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GEOMETRICALLY EXACT ANALYSIS OF SHELLS BY A MESHLESS APPROACH

Carlos Tiago*, **Paulo M. Pimenta**†

*Instituto Superior Técnico, Universidade Técnica de Lisboa
Av. Rovisco Pais, 1049-001, Lisboa, Portugal
e-mail: carlos.tiago@civil.ist.utl.pt

†Escola Politécnica, Universidade de São Paulo
Av. Prof. Almeida Prado, trav. 2, 83, São Paulo, Brazil
e-mail:ppimenta@usp.br

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Abstract. *There is a growing interest in the geometrically exact analysis of structures. The innate elegance of this king of formulations arises from the exact representation of the rotations. In this case, the rotation vector is parameterized by the Euler-Rodrigues formula. The internal power arises from the first Piola-Kirchhoff stress tensor and the deformation gradient. A consistent plane stress condition is imposed in a hyperelastic material to derive the appropriate (symmetric) constitutive operator [1].*

In the present work a hybrid method of analysis is proposed where the solution is obtained by the approximation of the generalized internal displacement fields through the Moving Least Squares (MLS) scheme and the generalized boundary tractions are interpolated by Lagrange polynomials. To completely eliminate shear-locking phenomenon a consistency requirement is imposed to the generalized internal displacement fields: the exact reproduction of the Kirchhoff-Love constraints.

An extension of the arc-length method that includes the generalized internal displacement fields, the generalized boundary tractions and the load parameter in the constraint equation of the hyperellipse is proposed to solve the resulting nonlinear problem. A consistent linearization procedure is performed, resulting a semi-definite system matrix which, for hyperelastic materials and conservative loadings, is always symmetric (even for configurations far from a equilibrium trajectory).

Differently from the standard Finite Element Methods (FEM), the resulting solution are (arbitrary) smooth generalized displacements and stress fields. Also, the representation of the initial configuration is exact, contrary the usual FEM, where a C^0 approximation of the original problem is made (usually by the assembly of flat elements).

1 INTRODUCTION

1.1 Historical background

The research on geometrically exact shell models was initiated by Simo and co-workers. The formulation and parametrization of the model was presented in [2], where the hypothesis of *one inextensible director*, used in the present work, was already considered. In the subsequent papers the linear and nonlinear computational aspects of the theory are dealt. Other perspectives were latter considered, like through-the-thickness stretch, plasticity constitutive model, time-stepping conserving algorithms for dynamical analysis and shell intersections problems.

Nevertheless, some drawbacks were still present like the need for complex configuration updates and the use of *assumed strain* methods to avoid the *shear locking* effect.

On the *twin* papers [3, 4] a unified theory for beams and shells, respectively, was presented. Here, the fundamental variable for parameterizing the rotation tensor is the rotation vector, delivering an expression for the tangent stiffness which is always symmetric[1] even far from the equilibrium path.

Implementation of this theory for beams was presented in [5], which was latter generalized to curved rods [6] and to accommodate warping and a genuine finite strain constitutive relation [7].

In the shell model implementation [1] a constitutive relation was derived based on a true plane stress condition. The generalization presented in [8] accommodates the thickness variation of the shell, thus allowing the use of a full three dimensional finite strain constitutive model.

The traditional version of the Finite Element Method[2] (FEM) is, invariably, the chosen numerical tool to discretize the unknown fields. However, some of the inconveniences of the FEM can be overcome by the use of meshfree discretizations, like (i) the need to explicitly set up incidences relations between nodes (in order to shape elements) and (ii) the lack of equilibrium between adjacent elements. Meshfree methods are nowadays a well establish tool to solve engineering problems. For reviews, see *e. g.* [9] and [10].

The first geometrically exact analysis using meshfree approximations was presented in [11]. The solution of beam problems was performed by using Moving Least Squares (MLS) to discretize the generalized displacements fields. Hence, the procedure can be considered an extension of the element free Galerkin (EFG) [12] for the geometrically exact analysis of structures.

1.2 Scope of the present work

In the present work an alternative method for the solution of shell is presented. Instead of the traditional FEM approach, the previous work [11] is now extended to shell analysis. Hence, a fresh approximation method is applied to the numerical solution of a, also recent, shell model.

In the FEM context the use of a initial curved elements is not imperative, as established in [13]. This behaviour can be explained by combining two sorts of reasons. On one hand, in the FEM the *geometry* is described by the elements and, on the other hand, in shell analysis *refined meshes* are usually required. Thus, the use of assembly of flat elements to model shells is usually acceptable.

[1]For hyperelastic materials and conservative loads, of course.

[2]By traditional version of the Finite Element Method we refer to the well known displacement model using nodal shape functions for approximation both the geometry and the generalized displacements fields and imposition of the essential boundary conditions through collocation. Non-conventional formulations (like hybrid, mixed or equilibrium) are not included here.

Unlike the FEM, in weak form based meshfree projections the whole shell is a *unique* domain[3], hence the consideration of initial curved geometries is essential.

A crucial enhancement in the geometrically exact shell formulation for the present work was the consideration on initially curved shells [14, 15]. Although developed and implemented in a FEM framework, the results can be straightforwardly incorporated in the present formulation. The consideration of possibly curved shells is performed by a simple mapping from the *plane* reference configuration to the *initial* form. All the computations are done over the plane reference configuration. The theoretical formulation presented in those works supplies a perfect basic theoretical background for the development of a meshfree formulation.

In order to circumvent the non-interpolation character of the approximations, which impairs the use of collocation for imposing the boundary conditions[4], a hybrid weak form suitable for meshless approximations is presented, which includes the internal virtual work, the external virtual work and the external complementary virtual work arising from the kinematic boundary.

The exact parametrization of the rotation tensor is made through Euler-Rodrigues formula. As *all* vectorial parameterizations of the rotation tensor, this closed-form solution has a limited range of application beyond which a singularity occurs. For circumvent this problem, a update Lagrangian formulation can be used, as in [11]. However, is not common to face this problem in shell analysis.

The only kinematical assumption is the plane section hypothesis of Reissner-Mindlin. The inextensibility of the director is complemented by a plane stress condition. This is imposed over the the constitutive model, which is the neo-Hookean material.

The internal virtual work is expressed by the first Piola-Kirchhoff stress tensor and the deformation gradient.

1.3 Notation and text organization

Throughout the text italic Latin or Greek lowercase letters ($a, b, \ldots \alpha, \beta, \ldots$) denote scalar quantities, bold italic Latin or Greek lowercase letters ($\boldsymbol{a}, \boldsymbol{b}, \ldots \boldsymbol{\alpha}, \boldsymbol{\beta}, \ldots$) denote vectors, bold italic Latin or Greek capital letters ($\boldsymbol{A}, \boldsymbol{B}, \ldots$) denote second-order tensors, bold Calligraphic Latin capital letters ($\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}}, \ldots$) denote third-order tensors and bold blackboard italic Latin capital letters ($\mathbb{A}, \mathbb{B}, \ldots$) denote forth-order tensors in a three dimensional Euclidian space. The same letter is used to identify the skew-symmetric second order tensors ($\boldsymbol{A}, \boldsymbol{B}, \ldots \boldsymbol{\Omega}, \boldsymbol{\Theta}, \ldots$) and their associated axial vector ($\boldsymbol{a}, \boldsymbol{b}, \ldots \boldsymbol{\omega}, \boldsymbol{\theta}, \ldots$).

The problem is presented in section 2. In section 3 the mappings of the initial configuration and the generalized displacements fields are introduced and the deformation and velocity gradients derived. The generalized stresses, the internal power and the external power are presented in section 4, followed by the proposed variational formulation of the problem in section 5. The linearization of the weak form is established in section 6. The suggested meshfree method and associated implementation issues are exhibited in sections 7 and 8.

2 THE MODEL PROBLEM

Consider the shell exhibited in figure 1, where three orthonormal right-handed coordinate systems are represented. $\boldsymbol{e}_i^r$ for the reference configuration, $\boldsymbol{e}_i^o$ for the initial configuration and $\boldsymbol{e}_i$ for the current configuration.

The reference plane is denoted by $\Omega^r \subset \mathbb{R}^2$. The contour of Ω^r is denoted by Γ^r, *i. e.*,

[3]Subdivisions are possible but not advisable, as the nature of the approximation is element free.

[4]In fact, with an appropriate change of coordinates this could also be archived, see [10].

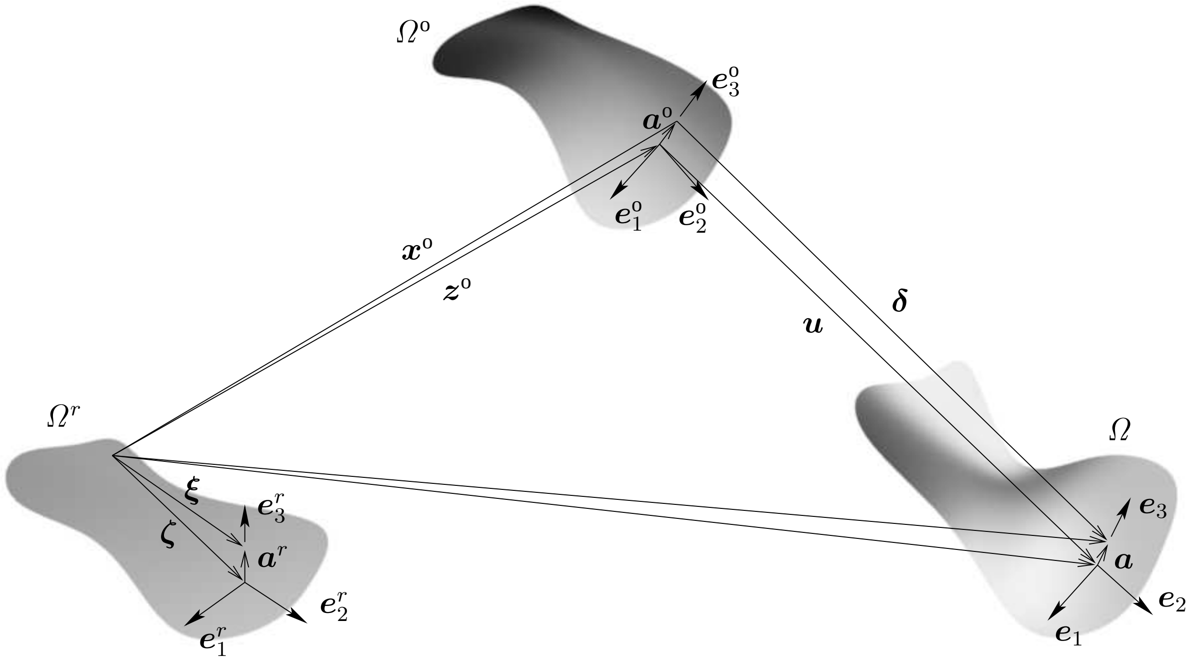

Figure 1: The reference, initial and current configurations of the shell.

$\Gamma^r = \partial\Omega^r$ and can be decomposed as $\Gamma_t^r \cup \Gamma_u^r = \Gamma^r$ and $\Gamma_t^r \cap \Gamma_u^r = \emptyset$, where Γ_t^r and Γ_u^r identify the static and kinematic boundaries. The volume is V^r and $H^r = [-h_b^r, h_t^r]$ is the shell thickness, both on the reference configuration. The endpoints of H^r are collected in the set $C^r = \{-h_b^r, h_t^r\}$, thus $C^r = \partial H^r$.

The reference configuration can be described by $\boldsymbol{\xi}$, which can be written as

$$\boldsymbol{\xi} = \boldsymbol{\zeta} + \boldsymbol{a}^r, \tag{1}$$

where $\boldsymbol{\zeta} = \xi_\alpha \boldsymbol{e}_\alpha^r$ defines the position of a material point over the middle plane of the reference configuration, Ω^r, and $\boldsymbol{a}^r = \zeta \boldsymbol{e}_3^r$ representes the component along the normal.

The position of the material points in the initial configuration, Ω^{o}, is

$$\boldsymbol{x}^{\mathrm{o}} = \boldsymbol{z}^{\mathrm{o}} + \boldsymbol{a}^{\mathrm{o}}, \tag{2}$$

where the middle surface, $\boldsymbol{z}^{\mathrm{o}}$, of the initial configuration $\Omega^o \in \mathbb{R}^3$ is defined by

$$\boldsymbol{z}^{\mathrm{o}} = \boldsymbol{z}^{\mathrm{o}}(\boldsymbol{\zeta}), \tag{3}$$

and the normal vector to the initial configuration is given by

$$\boldsymbol{a}^{\mathrm{o}} = \boldsymbol{Q}^{\mathrm{o}} \boldsymbol{a}^r, \tag{4}$$

where $\boldsymbol{Q}^{\mathrm{o}}$ is the initial rotation tensor.

