On the procedures to eliminate shear locking in meshless methods

Carlos Tiago(1) and Vitor Leitão(2)

Abstract: Shear-locking is a well studied phenomenon in the conventional displacement approach of the Finite Element Method (FEM). Although not very elegant and mathematically soundly, the reduced/selective integration technique is, in general, the chosen option. However, for approximations based on unstructured data, like the ones used on meshless methods, it is not possible to relay on a concept such as reduced integration due to the non-polynomial character of the approximation. Therefore it is necessary to develop new strategies to overcome this problem. In this work two proposals are reviewed and critically analyzed, namely a method based on a change of variables (which proves to be, in fact, a mixed approach) and the construction of the approximation fields based on the reproduction of the Kirchhoff assumption on the thin limit (it is shown that this procedure originate a rank deficiency in the equations system). The two procedures are applied to the Element Free Galerkin (EFG) method analysis of beams and its relative numerical performance is compared. For the sake of completeness, tests are also carried out with the traditional variables and approximation. The merits and drawbacks of each approach are discussed.

Keywords: shear–locking, meshless, consistency, mixed.

1 Introduction

The presence of locking (whether is shear, membrane or volumetric) in the numerical solutions solely based on the approximation of the generalized displacement fields can lead to totally erroneous solutions. In the displacement version of the FEM is usual to overcome such a problem resorting the reduced/selective integration schemes. It can be proved that this procedure is equivalent to a certain class of mixed formulations, see Malkus and Hughes [10]. However, this can hardly be extended to meshless methods due to the non-polynomial character of the approximations, e. g., Moving Least Squares (MLS), Reproducing Kernel Particle Methods (RKPM), natural neighbour co-ordinates (Sibson co-ordinates) or enriched Shepard functions (used in the h-p Cloud method). A poor integration of the weak form lead, in general, to unsatisfactory results. Thus, alternative procedures were proposed. In the following, a brief review is presented:

Change of variables. By a simple change of the independent variables, it is possible to construct a locking–free formulation, as proposed by Cho and Atluri [3]. This change does not increase the total number of degrees of freedom, although it can be interpreted as a mixed formulation.
Unequal order of interpolation. This idea is based on the field-consistency paradigm developed by Prathap [11] and was implemented in the meshless framework by Donning and Liu [6]. Of course, the original expression unequal order of interpolation has to be restated in the context of the meshless methods. Notice that, contrary the usual FEM interpolations, an unequal order of approximation does not provide consistent fields.

Increase of the degree of basis functions. It is well known that the increase of the degree of the interpolation functions can alleviate the locking effects in the traditional FEM. The equivalence in meshless methods is the increase of the number of the basis functions. This approach was followed by Duarte et al. [7, 4] to solve shear deformable beams and plates by the h-p cloud method.

Nodal integration. Several nodal schemes were devised for the underintegration of the weak form. Usually this procedure suffers from spurious singular modes, as noted by Beissel and Belytschko [2] and it requires some sort of stabilization. Wang and Chen [12] used a curvature smoothing to solve shear deformable beams and plates.

Mixed formulation. With a mixed formulation, based on independent approximations of some interior fields, the locking can be eliminated, as show by Dolbow and Belytschko [5].

As could be anticipated, the increase of the degree of basis functions does not eliminate completely the shear–locking, as can be seen in figure 5(b), page 1395 of [7]. Although the nodal integration sounds very appealing to use in conjunction with a meshless method, this approach can lead to singularities in the system matrix if special procedures, like the addition of stabilization terms to the energy functional, are not employed. Also the weights of nodal integration rule are, sometimes, based on Voronoy diagrams, which is, in fact, a sort of cell structure. The mixed formulation will work correctly as long as the Ladyzhenskaya–Babuška–Brezzi (LBB) stability criterion is satisfied, which is not always a trivial task to check. Also, there is an increase in the dimension on the problem. Therefore, in the present work the first two approaches are detailed and implemented in the EFG platform. Its relative performance is measured and compared through a plane Timoshenko beam example. This model problem is also used to illustrate the formulations. Extension to spatial frames and Reissner-Mindlin plates is straightforward.

