A modified variational principle for free vibration analysis suitable for meshless methods

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Keywords: variational principle; meshless methods; EFG; free vibration.

Abstract A Modified Variational Principle (MVP) is generalized for the free vibration analysis of solids. The proposed technique is particulary suitable for methods whereby the imposition of the essential boundary conditions (EBC) presents difficulties. Applications to two dimensional elasticity and thin plate bending benchmarks are made. The main advantages of the MVP presented here are the accuracy and the fact that no additional parameters or variables are required.

Introduction

The imposition of the EBC in methods, such as the Smooth Particle Hydrodynamics, the Reproducing Kernel Particle Methods (RKPM), the Moving Least Squares (MLS) and the *hp*-clouds functions, that use approximations which do not possess interpolatory character (Kronecker-delta property) is, usually, seen as cumbersome. The Lagrange Multiplier Method (LMM), the coupling with the Finite Element Method (FEM), the introduction of penalty parameters, the double change of coordinates or the MVP are some of the approaches that have been proposed to impose the EBC in the above referred methods. For a recent review on meshfree and particle methods see Liu *et al* [3].

In the classical work of Washizu [9] several MVP's of potential energy with relaxed continuity requirements are presented. This principles can be very useful in the formulation of meshless methods because they do not require the variations of the approximated field to vanish on the essential boundary. This was the technique used by Belytschko and co-workers [5], for static analysis, in the Element Free Galerkin (EFG) method. It was concluded that the use of a MVP results in a slightly less accurate method, but possess some other advantages, like providing a system matrix which is banded and positive definite.

In this work the MVP is generalised for the free vibration analysis. Application is made using the EFG but the principle remains valid for other methods, such as the RKPM or the Meshless Local Petrov-Galerkin (MLPG) method. Application to a plane two-dimensional elasticity problem and a thin plate bending problem is carried out.

The eigenvalue problem for solid mechanics

In this section the eigenvalue problem for two dimensional elasticity problems is addressed. The generalization for beams, plates and three-dimensional elasticity is straightforward. The resulting expressions are the same and only the definitions of the generalized stresses, $\boldsymbol{\sigma}$, generalized strains, $\boldsymbol{\epsilon}$, generalized displacements $\mathbf{u}$, generalized accelerations $\ddot{\mathbf{u}}$, generalized elasticity operator, $\mathbf{D}$, and differential operator, $\mathbf{L}$, have to be changed.

Let Ω be a open set with piecewise smooth boundary Γ. The boundary can be decomposed in the static part, Γ_t, and the kinematic part, Γ_u, thus $\Gamma = \overline{\Gamma_t \cup \Gamma_u}$, $\Gamma_t \cap \Gamma_u = \emptyset$ and $\overline{\Omega} = \Omega \cup \Gamma$. The equilibrium (in the absence of body forces), compatibility, constitutive equations are, respectively,

$$\mathbf{L}^T \boldsymbol{\sigma} = \rho \ddot{\mathbf{u}}, \tag{1a}$$

$$\boldsymbol{\epsilon} = \mathbf{L}\mathbf{u}, \tag{1b}$$

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\epsilon}. \tag{1c}$$

The (homogeneous) boundary conditions are given by

$$\mathbf{u} = \mathbf{0}, \tag{2a}$$

$$\mathbf{t} = \mathbf{0}, \tag{2b}$$

where $\mathbf{t} = \mathbf{N}\boldsymbol{\sigma}$ are the boundary tractions and $\mathbf{N}$ is a matrix that collects the unit outward normal components associated with the differential operator $\mathbf{L}$.