We assume the applied load vary linearly with a parameter, λ. Nevertheless, for simplicity, this dependance will be omitted in the following. The shell is under the action of a body forces, $\overline{\boldsymbol{b}}^{\mathrm{o}}$, per unit volume of the initial configuration and traction forces, $\overline{\boldsymbol{t}}^{\mathrm{o}}$, per unit area of the initial configuration on the top and bottom surfaces. Eventually, configuration dependant loads may be included. In the lateral surfaces the shell is subjected either prescribed tractions[5], $\overline{\boldsymbol{t}}^{\mathrm{o}}$, per unit

[5]No distinction in the notation is used for traction forces on the lateral surface and top and bottom surfaces. This identification can be inferred from the context.

area of the initial configuration, or imposed displacements. Due to the kinematical assumption, the displacements of a given point, $\boldsymbol{\zeta}$, on the lateral surface are not independent along ζ. The precise definition of the quantities to be imposed to explicitly prescribe the displacements is introduced latter.

3 KINEMATICS

3.1 The initial configuration and displacement field

The basis $\boldsymbol{e}_i^{\mathrm{o}}$ can be obtained[6] by

$$\boldsymbol{e}_1^{\mathrm{o}} = \frac{\boldsymbol{z}_{,1}^{\mathrm{o}}}{\|\boldsymbol{z}_{,1}^{\mathrm{o}}\|} \tag{5a}$$

$$\boldsymbol{e}_3^{\mathrm{o}} = \frac{\boldsymbol{z}_{,1}^{\mathrm{o}} \times \boldsymbol{z}_{,2}^{\mathrm{o}}}{\|\boldsymbol{z}_{,1}^{\mathrm{o}} \times \boldsymbol{z}_{,2}^{\mathrm{o}}\|} \tag{5b}$$

$$\boldsymbol{e}_2^{\mathrm{o}} = \boldsymbol{e}_3^{\mathrm{o}} \times \boldsymbol{e}_1^{\mathrm{o}} \tag{5c}$$

where $(\cdot)_{,\alpha} = \partial(\cdot)/\partial\xi_\alpha$.

The explicit evaluation of $\boldsymbol{e}_{i,\alpha}^{\mathrm{o}}$ is given by

$$\boldsymbol{e}_{1,\alpha}^{\mathrm{o}} = \frac{1}{\|\boldsymbol{z}_{,1}^{\mathrm{o}}\|}\left(\boldsymbol{I} - \boldsymbol{e}_1^{\mathrm{o}} \otimes \boldsymbol{e}_1^{\mathrm{o}}\right)\boldsymbol{z}_{,1\alpha}^{\mathrm{o}} \tag{6a}$$

$$\boldsymbol{e}_{3,\alpha}^{\mathrm{o}} = \frac{1}{\|\boldsymbol{z}_{,1}^{\mathrm{o}} \times \boldsymbol{z}_{,2}^{\mathrm{o}}\|}\left(\boldsymbol{I} - \boldsymbol{e}_3^{\mathrm{o}} \otimes \boldsymbol{e}_3^{\mathrm{o}}\right)\left(\boldsymbol{z}_{,1}^{\mathrm{o}} \times \boldsymbol{z}_{,2\alpha}^{\mathrm{o}} - \boldsymbol{z}_{,2}^{\mathrm{o}} \times \boldsymbol{z}_{,1\alpha}^{\mathrm{o}}\right) \tag{6b}$$

$$\boldsymbol{e}_{2,\alpha}^{\mathrm{o}} = \boldsymbol{e}_{,3\alpha}^{\mathrm{o}} \times \boldsymbol{e}_{,1}^{\mathrm{o}} + \boldsymbol{e}_{,3}^{\mathrm{o}} \times \boldsymbol{e}_{,1\alpha}^{\mathrm{o}} \tag{6c}$$

The initial rotation tensor can be expressed as

$$\boldsymbol{Q}^{\mathrm{o}} = \boldsymbol{e}_i^{\mathrm{o}} \otimes \boldsymbol{e}_i^{r}. \tag{7}$$

From figure 1 it is possible to conclude that the position of the material points on the deformed configuration, Ω, is

$$\boldsymbol{x} = \boldsymbol{z} + \boldsymbol{a}. \tag{8}$$

3.2 Deformation gradient due initial mapping

The initial deformation gradient, $\boldsymbol{F}^{\mathrm{o}}$, for the transformation between the reference plane, Ω^r, and the initial configuration, Ω^o, is given by

$$\begin{aligned}\boldsymbol{F}^{\mathrm{o}} &= \frac{\partial \boldsymbol{x}^{\mathrm{o}}}{\partial \boldsymbol{\xi}_\alpha} \otimes \boldsymbol{e}_\alpha^r + \frac{\partial \boldsymbol{x}^{\mathrm{o}}}{\partial \boldsymbol{\zeta}} \otimes \boldsymbol{e}_3^r = \left(\boldsymbol{z}_{,\alpha}^{\mathrm{o}} + \boldsymbol{Q}_{,\alpha}^{\mathrm{o}}\boldsymbol{Q}^{\mathrm{o}T}\boldsymbol{a}^{\mathrm{o}}\right) \otimes \boldsymbol{e}_\alpha^r + \boldsymbol{a}^{\mathrm{o}\prime} \otimes \boldsymbol{e}_3^r \\ &= \left(\boldsymbol{\eta}_\alpha^{\mathrm{o}} + \boldsymbol{K}_\alpha^{\mathrm{o}}\boldsymbol{a}^{\mathrm{o}}\right) \otimes \boldsymbol{e}_\alpha^r + \boldsymbol{Q}^{\mathrm{o}}\end{aligned} \tag{9}$$

where it was introduced the vector, $\boldsymbol{\eta}_\alpha^{\mathrm{o}}$, and the skew-symmetric tensor, $\boldsymbol{K}_\alpha^{\mathrm{o}}$,

$$\boldsymbol{\eta}_\alpha^{\mathrm{o}} = \boldsymbol{z}_{,\alpha}^{\mathrm{o}} - \boldsymbol{Q}^{\mathrm{o}}\boldsymbol{e}_\alpha^r, \qquad \boldsymbol{K}_\alpha^{\mathrm{o}} = \boldsymbol{Q}_{,\alpha}^{\mathrm{o}}\boldsymbol{Q}^{\mathrm{o}T} \tag{10}$$

and derivatives on scalar parameter, ζ, were denoted by $(\cdot)' = \partial(\cdot)/\partial\zeta$.

[6] The order by which the expressions of the basis vectors is presented aim, on the one hand, an easy identification of the involved operations and, on the other hand, shorten the expressions size.

The latter can be expressed as

$$\begin{aligned}\boldsymbol{K}_\alpha^{\mathrm{o}} &= \boldsymbol{Q}_{,\alpha}^{\mathrm{o}}\boldsymbol{Q}^{\mathrm{o}T} = \left(\boldsymbol{e}_{i,\alpha}^{\mathrm{o}} \otimes \boldsymbol{e}_i^r\right)\left(\boldsymbol{e}_j^{\mathrm{o}} \otimes \boldsymbol{e}_j^r\right)^T \\ &= \left(\boldsymbol{e}_{i,\alpha}^{\mathrm{o}} \otimes \boldsymbol{e}_i^r\right)\left(\boldsymbol{e}_j^r \otimes \boldsymbol{e}_j^{\mathrm{o}}\right) = \left(\boldsymbol{e}_{i,\alpha}^{\mathrm{o}} \otimes \boldsymbol{e}_j^{\mathrm{o}}\right)\left(\boldsymbol{e}_i^r \cdot \boldsymbol{e}_j^r\right) = \boldsymbol{e}_{i,\alpha}^{\mathrm{o}} \otimes \boldsymbol{e}_i^{\mathrm{o}}\end{aligned} \tag{11}$$

whose axial vector, $\boldsymbol{\kappa}_\alpha^{\mathrm{o}}$, is given by

$$\boldsymbol{\kappa}_\alpha^{\mathrm{o}} = \mathrm{Axial}\left(\boldsymbol{K}_\alpha^{\mathrm{o}}\right) = \left(\boldsymbol{e}_{2,\alpha}^{\mathrm{o}} \cdot \boldsymbol{e}_3^{\mathrm{o}}\right)\boldsymbol{e}_1^{\mathrm{o}} + \left(\boldsymbol{e}_{3,\alpha}^{\mathrm{o}} \cdot \boldsymbol{e}_1^{\mathrm{o}}\right)\boldsymbol{e}_2^{\mathrm{o}} + \left(\boldsymbol{e}_{1,\alpha}^{\mathrm{o}} \cdot \boldsymbol{e}_2^{\mathrm{o}}\right)\boldsymbol{e}_3^{\mathrm{o}}. \tag{12}$$

Defining the vector

$$\boldsymbol{\gamma}_\alpha^{\mathrm{o}} = \boldsymbol{\eta}_\alpha^{\mathrm{o}} + \boldsymbol{\kappa}_\alpha^{\mathrm{o}} \times \boldsymbol{a}^{\mathrm{o}} \tag{13}$$

the initial deformation gradient can be written as

$$\boldsymbol{F}^{\mathrm{o}} = \left(\boldsymbol{\eta}_\alpha^{\mathrm{o}} + \boldsymbol{\kappa}_\alpha^{\mathrm{o}} \times \boldsymbol{a}^{\mathrm{o}}\right) \otimes \boldsymbol{e}_\alpha^r + \boldsymbol{Q}^{\mathrm{o}} = \boldsymbol{\gamma}_\alpha^{\mathrm{o}} \otimes \boldsymbol{e}_\alpha^r + \boldsymbol{Q}^{\mathrm{o}}. \tag{14}$$

Hence, the generalized strains in the reference configuration, $\boldsymbol{\eta}_\alpha^{\mathrm{o}r}$ and $\boldsymbol{\kappa}_\alpha^{\mathrm{o}r}$, are

$$\boldsymbol{\eta}_\alpha^{\mathrm{o}r} = \boldsymbol{Q}^{\mathrm{o}T}\boldsymbol{\eta}_\alpha^{\mathrm{o}} = \boldsymbol{Q}^{\mathrm{o}T}\left(\boldsymbol{z}_{,\alpha}^{\mathrm{o}} - \boldsymbol{e}_\alpha^{\mathrm{o}}\right) = \boldsymbol{Q}^{\mathrm{o}T}\boldsymbol{z}_{,\alpha}^{\mathrm{o}} - \boldsymbol{e}_\alpha^r, \tag{15a}$$

$$\begin{aligned}\boldsymbol{\kappa}_\alpha^{\mathrm{o}r} &= \boldsymbol{Q}^{\mathrm{o}T}\boldsymbol{\kappa}_\alpha^{\mathrm{o}} = \boldsymbol{Q}^{\mathrm{o}T}\left(\left(\boldsymbol{e}_{2,\alpha}^{\mathrm{o}} \cdot \boldsymbol{e}_3^{\mathrm{o}}\right)\boldsymbol{e}_1^{\mathrm{o}} + \left(\boldsymbol{e}_{3,\alpha}^{\mathrm{o}} \cdot \boldsymbol{e}_1^{\mathrm{o}}\right)\boldsymbol{e}_2^{\mathrm{o}} + \left(\boldsymbol{e}_{1,\alpha}^{\mathrm{o}} \cdot \boldsymbol{e}_2^{\mathrm{o}}\right)\boldsymbol{e}_3^{\mathrm{o}}\right) \\ &= \left(\boldsymbol{e}_{2,\alpha}^{\mathrm{o}} \cdot \boldsymbol{e}_3^{\mathrm{o}}\right)\boldsymbol{e}_1^r + \left(\boldsymbol{e}_{3,\alpha}^{\mathrm{o}} \cdot \boldsymbol{e}_1^{\mathrm{o}}\right)\boldsymbol{e}_2^r + \left(\boldsymbol{e}_{1,\alpha}^{\mathrm{o}} \cdot \boldsymbol{e}_2^{\mathrm{o}}\right)\boldsymbol{e}_3^r.\end{aligned} \tag{15b}$$

