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A generalização para o caso 2D:

Seja
$$\begin{cases} x = f(\xi,\eta) \\ y = g(\xi,\eta) \end{cases}$$
No caso do M.E.F.
$$f(\xi,\eta) = \sum_{i=1}^{n} \psi_i(\xi,\eta)\, x_i$$
$$g(\xi,\eta) = \sum_{i=1}^{n} \psi_i(\xi,\eta)\, y_i$$
$n \equiv$ nº de nós do elemento

Supor que as funções $\begin{cases} \xi = r(x,y) \\ \eta = s(x,y) \end{cases}$ existem. Como avaliar $\dfrac{\partial r}{\partial x}, \dfrac{\partial r}{\partial y}, \dfrac{\partial s}{\partial x}$ e $\dfrac{\partial s}{\partial y}$?

Então
$$\begin{cases} x - f(\overbrace{r(x,y)}^{\xi}, \overbrace{s(x,y)}^{\eta}) = 0 \\ y - g(r(x,y), s(x,y)) = 0 \end{cases}$$

Logo
$$\begin{cases} \dfrac{\partial}{\partial x}(x-f) = 0 \\ \dfrac{\partial}{\partial x}(y-g) = 0 \end{cases} \quad \begin{cases} 1 - \dfrac{\partial f}{\partial r}\dfrac{\partial r}{\partial x} - \dfrac{\partial f}{\partial s}\dfrac{\partial s}{\partial x} = 0 \\ 0 - \dfrac{\partial g}{\partial r}\dfrac{\partial r}{\partial x} - \dfrac{\partial g}{\partial s}\dfrac{\partial s}{\partial x} = 0 \end{cases} \Leftrightarrow \underbrace{\begin{bmatrix} f_{,r} & f_{,s} \\ g_{,r} & g_{,s} \end{bmatrix}}_{\underline{J}} \begin{Bmatrix} r_{,x} \\ s_{,x} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} r_{,x} \\ s_{,x} \end{Bmatrix} = \underbrace{\underbrace{\begin{bmatrix} f_{,r} & f_{,s} \\ g_{,r} & g_{,s} \end{bmatrix}^{-1}}_{\underline{J}^{-1}} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}}_{1^{a} \text{ coluna de } \underline{J}^{-1}}$$

De igual modo
$$\begin{cases} \dfrac{\partial}{\partial y}(x-f) = 0 \\ \dfrac{\partial}{\partial y}(y-g) = 0 \end{cases} \Leftrightarrow \underbrace{\begin{bmatrix} f_{,r} & f_{,s} \\ g_{,r} & g_{,s} \end{bmatrix}}_{\underline{J}} \begin{Bmatrix} r_{,y} \\ s_{,y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} r_{,y} \\ s_{,y} \end{Bmatrix} = \underbrace{\begin{bmatrix} f_{,r} & f_{,s} \\ g_{,r} & g_{,s} \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}}_{2^{a} \text{ coluna de } \underline{J}^{-1}}$$

Nota:
$$\underline{J} = \begin{bmatrix} f_{,r} & f_{,s} \\ g_{,r} & g_{,s} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}$$

Sendo
$$\underline{J}^{-1} = \begin{bmatrix} f_{,r} & f_{,s} \\ g_{,r} & g_{,s} \end{bmatrix}^{-1} = \frac{1}{f_{,r}\, g_{,s} - f_{,s}\, g_{,r}} \begin{bmatrix} g_{,s} & -f_{,s} \\ -g_{,r} & f_{,r} \end{bmatrix}$$

então, voltando à notação inicial,
$$\begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \xi}{\partial y} \\ \dfrac{\partial \eta}{\partial x} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} = \frac{1}{\dfrac{\partial x}{\partial \xi}\dfrac{\partial y}{\partial \eta} - \dfrac{\partial x}{\partial \eta}\dfrac{\partial y}{\partial \xi}} \begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial x}{\partial \eta} \\ -\dfrac{\partial y}{\partial \xi} & \dfrac{\partial x}{\partial \xi} \end{bmatrix} \quad //$$

Estas derivadas só dependem da transformação $x = f(\xi,\eta)$ e $y = g(\xi,\eta)$