Modified Variational Principle for the eigenvalue problem

Using a constrained Lagrangian which includes the EBC through the use of Lagrange multipliers [8], the following variational form is obtained:

$$\int_\Omega \delta\boldsymbol{\epsilon}^T \boldsymbol{\sigma} \mathrm{d}\Omega - \int_{\Gamma_u} \delta\boldsymbol{\lambda}^T \mathbf{u} \mathrm{d}\Gamma_u - \int_{\Gamma_u} \boldsymbol{\lambda}^T \delta\mathbf{u} \mathrm{d}\Gamma_u + \int_\Omega \rho \delta\mathbf{u}^T \ddot{\mathbf{u}} \mathrm{d}\Omega = 0, \tag{3}$$

where $\boldsymbol{\lambda}^T = \left\{ \begin{array}{cc} \lambda_u & \lambda_v \end{array} \right\}$ represents the additional functions. Using the divergence theorem and integrating by parts, the weak form (3) can be written as

$$\int_\Omega \delta\mathbf{u}^T \left(-\mathbf{L}^T\boldsymbol{\sigma} + \rho\ddot{\mathbf{u}} \right) \mathrm{d}\Omega + \int_{\Gamma_t} \delta\mathbf{u}^T \mathbf{t} \mathrm{d}\Gamma_t + \int_{\Gamma_u} \delta\mathbf{u}^T \left(\mathbf{t} - \boldsymbol{\lambda} \right) \mathrm{d}\Gamma_u - \int_{\Gamma_u} \delta\boldsymbol{\lambda}^T \mathbf{u} \mathrm{d}\Gamma_u = 0. \tag{4}$$

For arbitrary variations of $\delta\mathbf{u}$ and $\delta\boldsymbol{\lambda}$ the Euler equations are

$$\mathbf{L}^T\boldsymbol{\sigma} - \rho\ddot{\mathbf{u}} = \mathbf{0} \quad \text{on} \quad \Omega, \tag{5a}$$

$$\mathbf{t} = \mathbf{0} \quad \text{on} \quad \Gamma_t, \tag{5b}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma_u, \tag{5c}$$

$$\boldsymbol{\lambda} = \mathbf{t} \quad \text{on} \quad \Gamma_u. \tag{5d}$$

Thus, besides the governing equations of the problem, (5d) allows to identify the Lagrange multipliers as the generalized tractions on the essential boundary. From the definition of tractions, the constitutive relationship (1c) and the compatibility equations (1b) the tractions can be expressed as

$$\mathbf{t} = \mathbf{N}\mathbf{D}\mathbf{L}\mathbf{u}. \tag{6}$$

For the free vibration problem the system responds in harmonic motion and the separation of variables technique can be used. The displacement field takes then the following form:

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}) \sin(\omega t + \varphi), \tag{7}$$

where ω is the circular frequency of the system and φ is the phase angle.

Substituting (6) and (7) in (3) it follows

$$\int_\Omega \delta(\mathbf{L}\,\mathbf{u})^T \mathbf{D}\,(\mathbf{L}\,\mathbf{u}) \mathrm{d}\Omega - \int_{\Gamma_u} \delta(\mathbf{L}\,\mathbf{u})^T \mathbf{D}^T \mathbf{N}^T \mathbf{u} \mathrm{d}\Gamma_u - \int_{\Gamma_u} \delta\mathbf{u}^T \mathbf{N}\mathbf{D}\mathbf{L}\mathbf{u} \mathrm{d}\Gamma_u - \omega^2 \int_\Omega \delta\mathbf{u}^T \rho \mathbf{u} \mathrm{d}\Omega = 0. \tag{8}$$

Discrete form of the Modified Variational Principle

Using the MLS approximation, the displacement field can assume the usual form:

$$\tilde{\mathbf{u}} = \boldsymbol{\Phi}\,\mathbf{U}, \tag{9}$$

Mode	ADINA	EFG-MVP	EFG-LMM	MLPG	NBNM
1	822.13	822.36	822.70	919.47	844.19
2	4931.95	4937.64	4935.83	5732.42	5051.21
10	63971.90	64244.22	64028.25	68681.87	64085.90
20	161114.00	161360.06	161214.88	—	—

Table 1: Plane stress cantilever beam: natural frequencies [Hz] for a coarse node distribution ($21 \times 3 = 63$ nodes).