It is now possible to define the vector $\boldsymbol{\gamma}_\alpha^{\mathrm{o}r}$ through

$$\boldsymbol{\gamma}_\alpha^{\mathrm{o}r} = \boldsymbol{Q}^{\mathrm{o}T}\boldsymbol{\gamma}_\alpha^{\mathrm{o}} = \boldsymbol{Q}^{\mathrm{o}T}\left(\boldsymbol{\eta}_\alpha^{\mathrm{o}} + \boldsymbol{\kappa}_\alpha^{\mathrm{o}} \times \boldsymbol{a}^{\mathrm{o}}\right) = \boldsymbol{Q}^{\mathrm{o}T}\boldsymbol{\eta}_\alpha^{\mathrm{o}} + \left(\boldsymbol{Q}^{\mathrm{o}T}\boldsymbol{\kappa}_\alpha^{\mathrm{o}}\right) \times \left(\boldsymbol{Q}^{\mathrm{o}T}\boldsymbol{a}^{\mathrm{o}}\right) = \boldsymbol{\eta}_\alpha^{\mathrm{o}r} + \boldsymbol{\kappa}_\alpha^{\mathrm{o}r} \times \boldsymbol{a}^r. \tag{16}$$

Hence, the initial deformation gradient, as a function of the generalized back-rotated strains, assumes the form

$$\begin{aligned}\boldsymbol{F}^{\mathrm{o}} &= \boldsymbol{\gamma}_\alpha^{\mathrm{o}} \otimes \boldsymbol{e}_\alpha^r + \boldsymbol{Q}^{\mathrm{o}} = \left(\boldsymbol{Q}^{\mathrm{o}}\boldsymbol{Q}^{\mathrm{o}T}\boldsymbol{\gamma}_\alpha^{\mathrm{o}}\right) \otimes \boldsymbol{e}_\alpha^r + \boldsymbol{Q}^{\mathrm{o}} \\ &= \boldsymbol{Q}^{\mathrm{o}}\left(\boldsymbol{I} + \left(\boldsymbol{Q}^{\mathrm{o}T}\boldsymbol{\gamma}_\alpha^{\mathrm{o}}\right) \otimes \boldsymbol{e}_\alpha^r\right) = \boldsymbol{Q}^{\mathrm{o}}\left(\boldsymbol{I} + \boldsymbol{\gamma}_\alpha^{\mathrm{o}r} \otimes \boldsymbol{e}_\alpha^r\right) = \boldsymbol{Q}^{\mathrm{o}}\boldsymbol{F}^{\mathrm{o}r}\end{aligned} \tag{17}$$

where it was introduced the initial back-rotated tensor $\boldsymbol{F}^{\mathrm{o}r}$,

$$\boldsymbol{F}^{\mathrm{o}r} = \boldsymbol{I} + \boldsymbol{\gamma}_\alpha^{\mathrm{o}r} \otimes \boldsymbol{e}_\alpha^r. \tag{18}$$

This tensor may be rewritten as

$$\begin{aligned}\boldsymbol{F}^{\mathrm{o}r} &= \boldsymbol{I} + \boldsymbol{\gamma}_\alpha^{\mathrm{o}r} \otimes \boldsymbol{e}_\alpha^r = \boldsymbol{e}_i^r \otimes \boldsymbol{e}_i^r + \boldsymbol{\gamma}_\alpha^{\mathrm{o}r} \otimes \boldsymbol{e}_\alpha^r \\ &= \left(\boldsymbol{e}_\alpha^r + \boldsymbol{\gamma}_\alpha^{\mathrm{o}r}\right) \otimes \boldsymbol{e}_\alpha^r + \boldsymbol{e}_3^r \otimes \boldsymbol{e}_3^r = \boldsymbol{f}_\alpha^{\mathrm{o}r} \otimes \boldsymbol{e}_\alpha^r + \boldsymbol{f}_3^{\mathrm{o}r} \otimes \boldsymbol{e}_3^r = \boldsymbol{f}_i^{\mathrm{o}r} \otimes \boldsymbol{e}_i^r\end{aligned} \tag{19}$$

where

$$\boldsymbol{f}_\alpha^{\mathrm{o}r} = \boldsymbol{e}_\alpha^r + \boldsymbol{\gamma}_\alpha^{\mathrm{o}r}, \tag{20a}$$

$$\boldsymbol{f}_3^{\mathrm{o}r} = \boldsymbol{e}_3^r. \tag{20b}$$

3.3 Total deformation gradient

From figure 1 the middle surface of the current configuration is

$$\boldsymbol{z} = \boldsymbol{z}^{\mathrm{o}} + \boldsymbol{u} \tag{21}$$

and we may introduce the effective rotation tensor, $\boldsymbol{Q}^{\mathrm{e}}$, between the initial and current configurations

$$\boldsymbol{a} = \boldsymbol{Q}^{\mathrm{e}}\boldsymbol{a}^{\mathrm{o}} \tag{22}$$

where

$$\boldsymbol{Q}^{\mathrm{e}} = \boldsymbol{I} + h_1(\theta)\,\boldsymbol{\Theta} + h_2(\theta)\,\boldsymbol{\Theta}^2, \tag{23}$$

is the Euler-Rodrigues rotation tensor, $\boldsymbol{\Theta}$ is the skew-symmetric tensor whose axial vector is $\boldsymbol{\theta}$, $\theta = \|\boldsymbol{\theta}\|$ is the rotation angle and

$$h_1(\theta) = \frac{\sin\theta}{\theta} \qquad\qquad h_2(\theta) = \frac{1}{2}\left[\frac{\sin(\theta/2)}{\theta/2}\right]^2 \tag{24}$$

are trigonometric functions. Hence the relation $\boldsymbol{e}_i = \boldsymbol{Q}^{\mathrm{e}}\boldsymbol{e}_i^{\mathrm{o}}$ holds.

Taking into account (4),

$$\boldsymbol{a} = \boldsymbol{Q}^{\mathrm{e}}\boldsymbol{a}^{\mathrm{o}} = \boldsymbol{Q}^{\mathrm{e}}\boldsymbol{Q}^{\mathrm{o}}\boldsymbol{a}^{r} = \boldsymbol{Q}\boldsymbol{a}^{r} \tag{25}$$

where $\boldsymbol{Q} = \boldsymbol{Q}^{\mathrm{e}}\boldsymbol{Q}^{\mathrm{o}}$ describes the rotation from the plane reference configuration to the deformed configuration. Thus, $\boldsymbol{Q}$ is here denoted by the total rotation tensor.

The total deformation gradient, $\boldsymbol{F}$, is given by

$$\begin{aligned}\boldsymbol{F} &= \frac{\partial\boldsymbol{x}}{\partial\boldsymbol{\xi}_\alpha}\otimes\boldsymbol{e}_\alpha^r + \frac{\partial\boldsymbol{x}}{\partial\boldsymbol{\zeta}}\otimes\boldsymbol{e}_3^r = (\boldsymbol{z}+\boldsymbol{a})_{,\alpha}\otimes\boldsymbol{e}_\alpha^r + (\boldsymbol{z}+\boldsymbol{a})'\otimes\boldsymbol{e}_3^r \\ &= (\boldsymbol{\eta}_\alpha + \boldsymbol{K}_\alpha\boldsymbol{a})\otimes\boldsymbol{e}_\alpha^r + \boldsymbol{Q}\end{aligned} \tag{26}$$

where the skew-symmetric tensor $\boldsymbol{K}_\alpha = \boldsymbol{Q}_{,\alpha}\boldsymbol{Q}^T$ and the generalized strain vector

$$\boldsymbol{\eta}_\alpha = \boldsymbol{z}^{\mathrm{o}}_{,\alpha} + \boldsymbol{u}_{,\alpha} - \boldsymbol{e}_\alpha \tag{27}$$

were introduced.

The latter can be expressed as

$$\boldsymbol{K}_\alpha = \boldsymbol{Q}_{,\alpha}\boldsymbol{Q}^T = (\boldsymbol{Q}^{\mathrm{e}}\boldsymbol{Q}^{\mathrm{o}})_{,\alpha}(\boldsymbol{Q}^{\mathrm{e}}\boldsymbol{Q}^{\mathrm{o}})^T = \boldsymbol{K}_\alpha^{\mathrm{e}} + \boldsymbol{Q}^{\mathrm{e}}\boldsymbol{K}_\alpha^{\mathrm{o}}\boldsymbol{Q}^{\mathrm{e}T} \tag{28}$$

where the skew-symmetric tensors $\boldsymbol{K}_\alpha^{\mathrm{e}} = \boldsymbol{Q}^{\mathrm{e}}_{,\alpha}\boldsymbol{Q}^{\mathrm{e}T}$ and $\boldsymbol{K}_\alpha^{\mathrm{o}} = \boldsymbol{Q}^{\mathrm{o}}_{,\alpha}\boldsymbol{Q}^{\mathrm{o}T}$ were introduced.

The axial vector $\boldsymbol{K}_\alpha$ is given by $\boldsymbol{\kappa}_\alpha = \mathrm{Axial}(\boldsymbol{K}_\alpha) = \boldsymbol{\kappa}_\alpha^{\mathrm{e}} + \boldsymbol{Q}^{\mathrm{e}}\boldsymbol{\kappa}_\alpha^{\mathrm{o}}$ where the effective curvature vector is $\boldsymbol{\kappa}_\alpha^{\mathrm{e}} = \mathrm{Axial}(\boldsymbol{K}_\alpha^{\mathrm{e}}) = \boldsymbol{\Gamma}\boldsymbol{\theta}_{,\alpha}$ and the tensor $\boldsymbol{\Gamma}$ is given by

$$\boldsymbol{\Gamma} = \boldsymbol{I} + h_2(\theta)\,\boldsymbol{\Theta} + h_3(\theta)\,\boldsymbol{\Theta}^2 \tag{29}$$

with

$$h_3(\theta) = \frac{1 - h_1(\theta)}{\theta^2}. \tag{30}$$

The generalized back-rotated strains are given by

$$\boldsymbol{\eta}_\alpha^r = \boldsymbol{Q}^T\boldsymbol{\eta}_\alpha = \boldsymbol{Q}^T\left(\boldsymbol{z}^{\mathrm{o}}_{,\alpha} + \boldsymbol{u}_{,\alpha} - \boldsymbol{e}_\alpha\right) = \boldsymbol{Q}^T\left(\boldsymbol{z}^{\mathrm{o}}_{,\alpha} + \boldsymbol{u}_{,\alpha}\right) - \boldsymbol{e}_\alpha^r = \boldsymbol{Q}^T\boldsymbol{z}_{,\alpha} - \boldsymbol{e}_\alpha^r, \tag{31a}$$

$$\boldsymbol{\kappa}_\alpha^r = \boldsymbol{Q}^T\boldsymbol{\kappa}_\alpha = \boldsymbol{Q}^T\left(\boldsymbol{\kappa}_\alpha^{\mathrm{e}} + \boldsymbol{Q}^{\mathrm{e}}\boldsymbol{\kappa}_\alpha^{\mathrm{o}}\right) = \boldsymbol{\kappa}_\alpha^{\mathrm{e}r} + \boldsymbol{\kappa}_\alpha^{\mathrm{o}r}. \tag{31b}$$

An alternative form of expressing (31b) is

$$\boldsymbol{\kappa}_\alpha^r = \boldsymbol{Q}^T \boldsymbol{\kappa}_\alpha = \boldsymbol{Q}^T \left(\boldsymbol{\kappa}_\alpha^{\mathrm{e}} + \boldsymbol{Q}^{\mathrm{e}} \boldsymbol{\kappa}_\alpha^{\mathrm{o}}\right) = \boldsymbol{Q}^{\mathrm{o}T} \left(\boldsymbol{\Gamma}^T \boldsymbol{\theta}_{,\alpha} + \boldsymbol{\kappa}_\alpha^{\mathrm{o}}\right). \tag{32}$$

Substituting this results in the deformation gradient expression

$$\begin{aligned} \boldsymbol{F} &= (\boldsymbol{\eta}_\alpha + \boldsymbol{K}_\alpha \boldsymbol{a}) \otimes \boldsymbol{e}_\alpha^r + \boldsymbol{Q} = \boldsymbol{Q}\left(\boldsymbol{Q}^T (\boldsymbol{\eta}_\alpha + \boldsymbol{K}_\alpha \boldsymbol{a}) \otimes \boldsymbol{e}_\alpha^r + \boldsymbol{I}\right) \\ &= \boldsymbol{Q}\left(\boldsymbol{I} + (\boldsymbol{\eta}_\alpha^r + (\boldsymbol{\kappa}_\alpha^r \times \boldsymbol{a}^r)) \otimes \boldsymbol{e}_\alpha^r\right) = \boldsymbol{Q}\left(\boldsymbol{I} + \boldsymbol{\gamma}_\alpha^r \otimes \boldsymbol{e}_\alpha^r\right) \end{aligned} \tag{33}$$

where it was defined the vector

$$\boldsymbol{\gamma}_\alpha^r = \boldsymbol{\eta}_\alpha^r + \boldsymbol{\kappa}_\alpha^r \times \boldsymbol{a}^r. \tag{34}$$