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Assim,

$$\frac{\partial \omega(\xi,\eta)}{\partial x} = \frac{\partial \omega}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \omega}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial \omega(\xi,\eta)}{\partial y} = \frac{\partial \omega}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial \omega}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$\Leftrightarrow \begin{Bmatrix} \frac{\partial \omega}{\partial x} \\ \frac{\partial \omega}{\partial y} \end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix}}_{\text{estas derivadas parciais foram avaliadas na página anterior}} \begin{Bmatrix} \frac{\partial \omega}{\partial \xi} \\ \frac{\partial \omega}{\partial \eta} \end{Bmatrix} \qquad (1)$$

Alternativa:

Sabemos também que

$$\frac{\partial \omega(\overbrace{f(\xi,\eta)}^{x}, \overbrace{g(\xi,\eta)}^{y})}{\partial \xi} = \frac{\partial \omega}{\partial f}\frac{\partial f}{\partial \xi} + \frac{\partial \omega}{\partial g}\frac{\partial g}{\partial \xi}$$

$$\frac{\partial \omega(f(\xi,\eta), g(\xi,\eta))}{\partial \eta} = \frac{\partial \omega}{\partial f}\frac{\partial f}{\partial \eta} + \frac{\partial \omega}{\partial g}\frac{\partial g}{\partial \eta}$$

$$\Leftrightarrow \begin{Bmatrix} \frac{\partial \omega}{\partial \xi} \\ \frac{\partial \omega}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial \xi} & \frac{\partial g}{\partial \xi} \\ \frac{\partial f}{\partial \eta} & \frac{\partial g}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial \omega}{\partial f} \\ \frac{\partial \omega}{\partial g} \end{Bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{Bmatrix} \frac{\partial \omega}{\partial x} \\ \frac{\partial \omega}{\partial y} \end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}^{-1}}_{\underline{J}^{-T}} \begin{Bmatrix} \frac{\partial \omega}{\partial \xi} \\ \frac{\partial \omega}{\partial \eta} \end{Bmatrix} \qquad (2)$$

Comparando (1) e (2), tem-se

$$\begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix} = \frac{1}{|\underline{J}|} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{bmatrix}$$

Note-se que a disposição dos elementos não diagonais é diferente

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Seja $\begin{cases} x = f(\xi,\eta) \\ y = g(\xi,\eta) \end{cases}$ as equações que definem implicitamente $\begin{cases} \xi = r(x,y) \\ \eta = s(x,y) \end{cases}$

Como avaliar $\dfrac{\partial^2 r}{\partial x^2}, \dfrac{\partial^2 r}{\partial x \partial y}, \dfrac{\partial^2 r}{\partial^2 y}, \dfrac{\partial^2 s}{\partial x^2}, \dfrac{\partial^2 s}{\partial x \partial y}$ e $\dfrac{\partial^2 s}{\partial y^2}$?

$$x - f(r(x,y), s(x,y)) = 0 \Leftrightarrow \frac{\partial^2}{\partial x^2}(x - f) = 0 \Leftrightarrow -\frac{\partial}{\partial x}\left(f_{,r}\, r_{,x} + f_{,s}\, s_{,x}\right) = 0 \Leftrightarrow$$

$$\Leftrightarrow -\left(\frac{\partial f_{,r}}{\partial x} r_{,x} + f_{,r}\frac{\partial r_{,x}}{\partial x} + \frac{\partial f_{,s}}{\partial x} s_{,x} + f_{,s}\frac{\partial s_{,x}}{\partial x}\right) = 0 \Leftrightarrow$$

$$\Leftrightarrow -\left(\left(f_{,rr}\, r_{,x} + f_{,rs}\, s_{,x}\right) r_{,x} + \left(f_{,sr}\, r_{,x} + f_{,ss}\, s_{,x}\right) s_{,x} + f_{,r}\, r_{,xx} + f_{,s}\, s_{,xx}\right) = 0$$

$$\Leftrightarrow f_{,rr}(r_{,x})^2 + 2 f_{,rs}\, s_{,x} r_{,x} + f_{,ss}(s_{,x})^2 + f_{,r}\, r_{,xx} + f_{,s}\, s_{,xx} = 0 \quad (1)$$

$$y - g(r(x,y), s(x,y)) = 0 \Leftrightarrow \frac{\partial^2}{\partial x^2}(y - g) = 0 \Leftrightarrow$$

$$\Leftrightarrow g_{,rr}(r_{,x})^2 + 2 g_{,rs}\, s_{,x} r_{,x} + g_{,ss}(s_{,x})^2 + g_{,r}\, r_{,xx} + g_{,s}\, s_{,xx} = 0 \quad (2)$$