Mode	ADINA	EFG-MVP	EFG-LMM	MLPG	NBNM
1	822.13	822.12	822.06	824.44	844.19
2	4931.95	4931.93	4931.44	5070.32	5051.21
10	63971.90	63976.24	63966.53	64937.83	64085.90
20	161114.00	161125.22	161123.13	—	—

Table 2: Plane stress cantilever beam: natural frequencies [Hz] for a fine node distribution ($51 \times 6 = 306$ nodes).

where

$$\tilde{\mathbf{u}} = \left\{ \begin{array}{c} \tilde{u} \\ \tilde{v} \end{array} \right\}, \mathbf{\Phi} = \left[\begin{array}{cccccc} \phi_1 & 0 & \dots & \phi_n & 0 \\ 0 & \phi_1 & \dots & 0 & \phi_n \end{array} \right]$$

and $\mathbf{U}^T = \left\{ \begin{array}{cccccc} U_1 & V_1 & \dots & U_n & V_n \end{array} \right\}$. The MLS approximants are described by the authors in [8]. Belytschko *et al* used orthogonal basis functions for constructing the approximants [5] which improves the accuracy of the results.

Using the approximation (9) for the fields $\mathbf{u}$ on (8), the discretized form of the eigenvalue problem is given by

$$\left(\mathbf{K} - \omega^2 \mathbf{M}\right) \mathbf{U} = \mathbf{0}, \tag{10}$$

for arbitrary variations of $\delta \mathbf{U}$, where

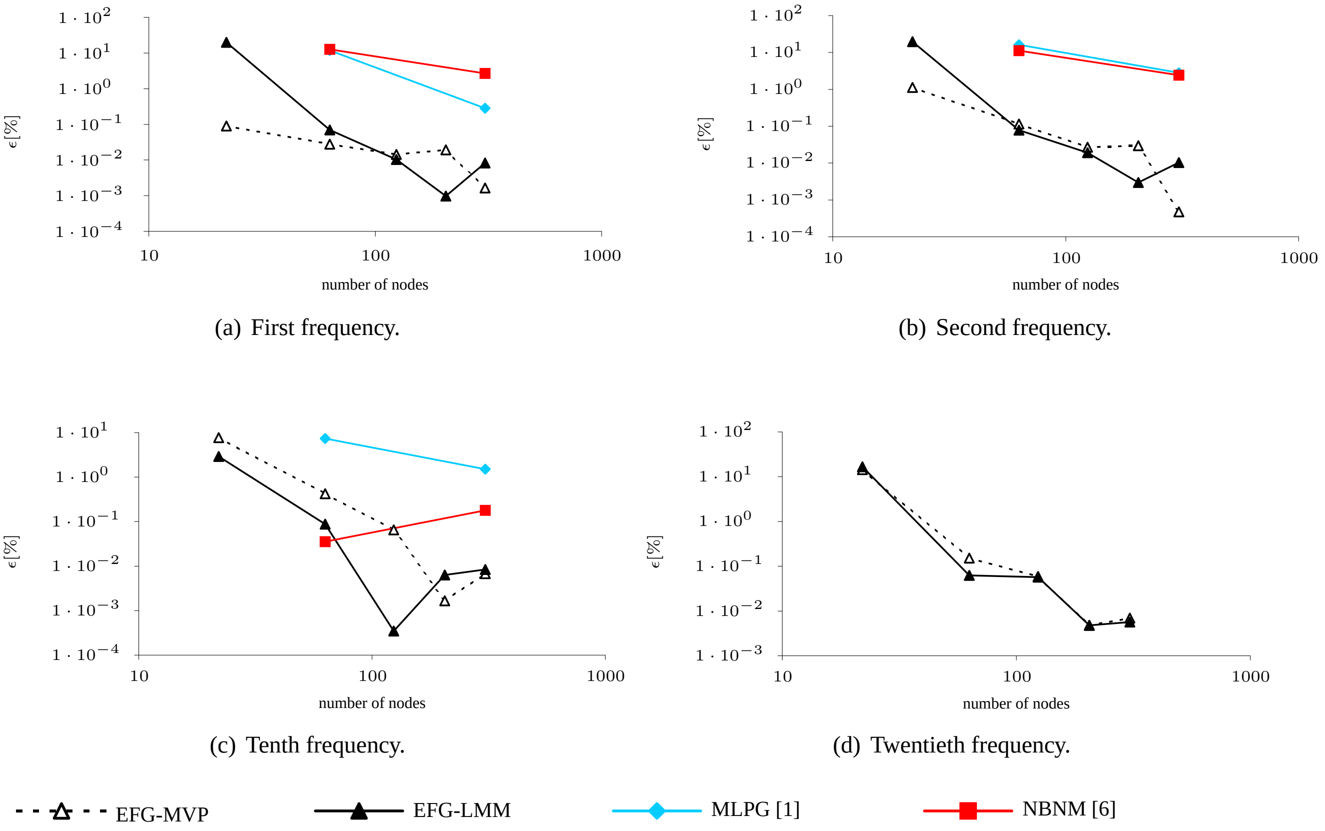
$$\mathbf{K} = \int_{\Omega} (\mathbf{L}\,\mathbf{\Phi})^T \mathbf{D} (\mathbf{L}\,\mathbf{\Phi}) \mathrm{d}\Omega - \int_{\Gamma_u} (\mathbf{L}\,\mathbf{\Phi})^T \mathbf{D}^T \mathbf{N}^T \Phi \mathrm{d}\Gamma_u - \int_{\Gamma_u} \Phi^T \mathbf{N} \mathbf{D} (\mathbf{L}\,\mathbf{\Phi}) \mathrm{d}\Gamma_u$$