It is still possible to write $\boldsymbol{F} = \boldsymbol{Q}\boldsymbol{F}^r$, where $\boldsymbol{F}^r = \boldsymbol{I} + \boldsymbol{\gamma}_\alpha^r \otimes \boldsymbol{e}_\alpha^r$ is the total back-rotated deformation gradient. This tensor can be written as

$$\boldsymbol{F}^r = \boldsymbol{I} + \boldsymbol{\gamma}_\alpha^r \otimes \boldsymbol{e}_\alpha^r = \boldsymbol{e}_i^r \otimes \boldsymbol{e}_i^r + \boldsymbol{\gamma}_\alpha^r \otimes \boldsymbol{e}_\alpha^r = (\boldsymbol{e}_\alpha^r + \boldsymbol{\gamma}_\alpha^r) \otimes \boldsymbol{e}_\alpha^r + \boldsymbol{e}_3^r \otimes \boldsymbol{e}_3^r = \boldsymbol{f}_i^r \otimes \boldsymbol{e}_i^r \tag{35}$$

where

$$\boldsymbol{f}_\alpha^r = \boldsymbol{e}_\alpha^r + \boldsymbol{\gamma}_\alpha^r, \tag{36a}$$

$$\boldsymbol{f}_3^r = \boldsymbol{e}_3^r. \tag{36b}$$

The total deformation gradient may also be expressed by

$$\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\xi}_\alpha} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{x}^{\mathrm{o}}} \frac{\partial \boldsymbol{x}^{\mathrm{o}}}{\partial \boldsymbol{x}} = \boldsymbol{F}^{\mathrm{e}} \boldsymbol{F}^{\mathrm{o}} \tag{37}$$

from where it can be concluded

$$\boldsymbol{F}^{\mathrm{e}} = \boldsymbol{F} \boldsymbol{F}^{\mathrm{o}-1}. \tag{38}$$

It is possible to evaluate explicitly $\boldsymbol{F}^{\mathrm{o}-1}$ by

$$\boldsymbol{F}^{\mathrm{o}-1} = \left(\boldsymbol{Q}^{\mathrm{o}} \boldsymbol{F}^{\mathrm{o}r}\right)^{-1} = \boldsymbol{F}^{\mathrm{o}r-1} \boldsymbol{Q}^{\mathrm{o}-1} = \frac{1}{J^{\mathrm{o}}} \left(\boldsymbol{g}_i^{\mathrm{o}r} \otimes \boldsymbol{e}_i^r\right)^T \boldsymbol{Q}^{\mathrm{o}T} = \frac{1}{J^{\mathrm{o}}} \left(\boldsymbol{e}_i^r \otimes \boldsymbol{g}_i^{\mathrm{o}r}\right) \boldsymbol{Q}^{\mathrm{o}T} \tag{39}$$

where $J^{\mathrm{o}} = \det \boldsymbol{F}^{\mathrm{o}r}$[7] is given by

$$J^{\mathrm{o}} = \det\left(\boldsymbol{F}^{\mathrm{o}r}\right) = \det\left(\boldsymbol{f}_i^{\mathrm{o}r} \otimes \boldsymbol{e}_i^r\right) = \boldsymbol{f}_1^{\mathrm{o}r} \cdot \boldsymbol{f}_2^{\mathrm{o}r} \times \boldsymbol{f}_3^{\mathrm{o}r} \tag{40}$$

and

$$\boldsymbol{g}_1^{\mathrm{o}r} = \boldsymbol{f}_2^{\mathrm{o}r} \times \boldsymbol{f}_3^{\mathrm{o}r}, \tag{41a}$$

$$\boldsymbol{g}_2^{\mathrm{o}r} = \boldsymbol{f}_3^{\mathrm{o}r} \times \boldsymbol{f}_1^{\mathrm{o}r}, \tag{41b}$$

$$\boldsymbol{g}_3^{\mathrm{o}r} = \boldsymbol{f}_1^{\mathrm{o}r} \times \boldsymbol{f}_2^{\mathrm{o}r}. \tag{41c}$$

Hence, from (38) the effective deformation gradient is

$$\begin{aligned} \boldsymbol{F}^{\mathrm{e}} &= \boldsymbol{F}\boldsymbol{F}^{\mathrm{o}-1} = \boldsymbol{Q}\left(\boldsymbol{f}_i^r \otimes \boldsymbol{e}_i^r\right) J^{\mathrm{o}-1} \left(\boldsymbol{e}_j^r \otimes \boldsymbol{g}_j^{\mathrm{o}r}\right) \boldsymbol{Q}^{\mathrm{o}T} = \boldsymbol{Q} J^{\mathrm{o}-1} \left(\boldsymbol{f}_i^r \otimes \boldsymbol{g}_j^{\mathrm{o}r}\right) \delta_{ij} \boldsymbol{Q}^{\mathrm{o}T} \\ &= \boldsymbol{Q}\left(\boldsymbol{f}_i^{\mathrm{e}r} \otimes \boldsymbol{e}_i^r\right) \boldsymbol{Q}^{\mathrm{o}T} = \boldsymbol{Q}\left(\boldsymbol{f}_i^{\mathrm{e}r} \otimes \left(\boldsymbol{Q}^{\mathrm{o}} \boldsymbol{e}_i^r\right)\right) = \boldsymbol{Q}\left(\boldsymbol{f}_i^{\mathrm{e}r} \otimes \boldsymbol{e}_i^{\mathrm{o}}\right) \end{aligned} \tag{42}$$

[7] The superscript r in J^{o} was suppressed because $\det\left(\boldsymbol{F}^{\mathrm{o}}\right) = \det\left(\boldsymbol{F}^{\mathrm{o}r}\right)$ and, therefore, $J^{\mathrm{o}} = J^{\mathrm{o}r}$.

where

$$\boldsymbol{f}_j^{\mathrm{er}} = J^{\mathrm{o}-1} \boldsymbol{f}_i^r \left(\boldsymbol{g}_i^{\mathrm{or}} \cdot \boldsymbol{e}_j^r \right). \tag{43}$$

This equation can be particularized to the circumstance that $j = \alpha$ and $j = 3$. It can be proved that

$$\boldsymbol{f}_3^{\mathrm{er}} = J^{\mathrm{o}-1} \boldsymbol{f}_i^r \left(\boldsymbol{g}_i^{\mathrm{or}} \cdot \boldsymbol{e}_3^r \right) = \boldsymbol{f}_3^r = \boldsymbol{e}_3^r, \tag{44a}$$

$$\boldsymbol{f}_\alpha^{\mathrm{er}} = J^{\mathrm{o}-1} \boldsymbol{f}_i^r \left(\boldsymbol{g}_i^{\mathrm{or}} \cdot \boldsymbol{e}_\alpha^r \right) = J^{\mathrm{o}-1} \boldsymbol{f}_\beta^r \left(\boldsymbol{g}_\beta^{\mathrm{or}} \cdot \boldsymbol{e}_\alpha^r \right) + J^{\mathrm{o}-1} \boldsymbol{f}_3^r \left(\boldsymbol{g}_3^{\mathrm{or}} \cdot \boldsymbol{e}_\alpha^r \right) = J^{\mathrm{o}-1} \boldsymbol{f}_\beta^r \left(\boldsymbol{g}_\beta^{\mathrm{or}} \cdot \boldsymbol{e}_\alpha^r \right). \tag{44b}$$

A effective strain deformation vector may now be introduced as

$$\boldsymbol{\gamma}_\alpha^{\mathrm{er}} = \boldsymbol{f}_\alpha^{\mathrm{er}} - \boldsymbol{e}_\alpha^r = J^{\mathrm{o}-1} \left(\boldsymbol{e}_\alpha^r \cdot \boldsymbol{g}_\beta^{\mathrm{or}} \right) \left(\boldsymbol{e}_\beta^r + \boldsymbol{\gamma}_\beta^r \right) - \boldsymbol{e}_\alpha^r. \tag{45}$$

3.4 Velocity gradient

The velocity gradient, *i. e.*, the time variation of the total displacements gradient is given by

$$\begin{aligned} \dot{\boldsymbol{F}} &= \dot{\overline{(\boldsymbol{Q}\boldsymbol{F}^r)}} = \dot{\boldsymbol{Q}}\boldsymbol{F}^r + \boldsymbol{Q}\dot{\boldsymbol{F}}^r = \dot{\boldsymbol{Q}} \left(\boldsymbol{I} + \boldsymbol{\gamma}_\alpha^r \otimes \boldsymbol{e}_\alpha^r \right) + \boldsymbol{Q} \dot{\overline{(\boldsymbol{I} + \boldsymbol{\gamma}_\alpha^r \otimes \boldsymbol{e}_\alpha^r)}} \\ &= \boldsymbol{\Omega}\boldsymbol{Q} \left(\boldsymbol{I} + \boldsymbol{\gamma}_\alpha^r \otimes \boldsymbol{e}_\alpha^r \right) + \boldsymbol{Q} \left(\dot{\boldsymbol{I}} + \dot{\boldsymbol{\gamma}}_\alpha^r \otimes \boldsymbol{e}_\alpha^r + \boldsymbol{\gamma}_\alpha^r \otimes \dot{\boldsymbol{e}}_\alpha^r \right) = \boldsymbol{\Omega}\boldsymbol{F} + \boldsymbol{Q} \left(\dot{\boldsymbol{\gamma}}_\alpha^r \otimes \boldsymbol{e}_\alpha^r \right) \end{aligned} \tag{46}$$

where the skew-symmetric tensor of the angular velocity was introduced

$$\boldsymbol{\Omega} = \dot{\boldsymbol{Q}}\boldsymbol{Q}^T. \tag{47}$$

Notice that

$$\boldsymbol{\Omega} = \dot{\overline{(\boldsymbol{Q}^{\mathrm{e}}\boldsymbol{Q}^{\mathrm{o}})}} \left(\boldsymbol{Q}^{\mathrm{e}}\boldsymbol{Q}^{\mathrm{o}} \right)^T = \left(\dot{\boldsymbol{Q}}^{\mathrm{e}}\boldsymbol{Q}^{\mathrm{o}} + \boldsymbol{Q}^{\mathrm{e}}\dot{\boldsymbol{Q}}^{\mathrm{o}} \right) \boldsymbol{Q}^{\mathrm{o}T}\boldsymbol{Q}^{\mathrm{e}T} = \dot{\boldsymbol{Q}}^{\mathrm{e}}\boldsymbol{Q}^{\mathrm{o}}\boldsymbol{Q}^{\mathrm{o}T}\boldsymbol{Q}^{\mathrm{e}T} = \dot{\boldsymbol{Q}}^{\mathrm{e}}\boldsymbol{Q}^{\mathrm{e}T}. \tag{48}$$

Moreover

$$\dot{\boldsymbol{\gamma}}_\alpha^r = \dot{\overline{(\boldsymbol{\eta}_\alpha^r + \boldsymbol{\kappa}_\alpha^r \times \boldsymbol{a}^r)}} = \dot{\boldsymbol{\eta}}_\alpha^r + \dot{\boldsymbol{\kappa}}_\alpha^r \times \boldsymbol{a}^r + \boldsymbol{\kappa}_\alpha^r \times \dot{\boldsymbol{a}}^r = \dot{\boldsymbol{\eta}}_\alpha^r + \dot{\boldsymbol{\kappa}}_\alpha^r \times \boldsymbol{a}^r. \tag{49}$$

The terms involving time variations can be set as functions of the generalized displacements. Hence,

$$\dot{\boldsymbol{\eta}}_\alpha^r = \dot{\overline{(\boldsymbol{Q}^T\boldsymbol{z}_{,\alpha} - \boldsymbol{e}_\alpha^r)}} = \dot{\boldsymbol{Q}}^T\boldsymbol{z}_{,\alpha} + \boldsymbol{Q}^T\dot{\boldsymbol{z}}_{,\alpha} - \dot{\boldsymbol{e}}_\alpha^r = \boldsymbol{Q}^T \left(\dot{\boldsymbol{u}}_{,\alpha} + \boldsymbol{Z}_{,\alpha}\boldsymbol{\Gamma}\dot{\boldsymbol{\theta}} \right) \tag{50a}$$

$$\dot{\boldsymbol{\kappa}}_\alpha^r = \dot{\overline{(\boldsymbol{\kappa}_\alpha^{\mathrm{er}} + \boldsymbol{\kappa}_\alpha^{\mathrm{or}})}} = \dot{\overline{(\boldsymbol{Q}^T\boldsymbol{\kappa}_\alpha^{\mathrm{e}})}} = \dot{\boldsymbol{Q}}^T\boldsymbol{\kappa}_\alpha^{\mathrm{e}} + \boldsymbol{Q}^T\dot{\boldsymbol{\kappa}}_\alpha^{\mathrm{e}} = \boldsymbol{Q}^T \left(\boldsymbol{\Gamma}_{,\alpha}\dot{\boldsymbol{\theta}} + \boldsymbol{\Gamma}\dot{\boldsymbol{\theta}}_{,\alpha} \right) \tag{50b}$$

where the equality $\boldsymbol{\omega}_{,\alpha} = \dot{\boldsymbol{\kappa}}_\alpha - \boldsymbol{\omega} \times \boldsymbol{\kappa}_\alpha$ was used.