Consider a straight beam along the $z$ axis. The usual sets of equilibrium and compatibility equations are

\[ V' + \bar{p} = 0, \quad (1a) \]
\[ V + M' + \bar{m} = 0. \quad (1b) \]

and

\[ \eta = u' - \theta, \quad (2a) \]
\[ \kappa = \theta'. \quad (2b) \]

respectively. Here $(\cdot)' = \frac{d(\cdot)}{dz}$. Constitutive relations are given by

\[ V = kGA\eta, \quad (3a) \]
\[ M = EI\kappa. \quad (3b) \]

Proceeding with this description of the motion and using the same approximations for the generalized displacements $u$ and $\theta$, gives rise to the shear–locking effect.
2 A mixed formulation

This approach was presented in the framework of the meshless local Petrov–Galerking (MLPG) method by Cho and Atluri [3]. The approximation of the generalized variables relayed on the Generalized Moving Least Squares (GMLS) and the essential boundary conditions were imposed via the penalty method. Here the same change of variables will be used and the corresponding equilibrium equations will be derived. The EFG formulation (employing Lagrange multipliers to impose the essential boundary conditions) is presented and implemented. Approximating directly the transversal displacement, $u$, and the transverse shear strain, $\eta$, from equations (2), the following redefinition of the curvature, $\kappa$, arises:

$$\kappa = u'' - \eta'.$$

Instead of the equilibrium equation (1a), the Euler-Lagrange equations of this functional reveal the equation

$$M'' + m' - p = 0.$$  

Of course, this equation is equivalent to equation (1a), because, from (1b), $M'' + m' = -V'$. By a similar reasoning the natural boundary condition associated with the shear force, $V$, can be justified.

The use of this new set of variables can be formulated in the EFG framework thought the following augmented weak form:

$$\delta W = \delta W^{int} - \delta W^{ext} = \int_{\Omega} (V \delta \eta + M \delta \kappa) \, d\Omega - \int_{\Omega} (\bar{p} \delta u + \bar{m} \delta \theta) \, d\Omega - \sum_i [P \delta u]_{\partial \Omega^V} - \sum_i [M \delta \theta]_{\partial \Omega^M}$$

$$= \int_{\Omega} (\delta \eta (V + M' + \bar{m}) + \delta u (M'' + \bar{m}' - \bar{p})) \, d\Omega + \sum_i [(\delta u' - \delta \eta) (n M - M')]_{\partial \Omega^M} + \sum_i [\delta u (n (-M' - \bar{m}) - \bar{F})]_{\partial \Omega^V}.$$  

(5)

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$$- \sum_i [\delta \lambda V (u - \bar{w})]_{\partial \Omega^V} - \sum_i [\delta \lambda M (\theta - \bar{\theta})]_{\partial \Omega^M}.$$  

(6)

Applying the usual linear expansion for the approximations of $\eta$ and $u$ (recall that $\lambda V$ and $\lambda M$ are scalar quantities, therefore do not require any discretization process) and its variations,

$$u = \psi^u u, \quad \delta u = \psi^u \delta u,$$

$$\eta = \psi^\eta \eta, \quad \delta \eta = \psi^\eta \delta \eta,$$

(7a)

(7b)

the global system of equations emerges in the standard form

$$\begin{bmatrix} K & G^T \\ G & 0 \end{bmatrix} \begin{bmatrix} \mathbf{d} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{q} \end{bmatrix},$$  

(8)
where

$$K = \int_{\Omega} \begin{bmatrix} \psi^T k G \psi + \psi^T E I \psi & -\psi^T E I \psi' & -\psi^T E I \psi'' \\ -\psi'^T E I \psi & \psi'^T E I \psi' & \psi'^T E I \psi'' \\ -\psi''^T E I \psi & -\psi''^T E I \psi' & \psi''^T E I \psi'' \end{bmatrix} \mathrm{d}\Omega, \quad (9a)$$

$$G = \begin{bmatrix} 0 & G_\eta^u \\ G_u^u & G_\eta^u \end{bmatrix}, \quad (9b)$$