$$\mathbf{M} = \int_{\Omega} \mathbf{\Phi}^T \rho \mathbf{\Phi} \mathrm{d}\Omega.$$

Notice that the last two terms of $\mathbf{K}$ are the transpose of each other, thus only one of them needs to be evaluated. The vibration frequencies and the modes shapes are obtained from the solution of the linear eigenvalue problem in (10).

Numerical examples

Plane Elasticity.A cantilever beam in plane stress state, Nagashima [6]($E = 2,1 \times 10^4\,\mathrm{kgf/mm^2}$, $\nu = 0,3$, $\rho = 8,0 \times 10^{-10}\,\mathrm{kgf\,s^2/mm^4}$, $t = 1\,\mathrm{mm}$, $D = 10\,\mathrm{mm}$ and $L = 100\,\mathrm{mm}$), has been analyzed with the present method. The results obtained with the MVP, EFG-MVP, are compared with the solutions obtained by the authors using the EFG and the LMM, EFG-LMM, (the formulation is given in [8]) and, when available, by Liu *et al* with the MLPG [1] and by Nagashima with the Node-By-Node Meshless (NBNM) method [6]. The ADINA [2] FEM code was used to generate a converged solution, considered numerically exact. This was achieved using 1 000 nine-noded Lagrangian elements for a total of 4221 nodes. The MLS nodal shape functions in the EFG solutions were generated using the cubic spline weight function with elliptic supports (the dimensions of the support in each direction were set equal to three times the nodal spacing in that direction). Linear basis were used, $\mathbf{p}^T = \{1 \quad x \quad y\}$, and ten integration points in each direction in each cell were used. The cell nodes coincide with the approximation nodes. Several meshes were used. Selected results are represented in tables 1 and 2 and Figure 1. The relative error, in percentage, is defined as $\epsilon = \left| \frac{\omega_{\mathrm{MESHLESS}} - \omega_{\mathrm{ADINA}}}{\omega_{\mathrm{ADINA}}} \right| \cdot 100$. The first ten vibration modes are represented in Figure 2 (the mesh is just for plotting purposes).



(a) First frequency.

(b) Second frequency.

(c) Tenth frequency.

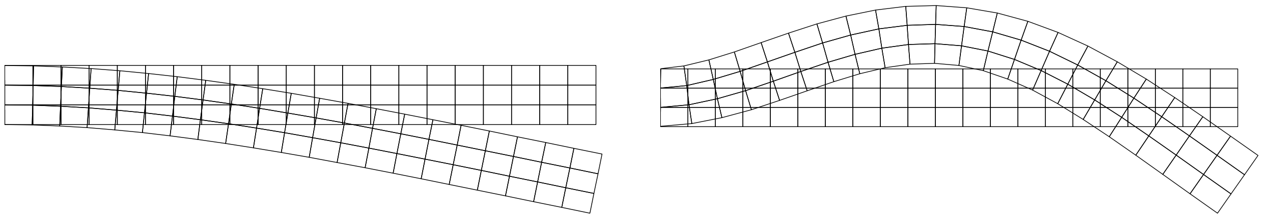
(d) Twentieth frequency.