The tensor $\boldsymbol{\Gamma}_{,\alpha}$

$$\boldsymbol{\Gamma}_{,\alpha} = h_2(\theta)\,\boldsymbol{\Theta}_{,\alpha} + h_3(\theta)\left(\boldsymbol{\Theta}\boldsymbol{\Theta}_{,\alpha} + \boldsymbol{\Theta}_{,\alpha}\boldsymbol{\Theta}\right) + h_4(\theta)\left(\boldsymbol{\theta} \cdot \boldsymbol{\theta}_{,\alpha}\right)\boldsymbol{\Theta} + h_5(\theta)\left(\boldsymbol{\theta} \cdot \boldsymbol{\theta}_{,\alpha}\right)\boldsymbol{\Theta}^2 \tag{51}$$

where $\boldsymbol{\Theta}_{,\alpha} = \mathrm{Skew}\left(\boldsymbol{\theta}_{,\alpha}\right)$ the trigonometric functions $h_4(\theta)$ and $h_5(\theta)$ are

$$h_4(\theta) = \frac{h_1(\theta) - 2h_2(\theta)}{\theta^2}, \qquad h_5(\theta) = \frac{h_2(\theta) - 3h_3(\theta)}{\theta^2}. \tag{52}$$

The generalized strains of the shell model can be collected in the vector

$$\boldsymbol{\varepsilon}^r = \begin{bmatrix} \boldsymbol{\varepsilon}_1 \\ \boldsymbol{\varepsilon}_2 \end{bmatrix} \qquad \text{where} \qquad \boldsymbol{\varepsilon}^r_\alpha = \begin{bmatrix} \boldsymbol{\eta}^r_\alpha \\ \boldsymbol{\kappa}^r_\alpha \end{bmatrix} = \begin{bmatrix} \boldsymbol{Q}^T \boldsymbol{z}_\alpha - \boldsymbol{e}^r_\alpha \\ \boldsymbol{Q}^{\mathrm{o}T} \left(\boldsymbol{\Gamma}^T \boldsymbol{\theta}_{,\alpha} + \boldsymbol{\kappa}^{\mathrm{o}}_\alpha \right) . \end{bmatrix} \tag{53}$$

The time variation of the generalized strains can be collected in a vector

$$\dot{\boldsymbol{\varepsilon}}^r = \begin{bmatrix} \dot{\boldsymbol{\varepsilon}}_1 \\ \dot{\boldsymbol{\varepsilon}}_2 \end{bmatrix} \qquad \text{where} \qquad \dot{\boldsymbol{\varepsilon}}^r_\alpha = \begin{bmatrix} \dot{\boldsymbol{\eta}}^r_\alpha \\ \dot{\boldsymbol{\kappa}}^r_\alpha \end{bmatrix} = \begin{bmatrix} \boldsymbol{Q}^T \left(\dot{\boldsymbol{u}}_{,\alpha} + \boldsymbol{Z}_{,\alpha} \boldsymbol{\Gamma} \dot{\boldsymbol{\theta}} \right) \\ \boldsymbol{Q}^T \left(\boldsymbol{\Gamma}_{,\alpha} \dot{\boldsymbol{\theta}} + \boldsymbol{\Gamma} \dot{\boldsymbol{\theta}}_{,\alpha} \right) . \end{bmatrix} \tag{54}$$

Introducing the generalized displacements vector, $\boldsymbol{d}$, given by

$$\boldsymbol{d} = \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{\theta} \end{bmatrix} \tag{55}$$

the time variation of the generalized strains can be recast in the compact form

$$\dot{\boldsymbol{\varepsilon}}^r = \boldsymbol{\Psi} \boldsymbol{\Delta} \dot{\boldsymbol{d}} \tag{56}$$

where $\boldsymbol{\Psi}$ and $\boldsymbol{\Delta}$ are defined in [1].

4 STATICS

4.1 Generalized stresses

The jacobian of the displacement field mapping is given by

$$J = \det \boldsymbol{F} = \det \left(\boldsymbol{F}^{\mathrm{e}} \boldsymbol{F}^{\mathrm{o}} \right) = \det \boldsymbol{F}^{\mathrm{e}} \det \boldsymbol{F}^{\mathrm{o}} = J^{\mathrm{e}} J^{\mathrm{o}} \tag{57}$$

where the effective jacobian is $J^{\mathrm{e}} = \det \boldsymbol{F}^{\mathrm{e}}$.

The effective fist Piola-Kirchhoff stress tensor is given by

$$\boldsymbol{P}^{\mathrm{e}} = J^{\mathrm{e}} \boldsymbol{T} \boldsymbol{F}^{\mathrm{e}-T} \tag{58}$$

where $\boldsymbol{T}$ is the Cauchy stress tensor. Solving for $\boldsymbol{T}$ in the previous equation and using (**??**), the total fist Piola-Kirchhoff stress tensor yields

$$\boldsymbol{P} = J \boldsymbol{T} \boldsymbol{F}^{-T} = J \left(J^{\mathrm{e}-1} \boldsymbol{P}^{\mathrm{e}} \boldsymbol{F}^{\mathrm{e}T} \right) \boldsymbol{F}^{-T} = J^{\mathrm{e}} J^{\mathrm{o}} \left(J^{\mathrm{e}-1} \boldsymbol{P}^{\mathrm{e}} \boldsymbol{F}^{\mathrm{e}T} \right) \left(\boldsymbol{F}^{\mathrm{e}} \boldsymbol{F}^{\mathrm{o}} \right)^{-T} = J^{\mathrm{o}} \boldsymbol{P}^{\mathrm{e}} \boldsymbol{F}^{\mathrm{o}-T}. \tag{59}$$

Taking into account (38) the latter equation can assume the form

$$\boldsymbol{P} = J \boldsymbol{T} \boldsymbol{F}^{-T} = J^{\mathrm{o}} \boldsymbol{P}^{\mathrm{e}} \left(\frac{1}{J^{\mathrm{o}}} \left(\boldsymbol{g}^{\mathrm{o}r}_i \otimes \boldsymbol{e}^r_i \right) \boldsymbol{Q}^{\mathrm{o}T} \right) = \boldsymbol{P}^{\mathrm{e}} \boldsymbol{Q}^{\mathrm{o}} \left(\boldsymbol{g}^{\mathrm{o}r}_i \otimes \boldsymbol{e}^r_i \right) . \tag{60}$$

As the effective first Piola-Kirchhoff stress tensor, $\boldsymbol{P}^{\mathrm{e}}$, can always be expressed as

$$\boldsymbol{P}^{\mathrm{e}} = \boldsymbol{\tau}^{\mathrm{e}}_i \otimes \boldsymbol{e}^o_i = \boldsymbol{Q} \boldsymbol{\tau}^{\mathrm{e}r}_i \otimes \boldsymbol{e}^o_i \tag{61}$$

equation (60) may be rewritten as

$$\begin{aligned} \boldsymbol{P} &= \left(\boldsymbol{Q} \boldsymbol{\tau}^{\mathrm{e}r}_i \otimes \boldsymbol{e}^o_i \right) \boldsymbol{Q}^{\mathrm{o}} \left(\boldsymbol{g}^{\mathrm{o}r}_j \otimes \boldsymbol{e}^r_j \right) = \left(\boldsymbol{Q} \boldsymbol{\tau}^{\mathrm{e}r}_i \otimes \boldsymbol{e}^r_i \right) \left(\boldsymbol{g}^{\mathrm{o}r}_j \otimes \boldsymbol{e}^r_j \right) \\ &= \left(\boldsymbol{Q} \boldsymbol{\tau}^{\mathrm{e}r}_i \otimes \boldsymbol{e}^r_j \right) \left(\boldsymbol{g}^{\mathrm{o}r}_j \cdot \boldsymbol{e}^r_i \right) = \boldsymbol{Q} \left(\boldsymbol{g}^{\mathrm{o}r}_j \cdot \boldsymbol{e}^r_i \right) \boldsymbol{\tau}^{\mathrm{e}r}_i \otimes \boldsymbol{e}^r_j = \boldsymbol{Q} \boldsymbol{\tau}^r_i \otimes \boldsymbol{e}^r_i \end{aligned} \tag{62}$$

where $\boldsymbol{\tau}_j^r = \left(\boldsymbol{g}_j^{\mathrm{o}r} \cdot \boldsymbol{e}_i^r\right) \boldsymbol{\tau}_i^{\mathrm{e}r}$ or

$$\boldsymbol{\tau}_\alpha^r = \left(\boldsymbol{g}_\alpha^{\mathrm{o}r} \cdot \boldsymbol{e}_\beta^r\right) \boldsymbol{\tau}_\beta^{\mathrm{e}r}, \tag{63a}$$
$$\boldsymbol{\tau}_3^r = \left(\boldsymbol{g}_\alpha^{\mathrm{o}r} \cdot \boldsymbol{e}_3^r\right) \boldsymbol{\tau}_3^{\mathrm{e}r} = \left(\boldsymbol{f}_1^{\mathrm{o}r} \cdot \boldsymbol{f}_2^{\mathrm{o}r} \times \boldsymbol{f}_3^{\mathrm{o}r}\right) \boldsymbol{\tau}_3^{\mathrm{e}r} = J^{\mathrm{o}} \boldsymbol{\tau}_3^{\mathrm{e}r}. \tag{63b}$$

After the imposition of the plane stress state, the stress vectors are denoted by "$\widetilde{()}$". Accordingly, equations (61) and (62) are modified to

$$\boldsymbol{P}^{\mathrm{e}} = \boldsymbol{Q} \widetilde{\boldsymbol{\tau}}_i^{\mathrm{e}r} \otimes \boldsymbol{e}_i^o \tag{64a}$$
$$\boldsymbol{P} = \boldsymbol{Q} \widetilde{\boldsymbol{\tau}}_i^r \otimes \boldsymbol{e}_i^r \tag{64b}$$

respectively.

4.2 Internal power

Resorting (59) and (38) and bearing in mind that $\dot{\boldsymbol{F}}^{\mathrm{o}} = \boldsymbol{O}$, the internal power per unit initial configuration volume is

$$\boldsymbol{P}^{\mathrm{e}} : \dot{\boldsymbol{F}}^{\mathrm{e}} = J^{\mathrm{o}-1} \boldsymbol{P} \boldsymbol{F}^{\mathrm{o}T} : \dot{\boldsymbol{F}} \boldsymbol{F}^{\mathrm{o}-1} = J^{\mathrm{o}-1} \boldsymbol{P} : \dot{\boldsymbol{F}}. \tag{65}$$

Taking into account (62) and (46) internal power per unit reference configuration volume is

$$\begin{aligned} \boldsymbol{P} : \dot{\boldsymbol{F}} &= \boldsymbol{P} : \left(\boldsymbol{\Omega} \boldsymbol{F} + \boldsymbol{Q} \left(\dot{\boldsymbol{\gamma}}_\alpha^r \otimes \boldsymbol{e}_\alpha^r\right)\right) = \boldsymbol{P} : \boldsymbol{\Omega} \boldsymbol{F} + \boldsymbol{P} : \boldsymbol{Q} \left(\dot{\boldsymbol{\gamma}}_\alpha^r \otimes \boldsymbol{e}_\alpha^r\right) \\ &= \boldsymbol{P} \boldsymbol{F}^T : \boldsymbol{\Omega} + \boldsymbol{Q} \widetilde{\boldsymbol{\tau}}_i^r \otimes \boldsymbol{e}_i^r : \boldsymbol{Q} \left(\dot{\boldsymbol{\gamma}}_\alpha^r \otimes \boldsymbol{e}_\alpha^r\right) = \widetilde{\boldsymbol{\tau}}_\alpha^r \cdot \dot{\boldsymbol{\gamma}}_\alpha^r \end{aligned} \tag{66}$$

where the symmetry condition $\boldsymbol{P} \boldsymbol{F}^T = \left(\boldsymbol{P} \boldsymbol{F}^T\right)^T$ was introduced.

