Resumo da notação

$u \equiv$ escalar

$\underset{\sim}{v} \equiv$ vector $\quad \underset{\sim}{v} = v_i \underset{\sim}{e}_i$

$\underline{T} \equiv$ tensor de 2ª ordem $\quad \underline{T} = \underset{\sim}{t}_j \otimes \underset{\sim}{e}_j = T_{ij} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$

onde $\underset{\sim}{t}_j = T_{ij} \underset{\sim}{e}_i$

Gradiente

i) de um campo escalar : $\quad \underset{\sim}{\nabla} u = u_{,i} \underset{\sim}{e}_i$

$$[\underset{\sim}{\nabla} u] = \begin{Bmatrix} u_{,1} \\ u_{,2} \\ u_{,3} \end{Bmatrix}$$

ii) de um campo vectorial : $\quad \underset{\sim}{\nabla} \underset{\sim}{v} = \underset{\sim}{v}_{,i} \otimes \underset{\sim}{e}_i = v_{i,j} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j$

$$[\underset{\sim}{\nabla} \underset{\sim}{v}] = [\underset{\sim}{v}_{,1} \;\; \underset{\sim}{v}_{,2} \;\; \underset{\sim}{v}_{,3}] = \begin{bmatrix} v_{1,1} & v_{1,2} & v_{1,3} \\ v_{2,1} & v_{2,2} & v_{2,3} \\ v_{3,1} & v_{3,2} & v_{3,3} \end{bmatrix}$$

Divergência

i) de um campo vectorial: $\quad div\, \underset{\sim}{v} = tr(\underset{\sim}{\nabla} \underset{\sim}{v}) = v_{i,i} = v_{1,1} + v_{2,2} + v_{3,3}$

ii) de um campo tensorial de 2ª ordem: $div\, \underline{T} = \underset{\sim}{t}_{j,j} = T_{ij,j} \underset{\sim}{e}_i$

$$[div\, \underline{T}] = \begin{Bmatrix} T_{11,1} + T_{12,2} + T_{13,3} \\ T_{21,1} + T_{22,2} + T_{23,3} \\ T_{31,1} + T_{32,2} + T_{33,3} \end{Bmatrix}$$

Laplaciano

i) de um campo escalar : $\nabla^2 u = div(\underset{\sim}{\nabla} u) = u_{,ii} = u_{,11} + u_{,22} + u_{,33}$

ii) de um campo vectorial : $\nabla^2 \underset{\sim}{v} = div(\underset{\sim}{\nabla} \underset{\sim}{v}) = v_{i,jj} \underset{\sim}{e}_i$

$$[\underset{\sim}{\nabla}^2 \underset{\sim}{v}] = \begin{Bmatrix} v_{1,11} + v_{1,22} + v_{1,33} \\ v_{2,11} + v_{2,22} + v_{2,33} \\ v_{3,11} + v_{3,22} + v_{3,33} \end{Bmatrix}$$

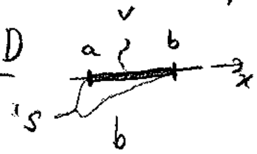
Integração por partes

$f(\underset{\sim}{x})$ e $g(\underset{\sim}{x})$ são funções escalares de variável vectorial

$$(fg)_{,i} = f_{,i}\, g + f\, g_{,i} \Leftrightarrow f_{,i}\, g = (fg)_{,i} - f\, g_{,i} \Leftrightarrow$$

$$\Leftrightarrow \int_V f_{,i}\, g \, dV = \int_V (fg)_{,i}\, dV - \int_V f\, g_{,i}\, dV$$

1D



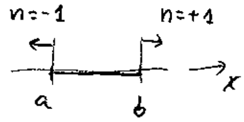
$$\int_a^b \frac{df}{dx} g\, dx = \int_a^b \frac{d}{dx}(fg)\, dx - \int_a^b f \frac{dg}{dx} dx \Leftrightarrow$$

$$\Leftrightarrow \int_a^b \frac{df}{dx} g\, dx = \left[f g \right]_a^b - \int_a^b f \frac{dg}{dx} dx \Leftrightarrow$$

$$\Leftrightarrow \int_a^b \frac{df}{dx} g\, dx = \left[n f g \right]_a + \left[n f g \right]_b - \int_a^b f \frac{dg}{dx} dx$$

Teorema fundamental do cálculo:

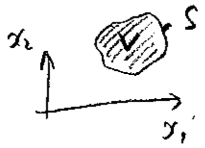
$$\int_a^b \frac{d}{dx}(f)\, dx = \left[f \right]_a^b$$



ou ainda

$$\int_a^b \frac{d}{dx}(f)\, dx = \left[n f \right]_a + \left[n f \right]_b$$

2D

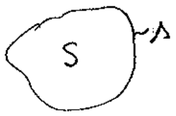


$$\int_V f_{,\alpha}\, g\, dV = \int_V (fg)_{,\alpha}\, dV - \int_V f\, g_{,\alpha}\, dV \Leftrightarrow$$

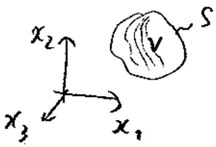
$$\Leftrightarrow \int_V f_{,\alpha}\, g\, dV = \int_S f g\, n_\alpha\, ds - \int_V f\, g_{,\alpha}\, dV$$

Teorema de Green:

$$\int_S f_{,\alpha}\, dS = \int_s f n_\alpha\, ds$$



3D



$$\int_V f_{,i}\, g\, dV = \int_V (fg)_{,i}\, dV - \int_V f\, g_{,i}\, dV \Leftrightarrow$$

$$\Leftrightarrow \int_V f_{,i}\, g\, dV = \int_S f g\, n_i\, dV - \int_V f\, g_{,i}\, dV$$

Teorema de Gauss

$$\int_V f_{,i}\, dV = \int_S f n_i\, dS$$

Generalizações destes resultados

$$\int_V \operatorname{div} \underset{\sim}{v}\, dV = \int_S \underset{\sim}{v} \cdot \underset{\sim}{n}\, dS \Leftrightarrow \int_V v_{i,i}\, dV = \int_S v_i n_i\, dS$$

(Esta última identidade é válida quer i seja um índice mudo ou livre)

$$\int_V \operatorname{div} \underline{T}\, dV = \int_S \underline{T}\, \underset{\sim}{n}\, dS \Leftrightarrow \int_V T_{ij,j}\, dV\, \underset{\sim}{e}_i = \int_S T_{ij} n_j\, dS\, \underset{\sim}{e}_i$$

ou ainda $\displaystyle\int_V T_{ij,j}\, dV = \int_S T_{ij} n_j\, dS$

Produto tensorial de vectores

$$(\underset{\sim}{a} \otimes \underset{\sim}{b})\, \underset{\sim}{x} = \underset{\sim}{a}\, (\underset{\sim}{b} \cdot \underset{\sim}{x})$$

Produto escalar entre tensores de 2ª ordem

$$\underline{A} : \underline{B} = \operatorname{tr}(\underline{A}^T \underline{B}) = A_{ij} B_{ij}$$

$$\underline{A} : \underline{B} = \operatorname{Sym}(\underline{A}) : \operatorname{Sym}(\underline{B}) + \operatorname{Skew}(\underline{A}) : \operatorname{Skew}(\underline{B})$$

onde $\operatorname{Sym} \underline{A} = \frac{1}{2}(\underline{A} + \underline{A}^T)$ e $\operatorname{Skew} \underline{A} = \frac{1}{2}(\underline{A} - \underline{A}^T)$

Note-se que $\underline{A} = \operatorname{Sym} \underline{A} + \operatorname{Skew} \underline{A}$

As equações de Navier - Stokes

i) Formulação em termos de pseudo-tensão

Domínio, Ω

Equilíbrio $\quad \operatorname{div} \underline{T} + \rho \underset{\sim}{b} = \rho (\dot{\underset{\sim}{u}} + (\underset{\sim}{\nabla} \underset{\sim}{u}) \underset{\sim}{u})$ (1)

Relação Constitutiva $\quad \underline{T} = 2\mu \underline{\varepsilon} - \tilde{p} \underline{I}$ (2)

Compatibilidade $\quad \underline{\varepsilon} = \frac{1}{2} (\underset{\sim}{\nabla} \underset{\sim}{u} + (\underset{\sim}{\nabla} \underset{\sim}{u})^T)$ (3)

$$\operatorname{div} \underset{\sim}{u} = 0 \qquad (4)$$

Condições de fronteira / iniciais

$\underset{\sim}{u} = \bar{\underset{\sim}{u}}$ em Γ_u (5)