Figure 1: Relative error of the natural frequencies [HZ] with *h*-refinement.

Thin Plates. Although the present formulation was developed for plane elasticity, the results are easily extended for thin plate bending problems. Consider a simply supported square plate ($E = 200 \times 10^9 \mathrm{N/m}^2$, $\nu = 0,3$, $\rho = 8 \times 10^3, \mathrm{kg/m}^3$, $t = 0.05\,\mathrm{m}$, $a = 10\,\mathrm{m}$). This example previously studied by Liu [4] using the EFG method. The analytical solution for this problem, neglecting the effect of the rotary inertia, is given by [7]. A quadratic complete basis was used,$\mathbf{p}^T = \left\{1 \quad x \quad y \quad x^2 \quad xy \quad y^2\right\}$. The cubic spline weight function is not a proper choice anymore, because the differential operator associated with the effective shear force includes third order partial derivatives. As this operator appears in the augmented form of the stiffness matrix when the present MVP is used, the nodal functions should posses, at least, third order continuity. Thus, the weight function used in the 2D examples in [8], with $s = 5$, is used. Regular meshes were used. The radius of the support of the weight function was made equal to 3.5 times the nodal spacing. The results obtained with the MVP and with the LMM for the first, third and sixth vibration frequencies are shown in Figure 3. The error is measured by $\epsilon = \left|\frac{\omega_{\text{MESHLESS}} - \omega_{\text{EXACT}}}{\omega_{\text{EXACT}}}\right| \cdot 100$.

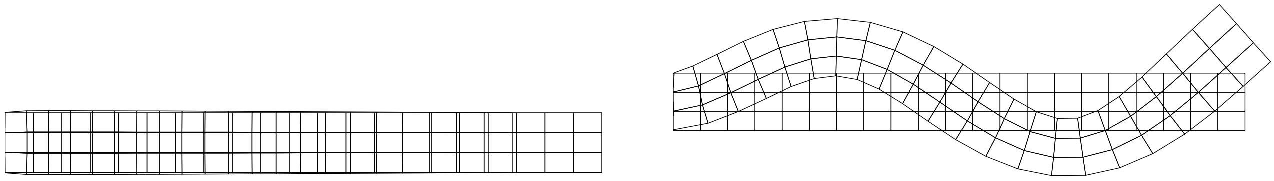
Conclusions

A modified variational principle was generalized for the free vibration analysis of solids and thin plates. The Element-Free Galerkin method was used and results show excellent agreement with the reference solutions. The EFG-MVP results for the plane elasticity problem reveal that the accuracy is very close to that of EFG-Lagrange Multiplier Method and better than that of the Meshless Local Petrov Galerkin and Node-by-node Meshless methods. For the thin plate bending problem however the accuracy of the EFG-LMM is consistently better than that of the EFG-MVP. This is due to the approximation of the effective shear force on the essential boundary that includes third order partial derivatives of the interior displacement field. This conclusion, for plate bending problems, replicates that of Belytschko and co-workers [5] for static analysis.



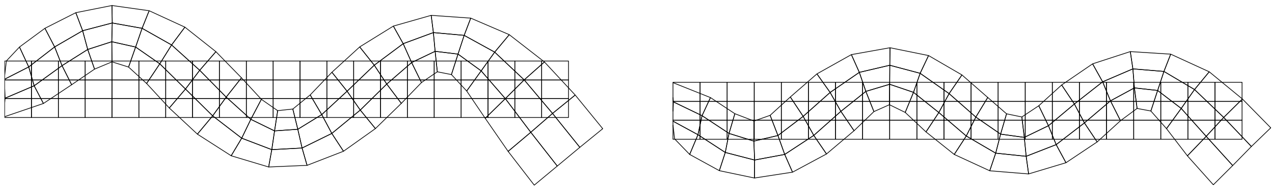
(a) First vibration mode.