Moreover, substituting (49) in (65) and (66) delivers

$$\boldsymbol{P} : \dot{\boldsymbol{F}} = \widetilde{\boldsymbol{\tau}}_\alpha^r \cdot \dot{\boldsymbol{\gamma}}_\alpha^r = \widetilde{\boldsymbol{\tau}}_\alpha^r \cdot \left(\dot{\boldsymbol{\eta}}_\alpha^r + \dot{\boldsymbol{\kappa}}_\alpha^r \times \boldsymbol{a}^r\right) = \widetilde{\boldsymbol{\tau}}_\alpha^r \cdot \dot{\boldsymbol{\eta}}_\alpha^r + \boldsymbol{a}^r \times \widetilde{\boldsymbol{\tau}}_\alpha^r \cdot \dot{\boldsymbol{\kappa}}_\alpha^r. \tag{67}$$

Noticing that $\mathrm{d}V^{\mathrm{o}} = J^{\mathrm{o}} \mathrm{d}V^r$ and using the former equation, the total internal power follows as

$$\begin{aligned} P_{\text{int}} &= \int_{V^{\mathrm{o}}} \boldsymbol{P}^{\mathrm{e}} : \dot{\boldsymbol{F}}^{\mathrm{e}} \mathrm{d}V^{\mathrm{o}} = \int_{V^r} J^{\mathrm{o}-1} \boldsymbol{P} : \dot{\boldsymbol{F}} J^{\mathrm{o}} \mathrm{d}V^r \\ &= \int_{\Omega^r} \int_{H^r} \left(\widetilde{\boldsymbol{\tau}}_\alpha^r \cdot \dot{\boldsymbol{\eta}}_\alpha^r + \boldsymbol{a}^r \times \widetilde{\boldsymbol{\tau}}_\alpha^r \cdot \dot{\boldsymbol{\kappa}}_\alpha^r\right) \mathrm{d}\zeta \mathrm{d}\Omega^r = \int_{\Omega^r} \left(\boldsymbol{n}_\alpha^r \cdot \dot{\boldsymbol{\eta}}_\alpha^r + \boldsymbol{m}_\alpha^r \cdot \dot{\boldsymbol{\kappa}}_\alpha^r\right) \mathrm{d}\Omega^r \end{aligned} \tag{68}$$

where the following stress resultants where introduced

$$\boldsymbol{n}_\alpha^r = \int_{H^r} \widetilde{\boldsymbol{\tau}}_\alpha^r \mathrm{d}\zeta, \qquad \boldsymbol{m}_\alpha^r = \int_{H^r} \boldsymbol{a}^r \times \widetilde{\boldsymbol{\tau}}_\alpha^r \mathrm{d}\zeta. \tag{69}$$

Collecting this generalized forces in the $\boldsymbol{\sigma}^r$ vector as

$$\boldsymbol{\sigma}^r = \begin{bmatrix} \boldsymbol{\sigma}_1^r \\ \boldsymbol{\sigma}_2^r \end{bmatrix} \quad \text{where} \quad \boldsymbol{\sigma}_\alpha^r = \begin{bmatrix} \boldsymbol{n}_\alpha^r \\ \boldsymbol{m}_\alpha^r \end{bmatrix} \tag{70}$$

the internal power can assume the compact form

$$P_{\text{int}} = \int_{\Omega^r} \boldsymbol{\sigma}^r \cdot \dot{\boldsymbol{\varepsilon}}^r \mathrm{d}\Omega^r. \tag{71}$$

4.3 External power

The external power may be expressed as

$$
\begin{aligned}
P_{\text{ext}} = & \int_{\Omega^{ot}} \overline{\boldsymbol{t}}^{o} \cdot \dot{\boldsymbol{x}} \mathrm{d}\Omega^{ot} + \int_{\Omega^{ob}} \overline{\boldsymbol{t}}^{o} \cdot \dot{\boldsymbol{x}} \mathrm{d}\Omega^{ob} + \int_{V^{o}} \overline{\boldsymbol{b}}^{o} \cdot \dot{\boldsymbol{x}} \mathrm{d}V^{o} \\
& + \int_{\Gamma_t^{o}} \int_{H^{o}} \overline{\boldsymbol{t}}^{o} \cdot \dot{\boldsymbol{x}} \mathrm{d}\zeta \mathrm{d}\Gamma_t^{o} + \int_{\Gamma_u^{o}} \int_{H^{o}} \boldsymbol{r}^{o} \cdot \dot{\boldsymbol{x}} \mathrm{d}\zeta \mathrm{d}\Gamma_u^{o}
\end{aligned}
\tag{72}
$$

where $\boldsymbol{r}^{o}$ are the reaction tractions on the kinematic boundary, per unit area of the initial configuration. The former equation can easily be rewritten in the reference configuration as

$$
\begin{aligned}
P_{\text{ext}} = & \int_{\Omega^{r}} \left(\overline{\boldsymbol{t}}^{t} \cdot \dot{\boldsymbol{x}} + \overline{\boldsymbol{t}}^{b} \cdot \dot{\boldsymbol{x}} + \int_{H^{r}} \overline{\boldsymbol{b}} \cdot \dot{\boldsymbol{x}} \mathrm{d}\zeta \right) \mathrm{d}\Omega^{r} \\
& + \int_{\Gamma_t^{r}} \int_{H^{r}} \overline{\boldsymbol{t}}^{l} \cdot \dot{\boldsymbol{x}} \mathrm{d}\zeta \mathrm{d}\Gamma_t^{r} + \int_{\Gamma_u^{r}} \int_{H^{r}} \boldsymbol{r} \cdot \dot{\boldsymbol{x}} \mathrm{d}\zeta \mathrm{d}\Gamma_u^{r}
\end{aligned}
\tag{73}
$$

if we introduce the definitions[8] $\overline{\boldsymbol{t}}^{ot} = J^{ot}\overline{\boldsymbol{t}}^{t}$, $\overline{\boldsymbol{t}}^{ob} = J^{ob}\overline{\boldsymbol{t}}^{b}$, $\overline{\boldsymbol{t}}^{ol} = J^{ol}\overline{\boldsymbol{t}}^{l}$ and $\overline{\boldsymbol{r}}^{o} = J^{ol}\overline{\boldsymbol{r}}$ with the notation for the transformation jacobians $\mathrm{d}V^{o} = J^{o}\mathrm{d}V^{r}$, $\mathrm{d}\Omega^{ot} = J^{ot}\mathrm{d}\Omega^{r}$, $\mathrm{d}\Omega^{ob} = J^{ob}\mathrm{d}\Omega^{r}$, $\mathrm{d}\Gamma_t^{o} = J^{ol}\mathrm{d}\Gamma_t^{r}$ and $\mathrm{d}\Gamma_u^{o} = J^{ol}\mathrm{d}\Gamma_u^{r}$.

Introducing (21) in (8) and performing time differentiation on both sides and then substituting the result in (73) yields

$$
\begin{aligned}
P_{\text{ext}} = & \int_{\Omega^{r}} \left(\overline{\boldsymbol{n}}^{\Omega} \cdot \dot{\boldsymbol{u}} + \overline{\boldsymbol{m}}^{\Omega} \cdot \boldsymbol{\omega} \right) \mathrm{d}\Omega^{r} + \int_{\Gamma_t^{r}} \left(\overline{\boldsymbol{n}}^{\Gamma} \cdot \dot{\boldsymbol{u}} + \overline{\boldsymbol{m}}^{\Gamma} \cdot \boldsymbol{\omega} \right) \mathrm{d}\Gamma_t^{r} \\
& + \int_{\Gamma_u^{r}} \left(\boldsymbol{n}^{\lambda} \cdot \dot{\boldsymbol{u}} + \boldsymbol{m}^{\lambda} \cdot \boldsymbol{\omega} \right) \mathrm{d}\Gamma_u^{r}
\end{aligned}
\tag{74}
$$

where,

$$
\overline{\boldsymbol{n}}^{\Omega} = \overline{\boldsymbol{t}}^{t} + \overline{\boldsymbol{t}}^{b} + \int_{H^{r}} \overline{\boldsymbol{b}} \mathrm{d}\zeta \qquad \overline{\boldsymbol{m}}^{\Omega} = \boldsymbol{a}^{t} \times \overline{\boldsymbol{t}}^{t} + \boldsymbol{a}^{b} \times \overline{\boldsymbol{t}}^{b} + \int_{H^{r}} \boldsymbol{a} \times \overline{\boldsymbol{b}} \mathrm{d}\zeta \tag{75a}
$$

$$
\overline{\boldsymbol{n}}^{\Gamma} = \int_{H^{r}} \overline{\boldsymbol{t}}^{l} \mathrm{d}\zeta \qquad \overline{\boldsymbol{m}}^{\Gamma} = \int_{H^{r}} \boldsymbol{a} \times \overline{\boldsymbol{t}}^{l} \mathrm{d}\zeta \tag{75b}
$$

$$
\overline{\boldsymbol{n}}^{\lambda} = \int_{H^{r}} \boldsymbol{r} \mathrm{d}\zeta \qquad \overline{\boldsymbol{m}}^{\lambda} = \int_{H^{r}} \boldsymbol{a} \times \boldsymbol{r} \mathrm{d}\zeta \tag{75c}
$$

cross-sectional generalized resultants, per unit length of the reference configuration, are introduced and the superscripts Ω^{r} and Γ_t^{r} were simplified to Ω and Γ, as no danger of misinterpretation exists.

It is possible to achieve a even compact form for the external power. By defining the vectors

$$
\overline{\boldsymbol{q}}^{\Omega} = \begin{bmatrix} \overline{\boldsymbol{n}}^{\Omega} \\ \overline{\boldsymbol{\mu}}^{\Omega} \end{bmatrix} \qquad \overline{\boldsymbol{q}}^{\Gamma} = \begin{bmatrix} \overline{\boldsymbol{n}}^{\Gamma} \\ \overline{\boldsymbol{\mu}}^{\Gamma} \end{bmatrix} \qquad \boldsymbol{q}^{\lambda} = \begin{bmatrix} \boldsymbol{n}^{\lambda} \\ \boldsymbol{\mu}^{\lambda} \end{bmatrix} \tag{76}
$$

the expression of the external power (74) reads

$$
P_{\text{ext}} = \int_{\Omega^{r}} \overline{\boldsymbol{q}}^{\Omega} \cdot \dot{\boldsymbol{d}} \mathrm{d}\Omega^{r} + \int_{\Gamma_t^{r}} \overline{\boldsymbol{q}}^{\Gamma} \cdot \dot{\boldsymbol{d}} \mathrm{d}\Gamma_t^{r} + \int_{\Gamma_u^{r}} \boldsymbol{q}^{\lambda} \cdot \dot{\boldsymbol{d}} \mathrm{d}\Gamma_u^{r}. \tag{77}
$$

[8] The superscript l stands for *lateral*.

Here, $\overline{\boldsymbol{q}}^{\Omega}$ is the vector resulting from the external loading along the shell middle plane per reference are unit, $\overline{\boldsymbol{q}}^{\Gamma}$ is the vector resulting from the external loading on the static boundary and $\boldsymbol{q}^{\lambda}$ is the vector resulting from the tractions on the kinematic boundary. Notice that $\overline{\boldsymbol{\mu}}^{\Omega} = \boldsymbol{\Gamma}^T\overline{\boldsymbol{m}}^{\Omega}$, $\overline{\boldsymbol{\mu}}^{\Gamma} = \boldsymbol{\Gamma}^T\overline{\boldsymbol{m}}^{\Gamma}$ and $\overline{\boldsymbol{\mu}}^{\lambda} = \boldsymbol{\Gamma}^T\overline{\boldsymbol{m}}^{\lambda}$ are pseudo-moments which are energetically conjugated with $\dot{\boldsymbol{\theta}}$. Notice that the true power conjugate of $\dot{\boldsymbol{\theta}}$ is not simply the moment resultants as usually happens on geometrically linear theories.