$(\mu \underset{\sim}{\nabla} \underset{\sim}{u} - \tilde{p} \underline{I}) \underset{\sim}{n} = \bar{\tilde{t}}$ em Γ_t (6)

$\underset{\sim}{u} = \underset{\sim}{u}_0$ em Ω para $t = t_0$ (7)

$\Gamma_u \cap \Gamma_t = \emptyset$

$\Gamma = \overline{\Gamma_t \cup \Gamma_u}$

$\bar{\Omega} = \Omega \cup \Gamma$

Forma fraca do problema

Impondo (1), (6) e (4) na forma fraca, vem

$$\int_\Omega \delta \underset{\sim}{u} \cdot \left(\operatorname{div} \underline{T} + \rho \underset{\sim}{b} - \rho (\dot{\underset{\sim}{u}} + (\underset{\sim}{\nabla} \underset{\sim}{u}) \underset{\sim}{u}) \right) d\Omega + \int_\Omega \delta \tilde{p} \operatorname{div} \underset{\sim}{u} \, d\Omega -$$

$$- \int_{\Gamma_t} \delta \underset{\sim}{u} \cdot \left((\mu \underset{\sim}{\nabla} \underset{\sim}{u} - \tilde{p} \underline{I}) \underset{\sim}{n} - \bar{\tilde{t}} \right) d\Gamma_t = 0 \qquad (8)$$

onde $\delta \underset{\sim}{u} \big|_{\Gamma_u} = \underset{\sim}{0}$ e $\underset{\sim}{u} \big|_{\Gamma_u} = \bar{\underset{\sim}{u}}$.

Dividindo (8) por ρ e introduzindo as variáveis

$$p = \frac{\tilde{p}}{\rho}, \quad \nu = \frac{\mu}{\rho} \quad e \quad \underset{\sim}{\bar{t}} = \frac{\underset{\sim}{\bar{\tilde{t}}}}{\rho} \quad \text{vem}$$

$$\int_\Omega \delta \underset{\sim}{u} \cdot \left(\frac{1}{\rho} \operatorname{div} \underline{T} + \underset{\sim}{b} - \left(\dot{\underset{\sim}{u}} + (\underset{\sim}{\nabla} \underset{\sim}{u}) \underset{\sim}{u} \right) \right) d\Omega + \int_\Omega \delta p \operatorname{div} \underset{\sim}{u} \, d\Omega -$$

$$- \int_{\Gamma_t} \delta \underset{\sim}{u} \cdot \left((\nu \underset{\sim}{\nabla} \underset{\sim}{u} - p \underline{I}) \underset{\sim}{n} - \underset{\sim}{\bar{t}} \right) d\Gamma_t = 0 \qquad (9)$$

As equações (2) e (3) são impostas localmente, por substituição directa em (9). Subs. (3) em (2) vem:

$$\underline{T} = - \tilde{p} \underline{I} + 2\mu \frac{1}{2} \left(\underset{\sim}{\nabla} \underset{\sim}{u} + (\underset{\sim}{\nabla} \underset{\sim}{u})^T \right) = - \tilde{p} \underline{I} + \mu \left(\underset{\sim}{\nabla} \underset{\sim}{u} + (\underset{\sim}{\nabla} \underset{\sim}{u})^T \right),$$

cuja divergência é dada por

$$\operatorname{div} \underline{T} = \operatorname{div}(-\tilde{p} \underline{I}) + \mu \operatorname{div}(\underset{\sim}{\nabla} \underset{\sim}{u}) + \mu \operatorname{div}\left((\underset{\sim}{\nabla} \underset{\sim}{u})^T\right) \qquad (10)$$

$$\operatorname{div}(-\tilde{p} \underline{I}) = (-\tilde{p}\, \delta_{ij})_{,j} \underset{\sim}{e}_i = -\tilde{p}_{,i} \underset{\sim}{e}_i = - \underset{\sim}{\nabla} \tilde{p}$$

$$\operatorname{div}(\underset{\sim}{\nabla} \underset{\sim}{u}) = \operatorname{div}(u_{i,j} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j) = (u_{i,j})_{,j} \underset{\sim}{e}_i = u_{i,jj} \underset{\sim}{e}_i = \underset{\sim}{\nabla}^2 \underset{\sim}{u}$$

$$\operatorname{div}\left((\underset{\sim}{\nabla} \underset{\sim}{u})^T\right) = \operatorname{div}\left((u_{i,j} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j)^T\right) = \operatorname{div}(u_{i,j} \underset{\sim}{e}_j \otimes \underset{\sim}{e}_i) =$$

$$= \operatorname{div}(u_{j,i} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j) = (u_{j,i})_{,j} \underset{\sim}{e}_i = (u_{j,j})_{,i} \underset{\sim}{e}_i =$$

$$= (\operatorname{div} \underset{\sim}{u})_{,i} \underset{\sim}{e}_i = \underset{\sim}{\nabla} (\operatorname{div} \underset{\sim}{u})$$

