Plate bending using \textit{hp-Clouds} and Trefftz-based enrichment

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\textbf{Abstract} The aim of the present work is to examine the performance of the \textit{hp-Cloud} method in the analysis of thin plates (Kirchhoff’s theory). The partition of unity (POU) is constructed using the Shepard functions. The enrichment is made through Trefftz functions, i.e., functions that are homogeneous solutions of the governing equation of the problem. As this approximation does not possess the Kronecker-delta property, the essential boundary conditions are relaxed and imposed via Lagrange multiplier method.

\textbf{Introduction} The mesh requirements usually associated with traditional finite element methods (FEM) may be reduced in, basically, two ways: by using boundary-type of formulations, such as boundary element methods (BEM) and Trefftz techniques, amongst others; or by establishing approximations based on nodes instead of elements. This latter approach is behind most of the meshless methods that have been presented in recent years such as SPH method, the DEM, the EFG method, and the RKPM. For a review see Belytschko \textit{et al} [2]. Using the concept of the partition of unity, other methods such as the Partition of Unity FEM [1] and the \textit{hp-Cloud} method [3] were devised. Except for the SPH method, which uses collocation, all other methods require a background cell structure in order to integrate the variational form of the problem. Alternative meshless collocation procedures (which totally avoid integrations) were used by the authors in the fields of Radial Basis Functions (RBF) [5] and Trefftz Methods [6] but will not be discussed here.
The $hp$-Cloud method may be seen to be more flexible [3] than most of the other meshless methods due to the possibility of improving accuracy, without remeshing, in two ways: $h$-type refinement whereby extra points (i.e. clouds centers) are added to the domain; $p$-type refinement whereby the approximate solution is enriched by adding specific functions (polynomials, trigonometric, etc) to some (or all) nodes. In this method, the partition of unity is constructed using the minimal set of monomials that result from the application of moving least squares (MLS) [2], i.e., the Shepard functions.

In a previous work of Garcia et al. [4] the $hp$-clouds method was applied to Mindlin’s thick plate model where shear locking was avoided by using $p$ refinement. In the present work the method is applied to the thin plate model. The thin plate model uses differential operators (in the expression of the internal energy) of higher order than those occurring for the thick plate and this poses some difficulties as compared to the thick plate implementation of the $hp$-clouds method. Enrichment is achieved with functions that satisfy a priori the Lagrange homogeneous equation. The final governing system is obtained by enforcing stationarity on a functional including the potential energy and Lagrange multipliers.

**Plate bending equations**

The main equations for the analysis of thin plates are now briefly described on the plate of uniform thickness, $t$, domain $\Omega$, kinematic boundary $\Gamma_w \bigcup \Gamma_{\partial w}$ and static boundary $\Gamma_M \bigcup \Gamma_{V_n}$, where the subscript $n$ stands for outside normal.

**Equilibrium:** Collecting the bending moments and shear forces in the vectors $M^T = \{ M_{xx} \ M_{yy} \ M_{xy} \}$ and $Q^T = \{ Q_x \ Q_y \}$, respectively, the equilibrium equations can be written as $L^T M - Q = 0$ and $\nabla^T Q + \vec{p} = 0$ where $\vec{p}$ is the transverse load, $L$ and $\nabla$ are the usual differential operators.

Applying the $\nabla^T$ to both sides of the first equation and substituting in the second it follows $(L \nabla)^T M + \vec{p} = 0$ and $(L \nabla)^T = \begin{bmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial y^2} & 2 \frac{\partial^2}{\partial x \partial y} \end{bmatrix}$.

**Compatibility:** Collecting the curvatures and shear strains in the vectors $\chi^T = \{ \chi_{xx} \ \chi_{yy} \ \chi_{xy} \}$ and $\gamma^T = \{ \gamma_{xz} \ \gamma_{yz} \}$, respectively, the compatibility equations can be written as $\chi = L \theta$ and $\gamma = \nabla w + \theta$ where $\theta^T = \{ \theta_x \ \theta_y \}$ is a vector containing the rotations. Neglecting the shear strains, $\gamma = 0$, the rotations vector and the curvatures are given by $\theta = -\nabla w$ and $\chi = -(L \nabla) w$.

**Constitutive relations:** Assuming a plane stress state for each fiber, the constitutive relations are given by $M = D \chi$ where $D$ represents the stiffness.