5 VARIATIONAL FORMULATION OF THE PROBLEM

5.1 A constrained weak form

The variation of the generalized strain vector was carried out on (49), hence

$$\delta\boldsymbol{\gamma}_{\alpha}^{r} = \delta\boldsymbol{\eta}_{\alpha}^{r} + \delta\boldsymbol{\kappa}_{\alpha}^{r} \times \boldsymbol{a}^{r}. \tag{78}$$

The variation of the generalized strains (31) are analogous to (50), hence

$$\delta\boldsymbol{\eta}_{\alpha}^{r} = \boldsymbol{Q}^T\left(\delta\boldsymbol{u}_{,\alpha} + \boldsymbol{Z}_{,\alpha}\boldsymbol{\Gamma}\delta\boldsymbol{\theta}\right), \tag{79a}$$

$$\delta\boldsymbol{\kappa}_{\alpha}^{r} = \boldsymbol{Q}^T\left(\boldsymbol{\Gamma}_{,\alpha}\delta\boldsymbol{\theta} + \boldsymbol{\Gamma}\delta\boldsymbol{\theta}_{,\alpha}\right). \tag{79b}$$

Resorting to (56)

$$\delta\boldsymbol{\varepsilon}^{r} = \boldsymbol{\Psi}\boldsymbol{\Delta}\delta\boldsymbol{d}. \tag{80}$$

In view of (68) the internal virtual work may, thus, be written as

$$\delta W_{\text{int}} = \int_{\Omega^r} \boldsymbol{\sigma}^r \cdot \delta\boldsymbol{\varepsilon}^r \mathrm{d}\Omega^r. \tag{81}$$

The external virtual work is

$$\delta W_{\text{ext}} = \int_{\Omega^r} \overline{\boldsymbol{q}}^{\Omega} \cdot \delta\boldsymbol{d}\mathrm{d}\Omega^r + \int_{\Gamma_t^r} \overline{\boldsymbol{q}}^{\Gamma} \cdot \delta\boldsymbol{d}\mathrm{d}\Gamma_t^r + \int_{\Gamma_u^r} \boldsymbol{q}^{\lambda} \cdot \delta\boldsymbol{d}\mathrm{d}\Gamma_u^r. \tag{82}$$

Notice the inclusion of the Virtual Work arising from the kinematic boundary, given by the projection of the generalized reactions on the virtual displacements.

The weak form of the equilibrium of the rod can be recast by the following virtual work principle

$$\delta W_{\text{int}} - \delta W_{\text{ext}} = 0, \quad \text{in } \Omega^r, \forall\delta\boldsymbol{d} \tag{83}$$

where $\delta\boldsymbol{d}$ stands for an infinitesimal perturbation of the generalized displacements field.

Let us now assume that the prescribed displacements are given as

$$\overline{\boldsymbol{d}} = \begin{bmatrix} \overline{\boldsymbol{u}} \\ \overline{\boldsymbol{\theta}} \end{bmatrix}, \tag{84}$$

i. e., we assume that the prescribed orientation of the kinematic part of the contour of the shell is already in terms of the Euler-Rodrigues parameters. In general, a rotation tensor can be used to prescribe the displacements. In this case an extraction procedure should be applied, see [16].

The weak imposition of the kinematic boundary conditions reads[9]

$$-\int_{\Gamma_u^r} \delta\boldsymbol{q}^{\lambda} \cdot \left(\boldsymbol{d} - \overline{\boldsymbol{d}}\right) \mathrm{d}\Gamma_u^r = 0, \quad \text{in } \Gamma_u^r, \forall\delta\boldsymbol{q}^{\lambda}. \tag{85}$$

[9] The convenience of the introduction of the minus sign is associated with (i) the attainment of a symmetric linearized weak form and (ii) the possibility of identifying $\boldsymbol{q}^{\lambda}$ with the generalized reaction force.

The combination of the Principle of Virtual Work (83) and the weak constraint imposition (85) gives the final weak form, which is the following hybrid functional

$$\delta W = 0, \quad \text{in } \Omega^r, \tag{86}$$

where

$$\begin{aligned}\delta W = &\int_{\Omega^r} \boldsymbol{\sigma}^r \cdot \delta\boldsymbol{\varepsilon}^r \mathrm{d}\Omega^r - \int_{\Omega^r} \overline{\boldsymbol{q}}^{\Omega} \cdot \delta\boldsymbol{d} \mathrm{d}\Omega^r - \int_{\Gamma_t^r} \overline{\boldsymbol{q}}^{\Gamma} \cdot \delta\boldsymbol{d} \mathrm{d}\Gamma_t^r \\ &- \int_{\Gamma_u^r} \boldsymbol{q}^{\lambda} \cdot \delta\boldsymbol{d} \mathrm{d}\Gamma_u^r - \int_{\Gamma_u^r} \delta\boldsymbol{q}^{\lambda} \cdot \left(\boldsymbol{d} - \overline{\boldsymbol{d}}\right) \mathrm{d}\Gamma_u^r.\end{aligned} \tag{87}$$

Combinations of variational statements were extensively used for generating generalized principles for linear analysis. Here the extension for nonlinear analysis is accomplished.

If the problem under analysis is conservative, the variational form could be derived from a constrained stationary potential energy principle.

Besides the usual requirements in order the integrals in (87) make sense, no additional restrictions are demanded. In particular, the usual $\delta\boldsymbol{d} = \boldsymbol{o}$ on the kinematic boundary points, Γ_u^r, is avoided in order to be able to use approximations not fulfilling the Kronecker-delta property.

5.2 Recovering the governing equations

In this section the governing equations will be derived from the derived weak form. Substituting (79) in (87) and integrating by parts in $\delta\boldsymbol{u}_{,\alpha}$ and $\left(\boldsymbol{\Gamma}\delta\boldsymbol{\theta}\right)_{,\alpha}$ yields

$$\begin{aligned}&-\int_{\Omega^r} \left(\left(\boldsymbol{n}_{\alpha,\alpha} + \overline{\boldsymbol{n}}^{\Omega}\right) \cdot \delta\boldsymbol{u} + \boldsymbol{\Gamma}^T \left(\boldsymbol{m}_{\alpha,\alpha} + \boldsymbol{z}_{,\alpha} \times \boldsymbol{n}_\alpha + \overline{\boldsymbol{m}}^{\Omega}\right) \cdot \delta\boldsymbol{\theta}\right) \mathrm{d}\Omega^r \\ &\quad + \int_{\Gamma_t^r} \left(\left(n_\alpha \boldsymbol{n}_\alpha - \overline{\boldsymbol{n}}^{\Gamma}\right) \cdot \delta\boldsymbol{u} + \left(n_\alpha \boldsymbol{\mu}_\alpha - \overline{\boldsymbol{\mu}}^{\Gamma}\right) \cdot \delta\boldsymbol{\theta}\right) \mathrm{d}\Gamma_t^r \\ &\quad + \int_{\Gamma_u^r} \left(\left(n_\alpha \boldsymbol{n}_\alpha - \boldsymbol{n}^{\lambda}\right) \cdot \delta\boldsymbol{u} + \left(n_\alpha \boldsymbol{\mu}_\alpha - \boldsymbol{\mu}^{\lambda}\right) \cdot \delta\boldsymbol{\theta}\right) \mathrm{d}\Gamma_u^r \\ &\quad - \int_{\Gamma_u^r} \left(\left(\boldsymbol{u} - \overline{\boldsymbol{u}}\right) \cdot \delta\boldsymbol{n}^{\lambda} + \left(\boldsymbol{\theta} - \overline{\boldsymbol{\theta}}\right) \cdot \delta\boldsymbol{\mu}^{\lambda}\right) \mathrm{d}\Gamma_u^r = 0,\end{aligned} \tag{88}$$

where n_α denotes the outward normal components[10].

The Euler-Lagrange equations of (88) are

$$\boldsymbol{n}_{\alpha,\alpha} + \overline{\boldsymbol{n}} = \boldsymbol{o} \quad n_\alpha \boldsymbol{n}_\alpha - \overline{\boldsymbol{n}}^{\Gamma} = \boldsymbol{o} \quad n_\alpha \boldsymbol{n}_\alpha - \boldsymbol{n}^{\lambda} = \boldsymbol{o} \quad \boldsymbol{u} - \overline{\boldsymbol{u}} = \boldsymbol{o} \tag{89a}$$

$$\boldsymbol{m}_{\alpha,\alpha} + \boldsymbol{z}_{,\alpha} \times \boldsymbol{n}_\alpha + \overline{\boldsymbol{m}} = \boldsymbol{o} \quad n_\alpha \boldsymbol{\mu}_\alpha - \overline{\boldsymbol{\mu}}^{\Gamma} = \boldsymbol{o} \quad n_\alpha \boldsymbol{\mu}_\alpha - \boldsymbol{\mu}^{\lambda} = \boldsymbol{o} \quad \boldsymbol{\theta} - \overline{\boldsymbol{\theta}} = \boldsymbol{o} \tag{89b}$$

on the domain, Ω^r, on the static boundary, Γ_t^r, and on the kinematic boundary, Γ_u^r, (the last two sets), respectively.

The sets of equations $(89)_{1,2}$ simply express the equilibrium on the domain and on the static boundary between the applied generalized forces and the internal generalized forces. These are the usual set of equations imposed in a weak sense in the traditional FEM (besides the pointwise imposition of essential boundary values). The set $(89)_3$ are the equilibrium equations on the kinematic boundary. This apparent contradiction is, in fact, what is being imposed: the equilibrium between the internal generalized forces and the independently approximated generalized reaction forces. The set $(89)_4$ is the compatibility on the kinematic boundary.

[10]Notice the subtle difference between the outward normal components, n_α, and the force resultants per unit length on the reference configuration, $\boldsymbol{n}_\alpha$.

6 LINEARIZATION OF THE WEAK FORM

6.1 Newton/Raphson's type of incremental/iterative process

For the solution of the weak form of the problem, stated by (87), within a Newton/Raphson's type of incremental/iterative process is crucial to explicitly know the exact tangent operator. This can be achieved by the consistent linearization of the weak form. Here this process must be performed not only on the generalized displacements, as usually is done, but also in the generalized reaction forces[11].

The incremental/iterative perturbation, Δ, of the Virtual Work statement (86) yields the linearization of the hybrid functional in $\boldsymbol{d}$ and $\boldsymbol{q}^\lambda$, *i. e.*, $\Delta\delta W$, where

$$\begin{aligned}\Delta\delta W = &\int_{\Omega^r} \left((\boldsymbol{\Psi}\boldsymbol{\Delta}\delta\boldsymbol{d})\cdot(\boldsymbol{D}\boldsymbol{\Psi}\boldsymbol{\Delta}\Delta\boldsymbol{d}) + (\boldsymbol{\Delta}\delta\boldsymbol{d})\cdot(\boldsymbol{G}\boldsymbol{\Delta}\Delta\boldsymbol{d}) - \delta\boldsymbol{d}\cdot\left(\boldsymbol{L}^{\Omega}\Delta\boldsymbol{d}\right)\right) \mathrm{d}\Omega^r \\ &- \int_{\Gamma_t^r} \delta\boldsymbol{d}\cdot\left(\boldsymbol{L}^{\Gamma}\Delta\boldsymbol{d}\right)\mathrm{d}\Gamma_t^r - \int_{\Gamma_u^r}\delta\boldsymbol{d}\cdot\Delta\boldsymbol{q}^\lambda \mathrm{d}\Gamma_u^r - \int_{\Gamma_u^r}\delta\boldsymbol{q}^\lambda\cdot\Delta\boldsymbol{d}\mathrm{d}\Gamma_u^r.\end{aligned} \tag{90}$$

Here $\boldsymbol{D}$, $\boldsymbol{G}$, $\boldsymbol{L}^\Omega$ and $\boldsymbol{L}^\Gamma$ are the constitutive, geometric, load on the domain and load on the static boundary generalized tangent stiffness operators. The definition of these can be found on [14, 15], except for the last term, which was not taken into account in those works. Nevertheless, its value can be inferred from $\boldsymbol{L}^\Omega$. For conservative loadings, these two latter matrix operators are always symmetric. For the common case of only applied load on the middle surface both of these operators are null.

Notice the two last terms on (90) do not depend on the generalized displacements themselves, but only on their virtual and incremental/iterative counterparts.