(b) Second vibration mode.



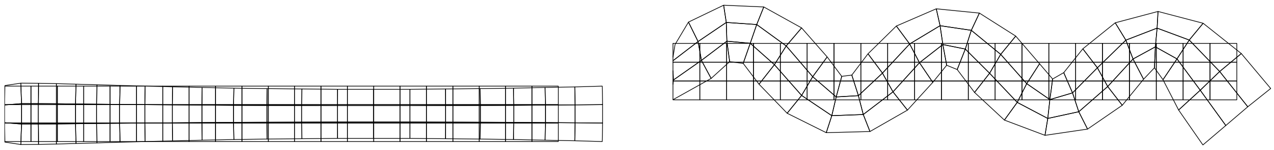
(c) Third vibration mode.

(d) Fourth vibration mode.



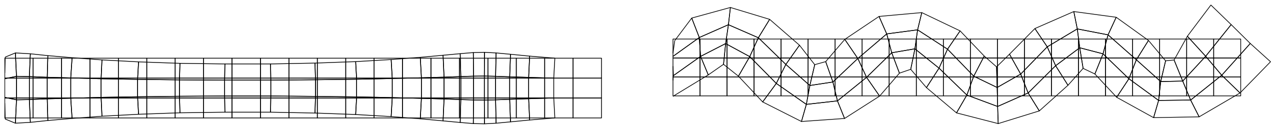
(e) Fifth vibration mode.

(f) Sixth vibration mode.



(g) Seventh vibration mode.

(h) Eighth vibration mode.



(i) Nineth vibration mode.

(j) Tenth vibration mode.

Figure 2: First ten vibration modes of plane stress cantilever beam. The discretization includes 21 × 3 nodes and a linear basis, $\mathbf{p}^T = \{1 \quad x \quad y\}$.

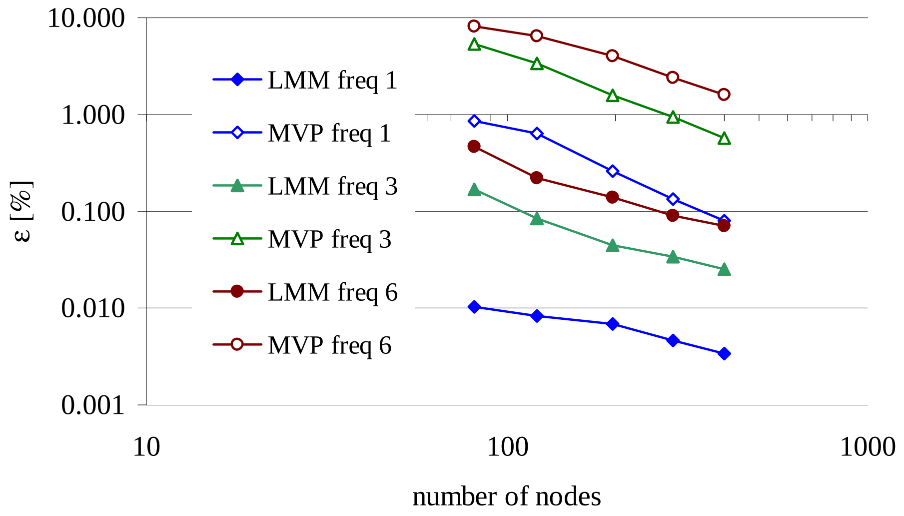


Figure 3: Relative error of the natural frequencies [HZ] with *h*-refinement.

The advantages of the EFG-MVP (for free vibration analysis) presented here are the fact that no additional parameters or variables are required. For instance the EFG-LMM requires an independent approximation in the EBC (choice of the number and position of nodes, nodal shape functions, a quadrature rule, etc) and the penalty method requires the imposition of the penalty parameter.

Acknowledgments

This work has been partially supported by FCT through projects "Financiamento Plurianual" and POCTI/33066/ECM/2000.

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