**Boundary conditions:** The two appropriate boundary conditions of each boundary point are $w = \vec{w}$ or $V_n = \nabla w$ and $\frac{\partial w}{\partial n} = \frac{\partial \vec{w}}{\partial n}$ or $M_n = \vec{M}_n$ where $V$ is the effective shear force.
**hp-Clouds approximations**

The notation used here follows closely the work of Garcia et al. [4]. In the following, only a brief description of the functions used is given. Consider an arbitrary set of \( N \) points \( x_\alpha \in \Omega, \alpha = 1, \ldots, N \). Associated with each of these points there is a open set called domain of influence or *cloud*, \( \omega_\alpha = \{ x \in \Omega : \| x - x_\alpha \| \leq h_\alpha \} \). These *clouds* are chosen in such a way that they form a finite open covering of the domain, \( \Omega_N = \{ \omega_\alpha \}_{\alpha = 1}^N \). Associated with the *clouds* there is a set of functions \( \mathcal{L}_N = \{ \psi_\alpha \}_{\alpha = 1}^N \) that form a partition of unity. In this work Shepard functions are used to build the POU:

\[
\psi_\alpha (x) = \frac{\mathcal{H}_\alpha (x)}{\sum_{\beta} \mathcal{H}_\beta (x)}, \quad \beta \in \{ \gamma : \mathcal{H}_\gamma (x) \neq 0 \} \tag{1}
\]

The weight function appearing in (1) depends only on the normalized radius or distance \( r_\alpha = \frac{\| x - x_\alpha \|}{h_\alpha} \) between the point \( x \) under consideration and the point \( x_\alpha \).

In this paper the following quartic spline was used:

\[
\mathcal{H}(r_\alpha) = \begin{cases} 
1 - 6r_\alpha^2 + 8r_\alpha^3 - 3r_\alpha^4 & \text{for } 1 > r \geq 0 \\
0 & \text{for } r \geq 1 
\end{cases} \tag{2}
\]

In the *hp-cloud* method, higher order approximations (than that given by the POU) are made possible through the enrichment of the POU functions (1) by a suitable set of functions, \( L_{\alpha i}(x), i = 1, \ldots, M_\alpha \). Thus, the approximation \( \tilde{u}(x) \) of \( u(x) \) on each point can be written as

\[
\tilde{u}(x) = \sum_{\alpha = 1}^N \psi_\alpha (x) \left\{ u_\alpha + \sum_{i=1}^{M_\alpha} L_{\alpha i}(x)b_{\alpha i} \right\} = \Phi u \tag{3}
\]

where \( u_\alpha \) and \( b_{\alpha i} \) are the unknown coefficients.

As the number of enrichment functions \( M_\alpha \) depends on the *cloud* under consideration, it is easy to use different \( p \) refinement on different parts of the domain.

Amongst the many possible enrichment functions, the one chosen in this work stems from the so-called T (or Trefftz)-functions which, for thin plates, may be given by

\[
w_h = \Re \left[ \bar{\zeta} \Phi + \chi \right], \tag{4}
\]

where \( \zeta = x + iy, \bar{\zeta} = x - iy \) and \( \Phi = \Phi (\zeta) \) and \( \chi = \chi (\zeta) \) are complex valued series. By choosing

\[
\Phi (\zeta) = \sum_{i=1}^M a_i (\zeta)^i \quad \text{and} \quad \chi (\zeta) = \sum_{i=2}^M b_i (\zeta)^i, \tag{5}
\]
the terms of equation (4) constitute the enrichment functions used in this work which may be written as
\[ L_\alpha(x) = \begin{bmatrix} L_1 & L_2 & \cdots & L_{M_\alpha} \end{bmatrix} \] where \( L_1 = \begin{bmatrix} r^2 \end{bmatrix} \) and \( L_i = \begin{bmatrix} r^2 \text{Re} (\zeta)^{i-1} & -r^2 \text{Im} (\zeta)^{i-1} & \text{Re} (\zeta)^i & -\text{Im} (\zeta)^i \end{bmatrix} \) with \( r^2 = x^2 + y^2 \) and \( i = 2, 3, \ldots, M_\alpha \).