Assim,

$$\operatorname{div} \underline{T} = -\tilde{p}_{,i} \underset{\sim}{e}_i + \mu \underset{\sim}{\nabla}^2 \underset{\sim}{u} + \mu \underset{\sim}{\nabla} (\operatorname{div} \underset{\sim}{u}) \qquad (11)$$

Impondo (4) no último termo vem $\mu \underset{\sim}{\nabla} (\underbrace{\operatorname{div} \underset{\sim}{u}}_{=0}) = 0$. Dividindo (11) por ρ tem-se

$$\frac{1}{\rho} \operatorname{div} \underline{T} = - \underset{\sim}{\nabla} p + \nu \operatorname{div}(\underset{\sim}{\nabla} \underset{\sim}{u}) = - \underset{\sim}{\nabla} p + \nu \underset{\sim}{\nabla}^2 \underset{\sim}{u} \qquad (12)$$

Finalmente, subs. (12) em (9), e multiplicando ambos os termos por (-1), tem-se: (3)

$$\int_\Omega \delta \underset{\sim}{u} \cdot \left(\dot{\underset{\sim}{u}} + (\underset{\sim}{\nabla} \underset{\sim}{u}) \underset{\sim}{u} - \nu \underset{\sim}{\nabla}^2 \underset{\sim}{u} + \underset{\sim}{\nabla} p - \underset{\sim}{b} \right) d\Omega - \int_\Omega \delta p \, \mathrm{div}\, \underset{\sim}{u} \, d\Omega +$$

$$+ \int_{\Gamma_t} \delta \underset{\sim}{u} \cdot \left((\nu \underset{\sim}{\nabla} \underset{\sim}{u} - p \underline{I}) \underset{\sim}{n} - \bar{\underset{\sim}{t}} \right) d\Gamma_t = 0 \qquad (13)$$

Comparar com equação apresentada na página 276 do livro de Huerta.

Notando que

$$\int_\Omega \delta \underset{\sim}{u} \cdot \nabla^2 \underset{\sim}{u} \, d\Omega = \int_\Omega \delta u_i \underset{\sim}{e}_i \cdot u_{j,kk} \underset{\sim}{e}_j \, d\Omega = \int_\Omega \delta u_i u_{j,kk} \underbrace{\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j}_{=\delta_{ij}} d\Omega =$$

$$= \int_\Omega \delta u_i u_{i,jj} \, d\Omega = \int_\Omega \left[(\delta u_i u_{i,j})_{,j} - \delta u_{i,j} u_{i,j} \right] d\Omega =$$

$$= \int_\Gamma \delta u_i u_{i,j} n_j \, d\Gamma - \int_\Omega \delta u_{i,j} u_{i,j} \, d\Omega =$$

$$= \int_\Gamma \delta \underset{\sim}{u} \cdot (\underset{\sim}{\nabla} \underset{\sim}{u}) \underset{\sim}{n} \, d\Gamma - \int_\Omega \underset{\sim}{\nabla} \delta \underset{\sim}{u} : \underset{\sim}{\nabla} \underset{\sim}{u} \, d\Omega \qquad (14)$$

$$\int_\Omega \delta \underset{\sim}{u} \cdot \underset{\sim}{\nabla} p \, d\Omega = \int_\Omega \delta u_i p_{,i} \, d\Omega = \int_\Omega \left((\delta u_i p)_{,i} - \delta u_{i,i} p \right) d\Omega =$$

$$= \int_\Gamma \delta \underset{\sim}{u} \cdot p \underset{\sim}{n} \, d\Gamma - \int_\Omega \mathrm{div}\, \delta \underset{\sim}{u} \, p \, d\Omega$$