De (1) e (2) vem

$$\begin{bmatrix} f_{,r} & f_{,s} \\ g_{,r} & g_{,s} \end{bmatrix} \begin{Bmatrix} r_{,xx} \\ s_{,xx} \end{Bmatrix} = -\begin{Bmatrix} f_{,rr}(r_{,x})^2 + 2 f_{,rs}\, s_{,x} r_{,x} + f_{,ss}(s_{,x})^2 \\ g_{,rr}(r_{,x})^2 + 2 g_{,rs}\, s_{,x} r_{,x} + g_{,ss}(s_{,x})^2 \end{Bmatrix}$$

$r_{,x}$ e $s_{,x}$ já foram identificados anteriormente.

Assim,

$$\begin{Bmatrix} \xi_{,xx} \\ \eta_{,xx} \end{Bmatrix} = \frac{1}{\left(\dfrac{\partial x}{\partial \xi}\dfrac{\partial y}{\partial \eta} - \dfrac{\partial x}{\partial \eta}\dfrac{\partial y}{\partial \xi}\right)^3} \begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial x}{\partial \eta} \\ -\dfrac{\partial y}{\partial \xi} & \dfrac{\partial x}{\partial \xi} \end{bmatrix} \begin{Bmatrix} -\dfrac{\partial^2 x}{\partial \xi^2}\left(\dfrac{\partial y}{\partial \eta}\right)^2 - 2\dfrac{\partial^2 x}{\partial \xi \partial \eta}\dfrac{\partial y}{\partial \eta}\left(-\dfrac{\partial y}{\partial \xi}\right) - \dfrac{\partial^2 x}{\partial \eta^2}\left(-\dfrac{\partial y}{\partial \xi}\right)^2 \\ -\dfrac{\partial^2 y}{\partial \xi^2}\left(\dfrac{\partial y}{\partial \eta}\right)^2 - 2\dfrac{\partial^2 y}{\partial \xi \partial \eta}\dfrac{\partial y}{\partial \eta}\left(-\dfrac{\partial y}{\partial \xi}\right) - \dfrac{\partial^2 y}{\partial \eta^2}\left(\dfrac{\partial y}{\partial \xi}\right)^2 \end{Bmatrix}$$

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De igual modo

$$\frac{\partial^2}{\partial y^2}(x-f)=0 \Leftrightarrow -\frac{\partial}{\partial y}\left(f_{,r}\, r_{,y}+f_{,s}\, s_{,y}\right)=0 \Leftrightarrow f_{,rr}(r_{,y})^2+2f_{,rs}\, s_{,y}\, r_{,y}+f_{,ss}(s_{,y})^2+f_{,r}\, r_{,yy}+$$

$$+f_{,s}\, s_{,yy}=0$$

$$\frac{\partial^2}{\partial y^2}(y-g)=0 \Leftrightarrow -\frac{\partial}{\partial y}\left(g_{,r}\, r_{,y}+g_{,s}\, s_{,y}\right)=0 \Leftrightarrow g_{,rr}(r_{,y})^2+2\, g_{,rs}\, s_{,y}\, r_{,y}+g_{,ss}(s_{,y})^2+g_{,r}\, r_{,yy}+$$

$$+g_{,s}\, s_{,yy}=0$$

Assim

$$\begin{bmatrix} f_{,r} & f_{,s} \\ g_{,r} & g_{,s} \end{bmatrix}\begin{Bmatrix} r_{,yy} \\ s_{,yy} \end{Bmatrix}=-\begin{Bmatrix} f_{,rr}(r_{,y})^2+2f_{,rs}\, s_{,y}\, r_{,y}+f_{,ss}(s_{,y})^2 \\ g_{,rr}(r_{,y})^2+2\, g_{,rs}\, s_{,y}\, r_{,y}+g_{,ss}(s_{,y})^2 \end{Bmatrix}$$

