The use of radial basis functions for one-dimensional structural analysis problems

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Abstract

The basic characteristic of the techniques generally known as “meshless methods” is the attempt to reduce or even to eliminate the need for a discretization (at least, not in the way normally associated with traditional finite element techniques) in the context of numerical solutions for boundary and/or initial value problems. The interest in meshless methods is relatively new and this is why, despite the existence of various applications of meshless techniques to several problems of mechanics (as well as to other fields), these techniques are still relatively unknown to engineers. Furthermore, and compared to traditional finite elements, it may be difficult to understand the physical meaning of the variables involved in the formulations.

As an attempt to clarify some aspects of the meshless techniques, and simultaneously to highlight the ease of use and the ease of implementation of the algorithms, applications are made, in this work, to one dimensional structural problems. The technique used here consists in the definition of a global approximation for a given variable of interest (in this case, components of the displacement field) by means of a superposition of a set of conveniently placed (in the domain and on the boundary) radial basis functions. Applications to beams on elastic foundation subject to static loads, the identification of critical (instability) modes and the identification of free vibration frequencies and modes are carried out.
1 Introduction

In recent years there has been a marked interest in the so-called meshless methods. The possibility of obtaining approximate solutions to various problems of mechanics (of engineering, in general) without the need for a mesh is quite appealing, in particular due to the reduction in time consumption and the time taken in preparing the data or analysing the results.


Another approach to meshless methods (and the one used in this work) derives from the early work of Hardy [11] on the use of radial basis functions (RBF) for interpolation problems which was later applied by Kansa [12] to the solution of partial differential problems arising in mechanics.

The basics of the technique used in this work were applied by Leitão [14] to the analysis of Kirchhoff plate bending problems.

In this work, applications are made of two approaches (symmetrical and non-symmetrical ones) of the collocation version of the radial basis functions method to one-dimensional problems of structural mechanics, namely, the analysis of beams on elastic foundations, the analysis of critical instability loads and the analysis of the natural frequencies (and modes) of one-dimensional structural elements.

2 Radial basis functions

Radial basis functions (RBFs) are all that exhibit radial symmetry, that is, may be seen to depend only (apart from some known parameters) on the distance \( r = \|x - x_j\| \) between the center of the function and a generic point \( x \). These functions may be generically represented in the form \( \phi(r) \).

For such a general definition it is not surprising that there exist infinite radial basis functions.

These functions may be classed into: globally supported and compactly supported ones depending on their supports, that is, whether they are defined on the whole domain or only on part of it.

Amongst the globally supported RBFs, the following types are probably the most used ones:
Multiquadrics (MQ) \( \sqrt{(x - x_j)^2 + c_j^2}, \ c_j > 0 \)

Reciprocal Multiquadrics (RMQ) \( \left( (x - x_j)^2 + c_j^2 \right)^{-\frac{1}{2}}, \ c_j > 0 \)

Gaussians (G) \( \exp(-cr^2), \ c_j > 0 \)

Thin-plate splines (TPS) \( r^{2\beta} \ln r, \ \beta \in \mathbb{N} \)

Compactly supported RBFS are, for example:
- Wu and Wendland, \((1 - r)^n p(r)\) where \(p(r)\) is a polynomial and \((1 - r)^n_+\) is 0 for \(r\) greater than the support;
- Buhmann, \(\frac{1}{r} + r^2 - \frac{4}{3}r^3 + 2r^2 \ln r\).

In Figures 1 to 4 globally supported RBFs for different \(c\) and \(\beta\) parameters are represented. Compactly supported RBFs are shown in Figure 5.
In a very brief manner, interpolation with RBFs may take the form:

\[ s(x) = \sum_{j=1}^{N} \alpha_j \phi(\|x - x_j\|) \]  \hspace{1cm} (1)

where \( f(x_i) \) is known for a series of points \( x_i \). This approximation is solved for the \( \alpha_j \) unknowns from the system of \( N \) linear equations of the type:

\[ s(x_i) = f(x_i) = \sum_{j=1}^{N} \alpha_j \phi(\|x_i - x_j\|) \]  \hspace{1cm} (2)

By using the same reasoning it is possible to extend the interpolation problem to that of finding the approximate solution of partial differential equations. This is made by applying the corresponding differential operators to the radial basis functions and then to use collocation at an appropriate set of boundary and domain points.

Collocation may be of two types: non-symmetrical or Kansa collocation and symmetrical or Hermite-like collocation. Details of both techniques may be found in [12] and [15], respectively for the non-symmetrical and the symmetrical collocation.

