) a simply supported (SS) rhombic plate of side *a* subjected to a uniform load $\bar{p}$ for the range of skew angles α 45º, 50º, 60º, 70º and 75º and for (II) a free-SS-free-SS plate where the free sides are of length *1.92a* and the SS sides are of length *a/cos(30º)*. Comparison is made with results obtained with a standard finite element package, SAP90 of Habibullah and Wilson (1990) using the DKQ element, of Batoz and Tahar (1982). The collocation points are placed at the positions corresponding to those of the Gauss points assuming the side has length 2. The number of unknowns is $4m_{máx}$ where m_{max} is the degree of the expansion.

Table 2 presents all the normalised values of the displacement and bending moments at the centre point for a number of skew angles. A reference solution, by Morley (1962), is also presented. Comparison with a DKQ model is represented in Figures 4 to 6 for α=75º.

Table 2: Displacement and bending moments for a simply supported plate (I)

		skew angle α (degrees)				
	m_{max}	45	50	60	70	75
displacement $wD10^3/(\bar{p}a^4)$	4	4.0631	3.4377	2.5602	0.9506	0.3775
	6	4.0624	3.7485	2.5601	0.9581	0.4070
	8	4.0624	3.8302	2.5601	0.9582	0.4078
	Reference	4.06	3.87	2.56	0.96	0.41
$M_{xx}10^2/(\bar{p}a^2)$	4	4.8033	4.0607	4.2567	2.7974	1.7295
	6	4.7920	4.6523	4.2536	2.8084	1.9073
	8	4.7897	4.8024	4.2534	2.8065	1.9074
	Reference	4.79	4.86	4.25	2.81	1.91
$M_{yy}10^2/(\bar{p}a^2)$	4	4.8035	3.9550	3.3316	1.8042	0.9861
	6	4.7922	4.3533	3.3292	1.8056	1.0845
	8	4.7911	4.4319	3.3291	1.8045	1.0846
	Reference	4.79	4.48	3.33	1.80	1.08

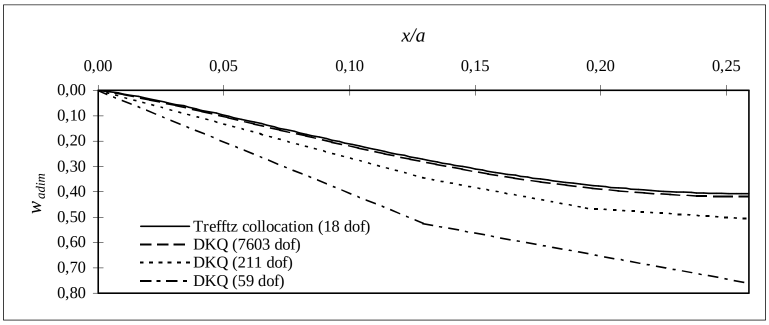

Figure 4:Variation of the normalized displacement $wD10^3/(\bar{p}a^4)$ along y = 0 for α=75º (I)

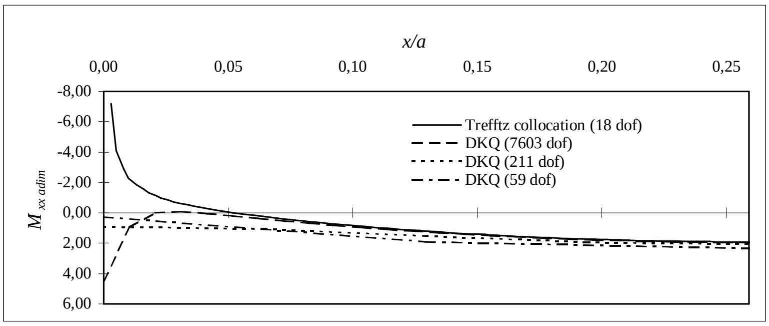

Figure 5: Variation of the normalized moment $M_{xx}10^2/(\bar{p}a^2)$ along y = 0 for α=75º (I)