Então,

$$\begin{Bmatrix} \dfrac{\partial^2 \xi}{\partial y^2} \\ \dfrac{\partial^2 \eta}{\partial y^2} \end{Bmatrix}=\frac{1}{\left(\dfrac{\partial x}{\partial \xi}\dfrac{\partial y}{\partial \eta}-\dfrac{\partial x}{\partial \eta}\dfrac{\partial y}{\partial \xi}\right)^3}\begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial x}{\partial \eta} \\ -\dfrac{\partial y}{\partial \xi} & \dfrac{\partial x}{\partial \xi} \end{bmatrix}\begin{Bmatrix} -\dfrac{\partial^2 x}{\partial \xi^2}\left(-\dfrac{\partial x}{\partial \eta}\right)^2-2\dfrac{\partial^2 x}{\partial \xi \partial \eta}\dfrac{\partial x}{\partial \xi}\left(-\dfrac{\partial x}{\partial \eta}\right)-\dfrac{\partial^2 x}{\partial \eta^2}\left(\dfrac{\partial x}{\partial \xi}\right)^2 \\ -\dfrac{\partial^2 y}{\partial \xi^2}\left(-\dfrac{\partial x}{\partial \eta}\right)^2-2\dfrac{\partial^2 y}{\partial \xi \partial \eta}\dfrac{\partial x}{\partial \xi}\left(-\dfrac{\partial x}{\partial \eta}\right)-\dfrac{\partial^2 y}{\partial \eta^2}\left(\dfrac{\partial x}{\partial \xi}\right)^2 \end{Bmatrix}$$

Por outro lado,

$$\frac{\partial^2}{\partial x \partial y}(x-f)=0 \Leftrightarrow -\frac{\partial}{\partial x}\left(f_{,r}\, r_{,y}+f_{,s}\, s_{,y}\right)=0 \Leftrightarrow$$

$$\Leftrightarrow \left(f_{,rr}\, r_{,x}+f_{,rs}\, s_{,x}\right) r_{,y}+\left(f_{,rs}\, r_{,x}+f_{,ss}\, s_{,x}\right) s_{,y}+f_{,r}\, r_{,xy}+f_{,s}\, s_{,xy}=0$$

$$\frac{\partial^2}{\partial x \partial y}(y-g)=0 \Leftrightarrow -\frac{\partial}{\partial x}\left(g_{,r}\, r_{,y}+g_{,s}\, s_{,y}\right)=0 \Leftrightarrow$$

$$\Leftrightarrow \left(g_{,rr}\, r_{,x}+g_{,rs}\, s_{,x}\right) r_{,y}+\left(g_{,rs}\, r_{,x}+g_{,ss}\, s_{,x}\right) s_{,y}+g_{,r}\, r_{,xy}+g_{,s}\, s_{,xy}=0$$

Assim

$$\begin{bmatrix} f_{,r} & f_{,s} \\ g_{,r} & g_{,s} \end{bmatrix}\begin{Bmatrix} r_{,xy} \\ s_{,xy} \end{Bmatrix}=-\begin{Bmatrix} f_{,rr}\, r_{,x}\, r_{,y}+f_{,rs}(s_{,x}\, r_{,y}+r_{,x}\, s_{,y})+f_{,ss}\, s_{,x}\, s_{,y} \\ g_{,rr}\, r_{,x}\, r_{,y}+g_{,rs}(s_{,x}\, r_{,y}+r_{,x}\, s_{,y})+g_{,ss}\, s_{,x}\, s_{,y} \end{Bmatrix}$$

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$$
\begin{Bmatrix} \dfrac{\partial^2 \xi}{\partial x \partial y} \\[2ex] \dfrac{\partial^2 \eta}{\partial x \partial y} \end{Bmatrix} = \frac{1}{\left(\dfrac{\partial x}{\partial \xi}\dfrac{\partial y}{\partial \eta} - \dfrac{\partial x}{\partial \eta}\dfrac{\partial y}{\partial \xi}\right)^3} \begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial x}{\partial \eta} \\[2ex] -\dfrac{\partial y}{\partial \xi} & \dfrac{\partial x}{\partial \xi} \end{bmatrix} \begin{Bmatrix} -\dfrac{\partial^2 x}{\partial \xi^2}\dfrac{\partial y}{\partial \eta}\left(-\dfrac{\partial x}{\partial \eta}\right) - \dfrac{\partial^2 x}{\partial \xi \partial \eta}\left(\left(-\dfrac{\partial y}{\partial \xi}\right)\left(-\dfrac{\partial x}{\partial \eta}\right) + \dfrac{\partial y}{\partial \eta}\dfrac{\partial x}{\partial \xi}\right) - \ldots \\[2ex] -\dfrac{\partial^2 y}{\partial \xi^2}\dfrac{\partial y}{\partial \eta}\left(-\dfrac{\partial x}{\partial \eta}\right) - \dfrac{\partial^2 y}{\partial \xi \partial \eta}\left(\left(-\dfrac{\partial y}{\partial \xi}\right)\left(-\dfrac{\partial x}{\partial \eta}\right) + \dfrac{\partial y}{\partial \eta}\dfrac{\partial x}{\partial \xi}\right) - \ldots \end{Bmatrix}
$$