In short, the non-symmetrical collocation is the application of the domain and boundary differential operators \( LI \) and \( LB \), respectively, to a set of \( N - M \) domain collocation points and \( M \) boundary collocation points.
From this, a system of linear equations of the following type may be obtained:

\[
LIu_h(x_i) = \sum_{k=1}^{N} \alpha_k LI\phi(||x_i - \varepsilon_k||)
\]

\[
LBu_h(x_i) = \sum_{k=1}^{N} \alpha_k LB\phi(||x_i - \varepsilon_k||)
\]

where the \(\alpha_k\) unknowns are determined from the satisfaction of the domain and boundary constraints at the collocation points.

The basic characteristic of the Hermite approach is the sequential application of the differential operators to each pair of collocation point-RBF center which gives rise to a symmetrical equation system wherever the positions of the collocation points and those of the RBFs coincide.

This approach may be described as follows:

\[
u_h(x) = \sum_{k=1}^{N-M} \alpha_k LI_x(\|x - \varepsilon_k\|) + \sum_{k=N-M+1}^{N} \alpha_k LB_x(\|x - \varepsilon_k\|)
\]

where \(LI\) and \(LB\) are, respectively, the domain and boundary differential operators, \(x\) is a generic point and \(\varepsilon_k\) represents the center of the \(k\)-th radial basis function.

The \(\alpha_k\) unknowns are obtained from the satisfaction of the domain and boundary constraints:

\[
LI_x^j u_h(x_j) = \sum_{k=1}^{N-M} \alpha_k LI_x^j LI_x^k \phi(||x_j - \varepsilon_k||) + \sum_{k=N-M+1}^{N} \alpha_k LB_x^j LB_x^k \phi(||x_j - \varepsilon_k||)
\]

for the domain collocation points and,

\[
LB_x^j u_h(x_j) = \sum_{k=1}^{N-M} \alpha_k LB_x^j LI_x^k \phi(||x_j - \varepsilon_k||) + \sum_{k=N-M+1}^{N} \alpha_k LB_x^j LB_x^k \phi(||x_j - \varepsilon_k||)
\]

for the boundary collocation points. In this expression the following definitions are used:

- \(L_x g(||x - \varepsilon||)\) is the function of \(\varepsilon\), when \(L\) is applied on \(g(||x - \varepsilon||)\) as a function of \(x\) and then evaluated at \(x = x_j\);
- \(L_x^j g(||x - \varepsilon||)\) is the function of \(x\), when \(L\) is applied on \(g(||x - \varepsilon||)\) as a function of \(\varepsilon\) and then evaluated at \(\varepsilon = \varepsilon_k\).

Both techniques require the appropriate definition of the differential operators \(LI\) and \(LB\).

At this stage, the Hermite approach is much more demanding than that of Kansa due to the dual application of the operators.
All the required terms may be computed beforehand (by using a symbolic maths program) and then stored and encoded in the program developed. In this work, use is made of the programming environment MATLAB[16].

4 Analysis of one-dimensional problems of structural analysis

The approaches described above are now applied to set of one-dimensional structural analysis problems. In all cases elastic constitutive relationships are considered. Application to other types of material behaviour was already carried out and presented elsewhere.

The first of these problems is that of a beam on an elastic foundation subjected to a static load. Then, the analysis of longitudinal and transversal eigen modes and eigen-frequencies for a beam is carried out. Finally, tests are made on the determination of elastic instability loads for a simply supported beam.

4.1 Simply supported beam on an elastic foundation subjected to a uniform load

Consider a beam on an elastic foundation subjected to a uniform load. The problem may be formulated in the following way: find the transversal displacement field \( w(x) \), for \( 0 < x < l \), such that

\[
EI \frac{d^4w}{dx^4} + k_w w = p, \quad \text{for } 0 < x < l
\]  

(8)

\[ w = 0 \quad \text{and} \quad -EI \frac{d^2w}{dx^2} = 0 \quad \text{for } x = \{0, l\}
\]  

(9)

where \( k_w \) is the foundation parameter per unit distance, \( p \) is the load per unit distance and \( EI \) is the stiffness of the beam.

The data used, for this analysis, is: \( I = 1,0 \text{ kN}\cdot\text{m}^2 \); \( l = 3,0 \text{ m} \); and \( p = 1,0 \text{ kN/m} \).

The exact solution is given by Timoshenko [17].