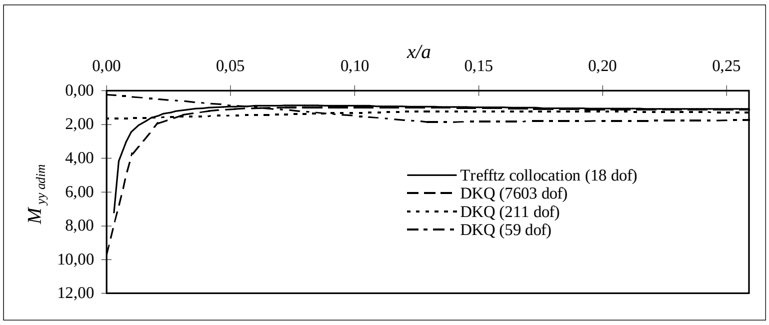

Figure 6: Variation of the normalized moment $M_{yy}10^2/(\bar{p}a^2)$ along y = 0 for α=75º (I)

The free-SS skew plate (II) subjected to a uniform load was also compared to a DKQ model and the results are represented in Figures 7 to 9. The Poisson ratio here is 0.2 and two subregions were considered the collocation being made at the interface with singular functions centered at both obtuse corners. The number of unknowns (degrees of freedom) is 32 and 7701, in the Trefftz model and in the DKQ model respectively.

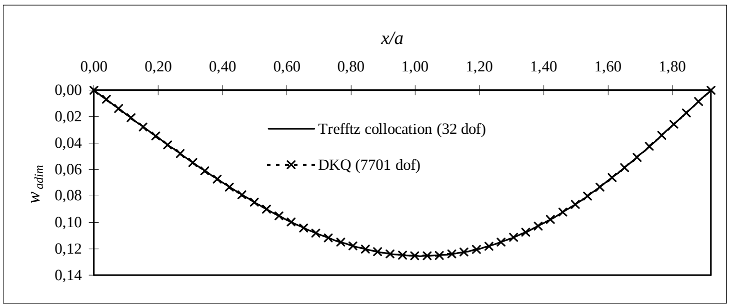

Figure 7: Variation of the normalized displacement, $wD/(\bar{p}a^4)$, at y = 0 (II)

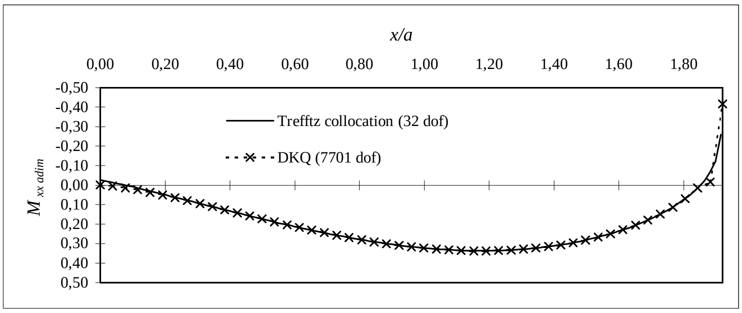

Figure 8: Variation of the normalized moment, $M_{xx}/(\bar{p}a^2)$, at y = 0 (II)

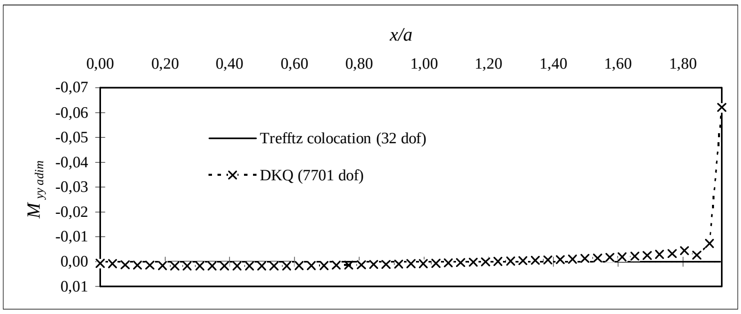

Figure 9: Variation of the normalized moment, $M_{yy}/(\bar{p}a^2)$, at y = 0 (II)

6 CONCLUSIONS

In this work a collocation formulation based on the indirect Trefftz method is further developed and applied to the analysis of bending of thin skewed plates. A set of singular solutions satisfying the differential equation as well as some of the boundary conditions is used which allows for the singular behaviour exhibited at the obtuse corners of the skewed plates to be adequately represented. Numerical tests on two skewed plates were

carried out and the results compared with other results available in the literature and also by using a finite element model based on the DKQ elements.
It should be emphasised the large number of degrees of freedom of the finite element model that has to be considered in order to obtain similarly accurate results to those of the Trefftz model. This is due to the use of non-singular (approximating) functions of the finite element model, which, obviously, makes it harder to simulate the singular behaviour at the obtuse corners of skewed plates.

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