7 A MESHFREE METHOD

7.1 Shear locking-free approximation functions

The approximation of the *six* generalized displacements fields over the *plane* reference system is made through MLS nodal functions. The use of this complex and computationally demanding, relatively to the polynomial nodal shape functions used by common FEM, functions is justified by their (i) reproducing properties and (ii) inherent prescribed continuity (which is limited by the basis and/or the bell-shaped weight function).

For the kinematic boundary, simple Lagrange polynomials can be used, but other options are also possible, like one-dimensional MLS, see [17].

Hence, consider the following approximations[12]

$$\boldsymbol{d} = \mathbf{\Phi}\mathbf{d} \qquad \boldsymbol{q}^\lambda = \mathbf{\Psi}\mathbf{q}^\lambda \tag{91a}$$

$$\delta\boldsymbol{d} = \mathbf{\Phi}\delta\mathbf{d} \qquad \delta\boldsymbol{q}^\lambda = \mathbf{\Psi}\delta\mathbf{q}^\lambda \tag{91b}$$

$$\Delta\boldsymbol{d} = \mathbf{\Phi}\Delta\mathbf{d} \qquad \Delta\boldsymbol{q}^\lambda = \mathbf{\Psi}\Delta\mathbf{q}^\lambda \tag{91c}$$

[11] Although the pure Newton/Raphson is a very robust method, the complete solution of certain problems can be greatly simplified by the resource of *ad hoc* schemes that combine the variables at stake. This subject will be dealt in detail in section 8.2, but, for using this methods, is also necessary to perform the linearization of the weak form in the load parameter, λ. As this parameter varies linearly with the load, the linearization task is trivial and, thus, not performed here.

[12] Notice the subtle difference between the matricial differential operator $\boldsymbol{\Psi}$ defined in [1] and the matrix $\mathbf{\Psi}$ that collects the approximation functions of the static boundary tractions.

for the real, virtual and incremental/iterative fields, respectively. It would be tempting to use the approximations

$$\mathbf{\Phi} = \begin{bmatrix} \phi_1^u \mathbf{I} & \mathbf{O} & \dots & \phi_n^u \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \phi_1^\theta \mathbf{I} & & \mathbf{O} & \phi_n^\theta \mathbf{I} \end{bmatrix} \qquad \mathbf{\Psi} = \begin{bmatrix} \psi_1^n \mathbf{I} & \mathbf{O} & \dots & \psi_n^n \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \psi_1^\mu \mathbf{I} & & \mathbf{O} & \psi_n^\mu \mathbf{I} \end{bmatrix} \tag{92}$$

where the possibility of using different functions for displacements and parameters of the Euler-Rodrigues formula was provided. However, this is not the most appropriate way due the particular geometry os shell structures.

The geometrically exact theories are especially interesting in the analysis of slender structures, where the change of the structural response due the variation of the configuration is important. As the shear deformation was taken into account, the shear-locking presence can be anticipated. In meshless methods, particulary for the ones relying in the use of MLS approximation, there isn't, in general, such concept as *reduced integration*, as the closed form solutions for integrals appearing in the generalized residual vector and the generalized tangent stiffness matrix are not known (even for linear problems).

Resorting the facilities of the meshless approximations to generate arbitrarily continuous functions, is very easy to chose such approximations that the Kirchhoff limit is exactly achieved, [18]. However, it was recently proved, see [19], that this procedure necessarily leads to a singular equation system, due to the linear dependencies between the approximation functions for the rotations. Moreover, except for the one dimensional case[13], the number of dependencies grow with the order of the basis (in the common case of polynomial basis are used). Nevertheless, if appropriate solvers are used, this problem can be easily overcome.

Let us now derive the Kirchhoff limit constraints for the particular notation of the present theory. If the distortion is negligible, then, from equation (27), $\boldsymbol{z}^{\mathrm{o}}_{,\alpha} + \boldsymbol{u}_{,\alpha} - \boldsymbol{e}_\alpha \cong \boldsymbol{o}$. If $\boldsymbol{z}^{\mathrm{o}}_{,\alpha} = \boldsymbol{o}$ and $\boldsymbol{Q} \cong \boldsymbol{I} + \mathbf{\Theta}$ the constraints $\theta_1 = u_{3,2}$ and $\theta_2 = -u_{3,1}$ emerge. Hence, the generalized displacement fields in $(91)_1$ should be as

$$\mathbf{\Phi} = \begin{bmatrix} \phi_1 & & & & & & & \phi_n & & & & & \\ & \phi_1 & & & & & & & \phi_n & & & & \\ & & \phi_1 & & & & & & & \phi_n & & & \\ & & & \phi_{1,2} & & & \dots & & & & \phi_{n,2} & & \\ & & & & \phi_{1,1} & & & & & & & \phi_{n,1} & \\ & & & & & \phi_1 & & & & & & & \phi_n \end{bmatrix}. \tag{93}$$

Notice the presence of *first* order derivatives on displacements and rotations in the $\mathbf{\Delta}$ differential operator. Accordingly, C^1 continuity is mandatory for the approximations. As MLS approximations are being used, this task is trivial to accomplish. In fact, usually, higher continuity is used in order to obtain continuous generalized stresses.

Frequently the measures of the error of the FEM are based on the discontinuities of the stress fields (i) between elements and (ii) on the static boundary. In the present formulation possible mesures of the error can be derived from (i) on the discontinuity of the generalized stresses on the static boundary, (ii) on the error between the independently approximated generalized stresses on the kinematic boundary and the same stresses evaluated from the domain approximation and (iii) on the error in the imposed displacements.

[13] In one-dimensional approximation only one dependency, per field, is introduced.

7.2 Discretized form of the residual vector and generalized tangent stiffness matrix

The use of the approximations (91) in the hybrid functional (87), after some algebraic manipulations, yields

$$\mathbf{R} = \mathbf{0} \qquad \forall \delta \mathbf{d}, \delta \mathbf{q}^{\lambda} \tag{94}$$

where $\mathbf{R}$ is the the residual vector

$$\mathbf{R} = \begin{bmatrix} \mathbf{P} + \mathbf{B}\mathbf{q}^{\lambda} \\ \mathbf{B}^T \mathbf{d} + \overline{\mathbf{q}} \end{bmatrix} \tag{95}$$

and

$$\mathbf{P} = \int_{\Omega^r} (\boldsymbol{\Delta\Phi})^T \, \boldsymbol{\Psi}^T \boldsymbol{\sigma}^r \mathrm{d}\Omega^r - \int_{\Omega^r} \boldsymbol{\Phi}^T \overline{\boldsymbol{q}}^{\Omega} \mathrm{d}\Omega^r - \int_{\Gamma_t^r} \boldsymbol{\Phi}^T \overline{\boldsymbol{q}}^{\Gamma} \mathrm{d}\Gamma_t^r, \tag{96a}$$

$$\mathbf{B} = - \int_{\Gamma_u^r} \boldsymbol{\Phi}^T \boldsymbol{\Psi} \mathrm{d}\Gamma_u^r, \tag{96b}$$

$$\overline{\mathbf{q}} = \int_{\Gamma_u^r} \boldsymbol{\Psi}^T \overline{\boldsymbol{d}} \mathrm{d}\Gamma_u^r. \tag{96c}$$

The use of the approximations (91) in the generalized tangent form (90), after some algebraic manipulations, yields

$$\mathbf{K}\Delta\mathbf{a} \qquad \forall \delta \mathbf{d}, \delta \mathbf{q}^{\lambda} \tag{97}$$

where

$$\mathbf{K} = \begin{bmatrix} \mathbf{S} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{0} \end{bmatrix} \qquad \text{and} \qquad \Delta\mathbf{a} = \begin{bmatrix} \Delta\mathbf{d} \\ \Delta\mathbf{q}^{\lambda} \end{bmatrix} \tag{98}$$

and $\mathbf{S}$ is the the generalized stiffness matrix

$$\begin{aligned} \mathbf{S} = & \int_{\Omega^r} \Big((\boldsymbol{\Delta\Phi})^T \, \boldsymbol{\Psi}^T \boldsymbol{D} \boldsymbol{\Psi} \, (\boldsymbol{\Delta\Phi}) + (\boldsymbol{\Delta\Phi})^T \boldsymbol{G} \, (\boldsymbol{\Delta\Phi}) - \boldsymbol{\Phi}^T \boldsymbol{L}^{\Omega} \boldsymbol{\Phi} \Big) \, \mathrm{d}\Omega^r \\ & - \int_{\Gamma_t^r} \boldsymbol{\Phi}^T \boldsymbol{L}^{\Gamma} \boldsymbol{\Phi} \mathrm{d}\Gamma_t^r . \end{aligned} \tag{99}$$

The identification of the location of the bifurcation points is made by the study of the eigenvalues of the discretized form of the generalized tangent stiffness matrix $(98)_1$. Notice that the dependencies introduces via the approximation should be taken into account, because they give rise to (numerically) null eigenvalues. Also, for each prescribed displacement a negative eigenvalue will appear. The rule for determining the exact number of null eigenvalues is given in [19].

In the frequent case were the shell middle surface is not smooth and is, in fact, an assembly of several smooth shells, it is also possible to analyze the all set by including continuity conditions on the intersections in the weak form (87). Of course, extra degrees of freedom will be associated to the intersection and the residual vector (95) and the generalized tangent form (98) will exhibit a somewhat complex form.

8 IMPLEMENTATION ASPECTS

8.1 Evaluation of the nodal approximation functions

The success of the presented method crucially depends on (i) the accuracy and (ii) the performance of the evaluation of the nodal functions.

The accuracy of the evaluation of the nodal functions is intimately linked to the discretization adopted and the size of the reference domain, because the moments matrix present in the normal system of equations, to be solved during the MLS functions evaluation, can be poorly conditioned. A very efficient way of solving this problem is the use of a local coordinate system *centered* at the sample (usually Gauss) point. In this way the performance is not affected and the moment matrix is always well conditioned[14]. In fact, the performance is slightly increased as the value of the basis function and its derivatives are always the same *independently* of the sample point considered.

As for the performance of the evaluation of the nodal functions, two aspects should be taken into account. On one hand, the inversion of the moments matrix and their derivatives should be avoided, as described in [20]. On the other hand, as the values of the nodal functions and their derivatives, at the integration points, are required many times along the incremental/iterative process, it is desirable to evaluate and store them at the beginning of the process.

8.2 A generalized arc-length method

The solution of the resulting nonlinear system of equations (94) is achieved by the use of an incremental/iterative approach. The full Newton/Raphson method should be combined with some (non-physical) constraint in order to trace the full loading path of the shell.

To be consistent with the approximations made, this constraint should not include only generalized displacements and loads, but also should render the generalized boundary tractions on the kinematic boundary. Therefore, the following arc-length constraint that nonlinearly relates the incremental/iterative generalized displacements, forces and load parameter with a certain constant, the arc-length Δl, is introduced

$$\Delta\mathbf{d}^T\mathbf{W}^{\boldsymbol{d}}\Delta\mathbf{d}+\Delta\boldsymbol{\theta}^T\mathbf{W}^{\boldsymbol{\theta}}\Delta\boldsymbol{\theta}+\Delta\mathbf{n}^{\lambda T}\mathbf{W}^{\mathbf{n}}\Delta\mathbf{n}^{\lambda}+\Delta\mathbf{m}^{\lambda T}\mathbf{W}^{\mathbf{m}}\Delta\mathbf{m}^{\lambda}+\psi^2\Delta\lambda^2=\Delta l^2 \tag{100}$$

where $\mathbf{W}$'s are weighting matrices which are, at least, positive semi-definite diagonals and ψ is also a scaling parameter. Thus, the Crisfield's method [21] was generalized in order to include the essential boundary reactions, resulting in a robust and fast procedure.

8.3 The initial configuration description

The initial configuration can be expressed by several ways. An obvious procedure, used for linear shell analysis in [22], is to resource MLS. In this way a sort of *isoparametric* approximation is performed. Nevertheless, there is no reason why the *exact* initial configuration should not be used. Thus, the term *geometrically exact* gains a new meaning in the present context.

9 CONCLUSIONS

A meshless method for the structural analysis of shells was presented. The shells can have an arbitrary initial configuration. A geometrically-exact approach was incorporated in a hybrid functional, so the essential boundary conditions are imposed via Lagrange multipliers. The constitutive tensor was derived from a three dimensional material law by a plane stress imposition and allows the consideration of finite strains. The MLS nodal functions used for the domain are shear-locking free. Several implementation aspects were discussed.

[14] Of course, the usual conditions on the number of points in the support and their disposition also apply here.

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