This beam was analysed with the Hermite approach, that is, from the approximation given in (5) with multiquadric RBF functions (MQ) whereby 13 functions were used for the domain and 4 at the boundary.

The application of this technique to the equilibrium equation, for example, results in:

\[
L I^2 L I_k \phi(\|x_j - \varepsilon_k\|) = \left( EI \frac{d^4}{dx^4} + k_w \right) \left( EI \frac{d^4}{d\varepsilon} + k_w \right) \phi(\|x_j - \varepsilon_k\|)
\]  

(10)

where

\[
L I = EI \frac{d^4}{dx^4} + k_w
\]  

(11)

is the domain differential operator.
Table 1: Simply supported beam on an elastic foundation: relative error for different values of the stiffness parameter, $\beta$, with MQ.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\frac{w_{RBF} - w_{exact}}{w_{max}} \times 100$</th>
<th>$\frac{M_{RBF} - M_{exact}}{M_{max}} \times 100$</th>
<th>$\frac{V_{RBF} - V_{exact}}{V_{max}} \times 100$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.279 · 10^{-3}</td>
<td>0.420 · 10^{-3}</td>
<td>88.378 · 10^{-3}</td>
</tr>
<tr>
<td>2</td>
<td>0.196 · 10^{-3}</td>
<td>-0.156</td>
<td>0.148</td>
</tr>
<tr>
<td>3</td>
<td>-0.208 · 10^{-3}</td>
<td>7.842 · 10^{-3}</td>
<td>58.665 · 10^{-3}</td>
</tr>
</tbody>
</table>

The results are compared with the exact solution in Figures 6 and 7 and in Table 1 for three values of parameter $\beta = \sqrt{\frac{k}{EI}}$ corresponding to different levels of relative stiffness between the beam and the foundation.

In order to compare different types of radial basis functions, the beam was tested for $k_w = 0$ by using multiquadric RBFs (MQ), reciprocal multiquadrics (RMQ) and gaussians (G) as shown earlier.

Table 2 summarizes the results obtained with the three different RBFs. The good accordance for the three different functions lead us to conclude that, at least for this problem, there are no strong advantages in the use of one or other of the functions.

### 4.2 Axial free vibrations: finding natural frequencies for a clamped beam

Consider now the case of a clamped beam subject to free vibration along its axis. The equation of movement may be given by:
Figure 7: Simply supported beam on an elastic foundation: bending moment for different values of the stiffness parameter $\beta$, with MQ.

Table 2: Simply supported beam on an elastic foundation: relative error for different families of radial basis functions for $(\beta = 0)$.

<table>
<thead>
<tr>
<th></th>
<th>MQ ($c = 3, 0$)</th>
<th>RMQ ($c = 3, 0$)</th>
<th>G ($c = 1, 0$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{w_{RBF} - w_{ex}}{w_{ex}} \times 100$</td>
<td>4,589 $\cdot$ $10^{-3}$</td>
<td>9,308 $\cdot$ $10^{-3}$</td>
<td>-5,288 $\cdot$ $10^{-3}$</td>
</tr>
<tr>
<td>$\frac{M_{RBF} - M_{ex}}{M_{ex}} \times 100$</td>
<td>3,796 $\cdot$ $10^{-3}$</td>
<td>7,692 $\cdot$ $10^{-3}$</td>
<td>-4,368 $\cdot$ $10^{-3}$</td>
</tr>
<tr>
<td>$\frac{V_{RBF} - V_{ex}}{V_{ex}} \times 100$</td>
<td>45,075 $\cdot$ $10^{-3}$</td>
<td>93,161 $\cdot$ $10^{-3}$</td>
<td>-54,426 $\cdot$ $10^{-3}$</td>
</tr>
</tbody>
</table>

where $m$ is the mass per unit distance and $EA$ is the axial stiffness.

By using the separation of variables technique, the spatial solution is given by:

$$EA \frac{\partial^2 u(x, t)}{\partial x^2} - m \frac{\partial^2 u(x, t)}{\partial t^2} = 0$$

(12)

where $m$ is the mass per unit distance and $EA$ is the axial stiffness.