**Governing system**

A generalization of the Principle of Minimum Potential Energy (which includes the kinematic boundary conditions through the inclusion of Lagrange multipliers [7]) was used. Note that, in this case, the solution is not obtained by minimization of the variational principle but by enforcing stationarity conditions.

\[
\Pi_I = \frac{1}{2} \int_\Omega \chi^T D \chi \, d\Omega - \int_\Omega p w \, d\Omega - \int_{\Gamma_{\text{V}}} \nabla_n w \, d\Gamma_{\text{V}} - \int_{\Gamma_{\text{M}}} \nabla_{\text{M}} \cdot \lambda_{\text{M}} \, \left( -\frac{\partial w}{\partial n} \right) \, d\Gamma_{\text{M}} \\
- \int_{\Gamma_{\text{M}}} \lambda_{\text{M}} \left( \frac{\partial w}{\partial n} - \frac{\partial w}{\partial n} \right) \, d\Gamma_{\text{M}}
\]

where \( \lambda^w \) and \( \lambda_{\text{M}} \) are Lagrange multipliers.

In this work the Lagrange multipliers \( \lambda^w \) and \( \lambda_{\text{M}} \) are expressed by linear combinations of the same Lagrange interpolants, \( N = \{N_1, N_2, \ldots\} : \)

\[
\lambda^w = N \lambda^w \quad \lambda_{\text{M}} = N \lambda_{\text{M}}
\]

were the column vectors \( \lambda^w \) and \( \lambda_{\text{M}} \) collect the associated weights.

Stationarity of the variational principle (6), \( \delta \Pi_I = 0 \), is:

\[
\frac{\partial \Pi_I}{\partial \mathbf{u}} \delta \mathbf{u} + \frac{\partial \Pi_I}{\partial \lambda^w} \delta \lambda^w + \frac{\partial \Pi_I}{\partial \lambda_{\text{M}}} \delta \lambda_{\text{M}} = 0
\]

which leads, for arbitrary variations of \( \delta \mathbf{u}, \delta \lambda^w \) and \( \delta \lambda_{\text{M}} \), to the standard form

\[
\begin{bmatrix}
K & G^w & G_{\lambda^w} \\
G_{\lambda^w}^T & 0 & G_{\lambda_{\text{M}}} \\
G_{\lambda_{\text{M}}}^T & \lambda_{\text{M}}^w & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{u} \\
\lambda^w \\
\lambda_{\text{M}}
\end{bmatrix}
= 
\begin{bmatrix}
f \\
\mathbf{q}^w \\
\mathbf{q}_{\text{M}}
\end{bmatrix}
\]

The domain integrals are
\[ K = \int_{\Omega} B^TDBd\Omega \]

\[ f = \int_{\Omega} \Phi^Tqdw + \int_{\Gamma_n} \Phi^T\nabla \Phi d\Gamma_n + \int_{\Gamma_{\partial \omega}} \left( -\frac{\partial \Phi^T}{\partial n} \right) M_n d\Gamma_m \]

where \( B = -(L \nabla)\Phi \) and the essential boundary conditions are

\[ \begin{align*}
    G^w &= -\int_{\Gamma_{\partial \omega}} \Phi^T N d\Gamma_{\partial \omega} \\
    q^w &= -\int_{\Gamma_{\partial \omega}} N^T \tilde{w} d\Gamma_{\partial \omega} \\
    G^{\partial w} &= -\int_{\Gamma_{\partial \omega}} \frac{\partial \Phi^T}{\partial n} N d\Gamma_{\partial \omega} \\
    q^{\partial w} &= -\int_{\Gamma_{\partial \omega}} N^T \frac{\partial \tilde{w}}{\partial n} d\Gamma_{\partial \omega}
\end{align*} \]

**Numerical examples**

The numerical tests here presented concern a square simply supported plate of which only a quarter was analysed due to symmetry. The following data was used: length, \( a = 2.0 \) m, thickness \( t = 0.1 \) m, Young’s modulus \( E = 3.0 \cdot 10^6 \) kN/m², Poisson’s ratio \( \nu = 0.3 \) and uniform load \( p = 10.0 \) kN/m. The same number of clouds was assumed for both directions and the background cell nodes (necessary for the integrations) coincide with the position of the center of the clouds. Gauss-Legendre quadrature with six points on each direction was used. For the interpolation of the Lagrange multipliers linear Lagrange shape functions were considered. The plate was analysed for different numbers of clouds with constant radius, \( h_\alpha = 2a \), and for the first two terms of the enrichment series, \( M_\alpha = 2 \). Results (normalized central displacement) are shown in Fig. 1.
Conclusions
The numerical implementation of the \(hp\)-clouds method to thin plates was presented. The possibility of accurately representing continuous derivatives of a prescribed order, thus generating \textit{a priori} smooth stress fields, and the use of Trefftz or T-functions as the enrichment functions were the main motivations for this work. Implementation aspects like the order of the quadrature used, the number of points used to impose the essential boundary conditions and the use of \(p\) refinement need further investigation.

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References