$$
\begin{matrix} \ldots - \dfrac{\partial^2 x}{\partial \eta^2}\left(-\dfrac{\partial y}{\partial \xi}\right)\dfrac{\partial x}{\partial \xi} \\[2ex] \ldots - \dfrac{\partial^2 y}{\partial^2 \eta^2}\left(-\dfrac{\partial y}{\partial \xi}\right)\dfrac{\partial x}{\partial \xi} \end{matrix}
$$

Assim,

$$
\frac{\partial^2 \omega}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \omega}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \omega}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \omega}{\partial \eta}\frac{\partial \eta}{\partial x}\right) = \left(\frac{\partial^2 \omega}{\partial \xi^2}\frac{\partial \xi}{\partial x} + \frac{\partial^2 \omega}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial x}\right)\frac{\partial \xi}{\partial x} + \left(\frac{\partial^2 \omega}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x} + \frac{\partial^2 \omega}{\partial \eta^2}\frac{\partial \eta}{\partial x}\right)\frac{\partial \eta}{\partial x} +
$$

$$
+ \frac{\partial \omega}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial \omega}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2} =
$$

$$
= \frac{\partial^2 \omega}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + 2\frac{\partial^2 \omega}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial x} + \frac{\partial^2 \omega}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{\partial \omega}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial \omega}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2}
$$

$\vdots$

$$
\begin{Bmatrix} \omega_{,x} \\ \omega_{,y} \\ \omega_{,xx} \\ \omega_{,xy} \\ \omega_{,yy} \end{Bmatrix} = \begin{bmatrix} \xi_{,x} & \eta_{,x} & \cdot & \cdot & \cdot \\ \xi_{,y} & \eta_{,y} & \cdot & \cdot & \cdot \\ \xi_{,xx} & \eta_{,xx} & (\xi_{,x})^2 & 2\xi_{,x}\eta_{,x} & (\eta_{,x})^2 \\ \xi_{,xy} & \eta_{,xy} & \xi_{,x}\xi_{,y} & \xi_{,x}\eta_{,y} + \xi_{,y}\eta_{,x} & \eta_{,x}\eta_{,y} \\ \xi_{,yy} & \eta_{,yy} & (\xi_{,y})^2 & 2\xi_{,y}\eta_{,y} & (\eta_{,y})^2 \end{bmatrix} \begin{Bmatrix} \omega_{,\xi} \\ \omega_{,\eta} \\ \omega_{,\xi\xi} \\ \omega_{,\xi\eta} \\ \omega_{,\eta\eta} \end{Bmatrix} \qquad (1)
$$

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Alternativa:

Evidentemente,

$$\begin{Bmatrix} w_{,\xi} \\ w_{,\eta} \\ w_{,\xi\xi} \\ w_{,\xi\eta} \\ w_{,\eta\eta} \end{Bmatrix} = \begin{bmatrix} x_{,\xi} & y_{,\xi} & \cdot & \cdot & \cdot \\ x_{,\eta} & y_{,\eta} & \cdot & \cdot & \cdot \\ x_{,\xi\xi} & y_{,\xi\xi} & (x_{,\xi})^2 & 2x_{,\xi}y_{,\xi} & (y_{,\xi})^2 \\ x_{,\xi\eta} & y_{,\xi\eta} & x_{,\eta}x_{,\xi} & x_{,\xi}y_{,\eta}+x_{,\eta}y_{,\xi} & y_{,\eta}y_{,\xi} \\ x_{,\eta\eta} & y_{,\eta\eta} & (x_{,\eta})^2 & 2x_{,\eta}y_{,\eta} & (y_{,\eta})^2 \end{bmatrix} \begin{Bmatrix} w_{,x} \\ w_{,y} \\ w_{,xx} \\ w_{,xy} \\ w_{,yy} \end{Bmatrix} \qquad (2)$$

Invertendo (2) e comparando com (1) obtém-se as derivadas requeridas.