$$d = \begin{bmatrix} \eta \\ u \end{bmatrix}, \quad (9c)$$

$$\lambda = \begin{bmatrix} \lambda^V \\ \lambda^M \end{bmatrix}, \quad (9d)$$

$$f = \begin{bmatrix} -\int_{\Omega} \psi^T \bar{m} \mathrm{d}\Omega - \sum_i \left[ \psi^T \bar{M} \right] \partial \omega^M \\ \int_{\Omega} \left( \psi^T \bar{p} + \psi'^T \bar{m} \right) \mathrm{d}\Omega + \sum_i \left[ \psi^T \bar{P} \right] \partial \omega^V + \sum_i \left[ \psi'^T \bar{M} \right] \partial \omega^M \end{bmatrix}, \quad (9e)$$

$$q = \begin{bmatrix} q^u \\ q^\theta \end{bmatrix}, \quad (9f)$$

and

$$G_\eta^u = \begin{bmatrix} \left[ -\psi^T \right]_{\partial \Omega^1}, & \left[ -\psi^T \right]_{\partial \Omega^2}, & \ldots \end{bmatrix}, \quad (10a)$$

$$G_u^u = \begin{bmatrix} \left[ \psi^u \right]_{\partial \Omega^1}, & \left[ \psi^u \right]_{\partial \Omega^2}, & \ldots \end{bmatrix}, \quad (10b)$$

$$G_\eta^u = \begin{bmatrix} \left[ \psi^u \right]_{\partial \Omega^1}, & \left[ \psi^u \right]_{\partial \Omega^2}, & \ldots \end{bmatrix}, \quad (10c)$$

$$q^u = \begin{bmatrix} -\bar{p} \big|_{\partial \Omega^1} & -\bar{p} \big|_{\partial \Omega^2} & \ldots \end{bmatrix}^T, \quad (10d)$$

$$q^\theta = \begin{bmatrix} -\bar{\theta} \big|_{\partial \Omega^1} & -\bar{\theta} \big|_{\partial \Omega^2} & \ldots \end{bmatrix}^T, \quad (10e)$$

for arbitrary variations $\delta d$ and $\delta \lambda$.

Inspection of the definition of the generalized stiffness matrix (9a) reveals the need of evaluating second order derivatives of the nodal approximation $\psi^u$. Moreover, this precludes the $C^1$ continuity in order to the respective integrals make sense. This is the main reason why this change of variables is not suitable for traditional FEM (however, it was possible to interpolate $u$ by the Hermite polinomials, thus duplicating the number of variables associated with this field). Conversely, for meshless approximations the imposition of $C^1$ continuity is a trivial task, the only drawback being the evaluation of second derivatives of the nodal functions. Another interesting fact about (9a) is the inclusion of the term $\psi'^T E I \psi''$, typical of formulations based on Kirchhoff-Love assumptions.

It can be argued that, in fact, the direct approximation of a displacement field, $u$, and a shear strain, $\eta$, is, in fact, a mixed formulation. Indeed it is, but without the disadvantage of the increase of the number of variables.
3 A consistency approach

When the thickness of the beam tends to zero, the shear strain also tends to zeros, $\eta \to 0$. Therefore, equation (2a) yields $u' - \theta = 0$ or

$$\theta = u'.$$

(11)

Equivalent consistency results can be obtained for curved beams, as been reported by Prathap [11] and Donning and Liu [6].

For approximations based on linear combinations of functions and the usual Timoshenko beam theory,

$$u = \psi^u u,$$

$$\theta = \psi^\theta \theta.$$  

(12a)

(12b)

Thus, equation (11) is expressed as

$$\psi^\theta \theta = \psi^u u.$$  

(13)

Hence, in the thin limit it is sufficient that the two following conditions holds:

$$\psi^\theta = \psi^u u,$$

$$\theta = u.$$  

(14a)

(14b)

The fist of this conditions is here imposed to construct the approximation for the rotation field, being the remainder of the EFG formulation (or for any other meshless method) exactly as in the classical form. This procedure was formulated by Donning and Liu [6] and revisited by Nukulchai et al. [8].