By using the separation of variables technique, the spatial solution is given by:

$$\frac{\partial^2 X(x)}{\partial x^2} + c^2 X(x) = 0 \quad \text{for } |0 < x < l|$$

(13)

$$X(0) = 0 \quad \text{and} \quad EA \frac{\partial X(x)}{\partial x} \bigg|_{x=l} = 0 \quad \text{for } x = \{0, l\}$$

(14)

where $c^2 = \omega^2 \frac{m}{EA}$ and $\omega$ is a natural axial frequency of the beam.
This problem will be solved with the non-symmetrical approach, represented by equations (3) and (4), which, when applied to the problem under analysis takes the form:

\[
\left\{ \begin{array}{c}
A(\phi) \\
C(\phi)
\end{array} \right\} + c^2 \left\{ \begin{array}{c}
B(\phi) \\
0
\end{array} \right\} \alpha = 0
\] (15)

The functionals \(A(\phi)\) and \(B(\phi)\), for a domain point \(i\), may be written as:

\[
A(\phi) = \left[ \frac{d^2}{dx^2} \phi(\|x_i - \varepsilon_1\|) \quad \frac{d^2}{dx^2} \phi(\|x_i - \varepsilon_2\|) \quad \ldots \quad \frac{d^2}{dx^2} \phi(\|x_i - \varepsilon_N\|) \right]
\] (16)

\[
B(\phi) = \left[ \phi(\|x_i - \varepsilon_1\|) \quad \phi(\|x_i - \varepsilon_2\|) \quad \ldots \quad \phi(\|x_i - \varepsilon_N\|) \right]
\] (17)

Functional \(C(\phi)\), at a boundary point \(j\), takes the form:

\[
C(\phi) = \left[ \begin{array}{cccc}
\phi(\|x_j - \varepsilon_1\|) & \phi(\|x_j - \varepsilon_2\|) & \ldots & \phi(\|x_j - \varepsilon_N\|) \\
EA \frac{d}{dx} \phi(\|x_j - \varepsilon_1\|) & EA \frac{d}{dx} \phi(\|x_j - \varepsilon_2\|) & \ldots & EA \frac{d}{dx} \phi(\|x_j - \varepsilon_N\|)
\end{array} \right]
\] (18)

By making zero the determinant of the first term of the first member in (15) the natural frequencies may be obtained.

The vibration mode, for frequency \(c_i\), may be obtained by directly replacing \(c_i\) in (15).

The analysis was carried out for three different discretizations (that is, number and position of RBFs): \(N_{RBF} = 4\), \(N_{RBF} = 7\) and \(N_{RBF} = 13\) and for the three types of functions previously used for the beam on elastic foundation, that is, MQ, RMQ e G.

The results obtained, for the first three fundamental frequencies are compared to the exact ones given by Clough and Penzien [18].

The following data was considered: \(m = 1,0\) kN·s²/m², \(l = 3,0\) m and \(EA = 1,0\) kN. The constant \(c\) was taken as \(c = l\), for the functions MQ and RMQ. For the gaussians, this parameter took the value 0,05.

The first three modes are represented (together with the exact solution) in Figure 8 for \(N_{RBF}^G = 13\).

4.3 Transversal free vibrations: finding natural frequencies for a simply supported beam

Consider now the case of finding the frequencies and vibration modes of a simply supported beam. The governing equation is:
\begin{center}

\begin{tabular}{|c|c|c|c|}
\hline
RBF's & 5 & 7 & 13 \\
\hline
MQ & 1.612726 & 1.579834 & 1.570894 \\
RMQ & 1.601579 & 1.579538 & 1.570983 \\
G & 1.574874 & 1.570926 & 1.570638 \\
\hline
\end{tabular}

\begin{tabular}{|c|c|c|c|}
\hline
RBF's & 5 & 7 & 13 \\
\hline
MQ & 4.771348 & 4.706318 & 4.711985 \\
RMQ & 4.902590 & 4.746071 & 4.712318 \\
G & 4.459906 & 4.732572 & 4.712472 \\
\hline
\end{tabular}

\begin{tabular}{|c|c|c|c|}
\hline
RBF's & 5 & 7 & 13 \\
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MQ & 7.552854 & 7.744331 & 7.852416 \\
RMQ & 7.898342 & 7.817227 & 7.851763 \\
G & 6.810501 & 7.690686 & 7.853753 \\
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\end{center}

Table 3: Clamped beam: normalized axial natural frequency, $\omega_1 \sqrt{\frac{m}{EA}}$.

Table 4: Clamped beam: normalized axial natural frequency, $\omega_2 \sqrt{\frac{m}{EA}}$.