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$$
\begin{Bmatrix} \omega_{,xxx} \\ \omega_{,xxy} \\ \omega_{,xyy} \\ \omega_{,yyy} \end{Bmatrix} =
\begin{array}{ccccccccc}
\omega_{,\xi} & \omega_{,\eta} & \omega_{,\xi\xi} & \omega_{,\xi\eta} & \omega_{,\eta\eta} & \omega_{,\xi\xi\xi} & \omega_{,\xi\xi\eta} & \omega_{,\xi\eta\eta} & \omega_{,\eta\eta\eta} \\
\end{array}
$$

$$
\left[\begin{array}{c|c|c|c|c|c|c|c|c}
\xi_{,xxx} & \eta_{,xxx} & 3\xi_{,x}\xi_{,xx} & 3\xi_{,x}\eta_{,xx} + 3\eta_{,x}\xi_{,xx} & 3\eta_{,x}\eta_{,xx} & (\xi_{,x})^3 & 3\eta_{,x}(\xi_{,x})^2 & 3(\eta_{,x})^2\xi_{,x} & (\eta_{,x})^3 \\
\hline
\xi_{,xxy} & \eta_{,xxy} & 2\xi_{,x}\xi_{,xy} + \xi_{,y}\xi_{,xx} & 2\xi_{,x}\eta_{,xy} + 2\eta_{,x}\xi_{,xy} + \xi_{,y}\eta_{,xx} + \eta_{,y}\xi_{,xx} & 2\eta_{,x}\eta_{,xy} + \eta_{,y}\eta_{,xx} & \xi_{,y}(\xi_{,x})^2 & 2\xi_{,y}\eta_{,x}\xi_{,x} + \eta_{,y}(\xi_{,x})^2 & \xi_{,y}(\eta_{,x})^2 + 2\eta_{,y}\eta_{,x}\xi_{,x} & \eta_{,y}(\eta_{,x})^2 \\
\hline
\xi_{,xyy} & \eta_{,xyy} & \xi_{,yy}\xi_{,x} + 2\xi_{,y}\xi_{,xy} & \xi_{,yy}\eta_{,x} + \eta_{,yy}\xi_{,x} + 2\xi_{,y}\eta_{,xy} + 2\eta_{,y}\xi_{,xy} & \eta_{,yy}\eta_{,x} + 2\eta_{,y}\eta_{,xy} & (\xi_{,y})^2\xi_{,x} & 2\eta_{,y}\xi_{,y}\xi_{,x} + (\xi_{,y})^2\eta_{,x} & 2\eta_{,y}\xi_{,y}\eta_{,x} + (\eta_{,y})^2\xi_{,x} & (\eta_{,y})^2\eta_{,x} \\
\hline
\xi_{,yyy} & \eta_{,yyy} & 3\xi_{,y}\xi_{,yy} & 3\xi_{,y}\eta_{,yy} + 3\eta_{,y}\xi_{,yy} & 3\eta_{,y}\eta_{,yy} & (\xi_{,y})^3 & 3\eta_{,y}(\xi_{,y})^2 & 3(\eta_{,y})^2\xi_{,y} & (\eta_{,y})^3
\end{array}\right]
\begin{Bmatrix} \omega_{,\xi} \\ \omega_{,\eta} \\ \omega_{,\xi\xi} \\ \omega_{,\xi\eta} \\ \omega_{,\eta\eta} \\ \omega_{,\xi\xi\xi} \\ \omega_{,\xi\xi\eta} \\ \omega_{,\xi\eta\eta} \\ \omega_{,\eta\eta\eta} \end{Bmatrix}
$$