A very important aspect is the consistency of the set of functions $\psi^\theta$. For the MLS/RKPM approximation this can easily be revealed owing the Lemma 3.3, namely the $m$-consistency condition II, due to Liu et al [9]. It is a far more general result of the so-called moving least-square reproducing kernel methods, but, in particular, it establishes that the first derivative of a MLS/RKPM approximation generated by a complete $m$-order, $\ell$ component polynomial basis and a weight function $\Phi \in C^m(\Omega)$ can reproduce polynomials of order $m - 1$. Accordingly, for reproducing constant curvatures the set of functions used for the displacement has to include, at least, all the monomials until the quadratic order.

However simple, the use of the approximation (14a) introduces dependencies in the system matrix. To demonstrate this is suffices to prove that the set of functions used to approximate $\theta$ is linearly dependant, i. e., $\psi^\theta \alpha = 0$ for some $\alpha_1, \alpha_2, \ldots, \alpha_n$ which are not all zero. Consider, in particular, the choice $\alpha_i = \beta$, $i = 1, 2, \ldots, n$ and $\beta \in V$ where $V = \{\beta | \beta \in \mathbb{R}, \beta \neq 0\}$. Owing to the fact that $\psi^u(z)$ is a partition of unity, i. e., $\sum_{i=1}^n \psi(z)_i = 1, \forall z \in \Omega$, then $\beta \sum_{i=1}^n \psi(z)_i = \beta$. Deriving both members in $z$ results $\beta \sum_{i=1}^n \frac{\partial \psi(z)_i}{\partial z} = 0$. Hence, the chosen set $\psi^\theta$ is linearly dependant, $\forall \beta \in V$. Moreover,

$$\sum_{i=1}^n \psi(z)_i = 0, \forall \beta \in V.$$  

(15)

which shows that $\psi^\theta$ is also a partition of nullity.

For the present problem this means that the approximation for the rotations has one deficiency for plane beams and two deficiencies for three dimensional beams and Reissner-Mindlin plates.
Remarkably this fact, to the authors knowledge, pass unnoticed until now. Nevertheless, the numerical implementation of the procedure with the use of appropriate solvers should be able to choose from the set of possible solutions an accurate one. In the present work the LAPACK [1] package was employed.

4 Numerical example

Consider a cantilever beam submitted solely to a uniform load. The values used for the analysis were: \( L = 1 \), \( EI = 1 \) and \( \bar{p} = 1 \). For the MLS approximations, the weight function is given by the equation (11), page 219 of reference [3] with \( s = 5 \) together with a \( p \) basis of complete cubic order. The accuracy of the solutions was measured by the relative \( L_2 \) error norm of \( u \), \( \eta \) and \( \theta \). To evidence that no reduced integration is used, 10 sample points of the Gauss-Legendre quadrature rule are employed between each node for carry out the integrals appearing on the weak form. This rule was also used to evaluate the error of the solutions. Besides the two methods presented in sections 2 and 3, the traditional approach in the generalized displacements as described in section 1 was also used.

The results in figure 1 evidence that: (i) for thick beams the traditional format gives the most accurate results for \((u, \theta)\), but for thin beams the locking effect is evident, (ii) both alternative formulations are free from locking and (iii) the consistency approach produces the best overall results.

5 Closing remarks

It is worth to emphasize that both of the techniques analyzed relay on the same properties of the meshless approximations: the higher order continuity and arbitrary consistency \(\text{i.e.}\), reproducing polynomials property). Moreover, both schemes require the evaluation of second order derivatives, which is a common feature of Kirchhoff-Love formulations. Thus, the computational cost of both approaches is, approximately, the same.

Contrary to the consistency method, the change of variables method can not be directly extended to problems where the compatibility equations (relating generalized displacements and generalized deformations) are nonlinear, \(\text{i.e.}\) geometrically nonlinear analysis.

However, special care has to be used for solving the resulting system from the consistency method. Also, when the number of null eigenvalues of the system matrix plays an important role (as the detection of a cross of a limit or bifurcation point in a geometrically nonlinear analysis) the rank deficiency caused by the approximation has to be taken into account.

For thick cross sections the usual generalized displacement formulation is the most accurate one, but only the alternative formulations preserves the accuracy and convergence properties in the thin limit.

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References

Figure 1: Results for cantilever beam.