Table 5: Clamped beam: normalized axial natural frequency, $\omega_3 \sqrt{\frac{m}{EA}}$.

\[
EI \frac{\partial^4 w(x, t)}{\partial x^4} - m \frac{\partial^2 w(x, t)}{\partial t^2} = 0 \quad (19)
\]

By using, once more, the separation of variables technique, the spatial part of the solution is given by:

\[
\frac{\partial^4 X(x)}{\partial x^4} - a^4 X(x) = 0 \quad (20)
\]

with the boundary conditions:

\[
w = 0 \quad \text{and} \quad -EI \frac{d^2 w}{dx^2} = 0 \quad \text{for} \ x = \{0, l\} \quad (21)
\]
where $a^4 = \omega^2 \frac{m}{EI}$ and $\omega$ is a natural transversal frequency of the beam.

The system of equations is, in this case, equivalent to that presented before (15). Instead of $c^2$, the term $-a^4$ now appears and the functional $A(\phi)$ is now defined, for a domain point $i$, by:

$$A(\phi) = \left[ \frac{d^4}{dx^4} \phi(\|x_i - \epsilon_1\|) \right]$$

Once more, the results obtained compare quite well with the exact ones, see Table 6 and Figure 9.

### 4.4 Simply supported beam: finding the instability loads

This problem may be represented by the following governing equation:
\[
\frac{\omega_{RBF}}{\pi^2 \sqrt{EI_{ml}}} = 1, 00019, 3, 99913, 8, 99467, 15, 9829, 25, 1337
\]

\[
\frac{\omega_{ex}}{\pi^2 \sqrt{EI_{ml}}} = 1, 4, 9, 16, 25
\]

\[
\varepsilon(\%) = \frac{\omega_{RBF} - \omega_{ex}}{\omega_{ex}} \cdot 100 = 0, 019, -0, 022, -0, 059, -0, 107, 0, 535
\]

Table 6: Simply supported beam: normalized transversal vibration frequencies, \( \pi^2 \sqrt{EI_{ml}} \), and relative error.

\[
EI \frac{\partial^4 w(x)}{\partial x^4} + P \frac{\partial^2 w(x)}{\partial x^2} = 0
\]

(23)

with the boundary conditions

\[
w = 0 \quad \text{and} \quad -EI \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{for} \quad x = \{0, l\}
\]

(24)

where \( P \) is the instability load for the beam.

In this case, the Hermite variant, given by equations (6) and (7), is used:
\[
\frac{P_{\text{RBF}}}{P_{\text{ex}}} = 1,00116 \quad 4,00027 \quad 8,97785 \quad 15,8701
\]

\[
\frac{P_{\text{RBF}}}{P_{\text{ex}}} = 1 \quad 4 \quad 9 \quad 16
\]

\[
\epsilon(\%) = \frac{P_{\text{RBF}} - P_{\text{ex}}}{P_{\text{ex}}} \cdot 100 = 0,116 \quad 0,007 \quad -0,246 \quad -0,811
\]

Table 7: Simply supported beam: normalized instability loads and relative error to the exact solution.

\[
\left[ \left\{ A(\phi) \right\} + k \left\{ B(\phi) \right\} + k^2 \left\{ C(\phi) \right\} \right] \alpha = 0 \quad (25)
\]

where \( k = \frac{P}{EI} \).
Functional \( A(\phi) \) now represents the terms due to the application (in sequence) of the differential operators related to \( \frac{d^4}{dx^4} \) (or \( \frac{d^4}{dx^4} \)), as well as with the boundary conditions (see definition at the end of section 4).
Functional \( B(\phi) \) represents the terms due to the application in sequence of \( \left\{ \frac{d^4}{dx^4}, k \frac{d^2}{dx^2}, \frac{d^4}{dx^4} \right\} \) and \( \left\{ k \frac{d^2}{dx^2}, \frac{d^4}{dx^4} \right\} \), and the boundary terms.
Functional \( C(\phi) \) is due to the remaining terms, \( k \frac{d^2}{dx^2} \) (or \( k \frac{d^2}{dx^2} \)) and boundary conditions.
The results obtained are compared, in Table 7, with the exact solution given by [17].

5 Conclusions

The main purpose of this work was that of contributing for the increase in the number and type of applications of the meshless techniques based on the use of radial basis functions as presented here.
The main advantages of the techniques is the ease of use and the ease of implementation. This was shown with applications to simple (and not so simple) problems of structural analysis.
Results show that collocation-based radial basis functions may be of interest for the one-dimensional types of structural analysis problems here documented.
Further work on the subject is now being conducted in order to assess, for as many problems as possible, the potential of the techniques.
6 Acknowledgements

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