$$
\begin{Bmatrix} \omega_{,\xi\xi\xi} \\ \omega_{,\xi\xi\eta} \\ \omega_{,\xi\eta\eta} \\ \omega_{,\eta\eta\eta} \end{Bmatrix}
=
\begin{array}{c}
\begin{array}{ccccccccc}
\omega_{,\xi} & \omega_{,\eta} & \omega_{,\xi\xi} & \omega_{,\xi\eta} & \omega_{,\eta\eta} & \omega_{,\xi\xi\xi} & \omega_{,\xi\xi\eta} & \omega_{,\xi\eta\eta} & \omega_{,\eta\eta\eta}
\end{array} \\
\begin{bmatrix}
x_{,\xi\xi\xi} & y_{,\xi\xi\xi} & 3x_{,\xi}x_{,\xi\xi} & 3y_{,\xi}x_{,\xi\xi}+3x_{,\xi}y_{,\xi\xi} & 3y_{,\xi}y_{,\xi\xi} & (x_{,\xi})^3 & 3(x_{,\xi})^2 y_{,\xi} & 3x_{,\xi}(y_{,\xi})^2 & (y_{,\xi})^3 \\
x_{,\xi\xi\eta} & y_{,\xi\xi\eta} & 2x_{,\xi}x_{,\xi\eta}+x_{,\eta}x_{,\xi\xi} & 2y_{,\xi}x_{,\xi\eta}+2x_{,\xi}y_{,\xi\eta}+y_{,\eta}x_{,\xi\xi}+x_{,\eta}y_{,\xi\xi} & 2y_{,\xi}y_{,\xi\eta}+y_{,\eta}y_{,\xi\xi} & x_{,\eta}(x_{,\xi})^2 & y_{,\eta}(x_{,\xi})^2+2x_{,\eta}x_{,\xi}y_{,\xi} & 2y_{,\eta}x_{,\xi}y_{,\xi}+x_{,\eta}(y_{,\xi})^2 & y_{,\eta}(y_{,\xi})^2 \\
x_{,\xi\eta\eta} & y_{,\xi\eta\eta} & x_{,\eta\eta}x_{,\xi}+2x_{,\eta}x_{,\xi\eta} & y_{,\eta\eta}x_{,\xi}+x_{,\eta\eta}y_{,\xi}+2y_{,\eta}x_{,\xi\eta}+2x_{,\eta}y_{,\xi\eta} & y_{,\eta\eta}y_{,\xi}+2y_{,\eta}y_{,\xi\eta} & (x_{,\eta})^2 x_{,\xi} & 2x_{,\eta}y_{,\eta}x_{,\xi}+(x_{,\eta})^2 y_{,\xi} & (y_{,\eta})^2 x_{,\xi}+2x_{,\eta}y_{,\eta}y_{,\xi} & (y_{,\eta})^2 y_{,\xi} \\
x_{,\eta\eta\eta} & y_{,\eta\eta\eta} & 3x_{,\eta}x_{,\eta\eta} & 3y_{,\eta}x_{,\eta\eta}+3x_{,\eta}y_{,\eta\eta} & 3y_{,\eta}y_{,\eta\eta} & (x_{,\eta})^3 & 3(x_{,\eta})^2 y_{,\eta} & 3x_{,\eta}(y_{,\eta})^2 & (y_{,\eta})^3
\end{bmatrix}
\end{array}
\begin{Bmatrix} \omega_{,x} \\ \omega_{,y} \\ \omega_{,xx} \\ \omega_{,xy} \\ \omega_{,yy} \\ \omega_{,xxx} \\ \omega_{,xxy} \\ \omega_{,xyy} \\ \omega_{,yyy} \end{Bmatrix}
$$

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Como avaliar $r_{,xxx}$, $r_{,xxy}$, $r_{,xyy}$, $r_{,yyy}$, $s_{,xxx}$, $s_{,xxy}$, $s_{,xyy}$ e $s_{,yyy}$?

$$\begin{cases} x - f = 0 \Leftrightarrow \dfrac{\partial}{\partial x^3}(x - f) = 0 \\[2ex] y - g = 0 \Leftrightarrow \dfrac{\partial}{\partial x^3}(y - g) = 0 \end{cases}$$

$$\begin{bmatrix} f_{,r} & f_{,s} \\ g_{,r} & g_{,s} \end{bmatrix} \begin{Bmatrix} r_{,xxx} \\ s_{,xxx} \end{Bmatrix} = - \begin{Bmatrix} f_{,sss}(s_{,x})^3 + 3 f_{,rss}\, r_{,x} (s_{,x})^2 + 3 f_{,rs}\, s_{,x}\, r_{,xx} + 3 f_{,rr}\, r_{,x}\, r_{,xx} + 3 f_{,ss}\, s_{,x}\, s_{,xx} + 3 f_{,rs}\, r_{,x}\, s_{,xx} + 3 f_{,rrs} (r_{,x})^2 s_{,x} + f_{,rrr} (r_{,x})^3 \\ \text{(semelhante, mas } f \rightarrow g \text{)} \end{Bmatrix}$$

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