tem-se, de (13),

$$\int_\Omega \delta\mathbf{u}\cdot\dot{\mathbf{u}}\,d\Omega+\int_\Omega \delta\mathbf{u}\cdot(\nabla\mathbf{u})\,\mathbf{u}\,d\Omega+\int_\Omega \nabla\delta\mathbf{u}:\nu\nabla\mathbf{u}\,d\Omega-\int_\Omega \operatorname{div}\delta\mathbf{u}\,p\,d\Omega-$$

$$-\int_\Omega \delta p\,\operatorname{div}\mathbf{u}\,d\Omega-\cancel{\int_{\Gamma_t}\delta\mathbf{u}\cdot(\nu\nabla\mathbf{u})\,\mathbf{n}\,d\Gamma_t}+\cancel{\int_{\Gamma_t}\delta\mathbf{u}\cdot p\,\mathbf{n}\,d\Gamma_t}-\int_\Omega \delta\mathbf{u}\cdot\mathbf{b}\,d\Omega+$$

$$+\cancel{\int_{\Gamma_t}\delta\mathbf{u}\cdot(\nu\nabla\mathbf{u})\,\mathbf{n}\,d\Gamma_t}-\cancel{\int_{\Gamma_t}\delta\mathbf{u}\cdot p\,\mathbf{n}\,d\Gamma_t}-\int_{\Gamma_t}\delta\mathbf{u}\cdot\bar{\mathbf{t}}\,d\Gamma_t=0\Leftrightarrow$$

$$\Leftrightarrow\int_\Omega \delta\mathbf{u}\cdot\dot{\mathbf{u}}\,d\Omega+\int_\Omega \delta\mathbf{u}\cdot(\nabla\mathbf{u})\,\mathbf{u}\,d\Omega+\int_\Omega \nabla\delta\mathbf{u}:\nu\nabla\mathbf{u}\,d\Omega-\int_\Omega \operatorname{div}\delta\mathbf{u}\,p\,d\Omega-$$

$$-\int_\Omega \delta p\,\operatorname{div}\mathbf{u}\,d\Omega-\int_{\Gamma_t}\delta\mathbf{u}\cdot\bar{\mathbf{t}}\,d\Gamma_t-\int_\Omega \delta\mathbf{u}\cdot\mathbf{b}\,d\Omega=0 \qquad (15)$$

Notas:

i) Nesta formulação a condição (4) é imposta ponderadamente -em (8)-, mas também é imposta directamente em (11).

ii) Análise dimensional das equações. Termos da Forma Forte

$\rho=\frac{m}{V}\quad\left[\frac{Kg}{m^3}\right]\equiv\frac{[N][S^2]}{[m^4]}$ $\qquad$ $\dot{\mathbf{u}}\equiv\frac{[m]}{[S^2]}\ ;\ \rho\dot{\mathbf{u}}=\frac{[N]}{[m^3]}$

$\underline{T}\equiv\frac{F}{L^2}\quad\left[\frac{N}{m^2}\right]$ $\qquad$ $\operatorname{div}\underline{T}\equiv\frac{[N]}{[m^3]}$ $\qquad$ $\nabla\mathbf{u}=\frac{1}{[m]}\frac{[m]}{[S]}=\frac{1}{[S]}$

$\rho\mathbf{b}=\left[\frac{N}{m^3}\right]\ ;\ \mathbf{b}=\left[\frac{m}{s^2}\right]$ $\qquad$ $(\nabla\mathbf{u})\mathbf{u}=\frac{1}{[S]}\frac{[m]}{[S]}=\frac{[m]}{[S^2]}$

$\underline{\varepsilon}\equiv\frac{1}{m}\frac{m}{s}=\frac{1}{s}$ $\qquad$ $\tilde{p}\equiv\frac{N}{m^2}$

$\mu\equiv\frac{Kg}{s\cdot m}=\frac{NS^2}{m}\frac{1}{s\cdot m}=\frac{NS}{m^2}$ $\qquad$ $\operatorname{div}\mathbf{u}=\frac{1}{m}\frac{m}{s}=\frac{1}{s}$

$\textcircled{4.A}$

$$\mu \underline{\underline{\xi}} \equiv \frac{N s}{m^2} \, \frac{1}{s} = \frac{N}{m^2}$$

$$\mu \underset{\sim}{\nabla} \underset{\sim}{u} \equiv \frac{N s}{m^2} \times \frac{1}{s} = \frac{N}{m^2}$$

$$\bar{\tilde{\underset{\sim}{t}}} \equiv \frac{N}{m^2}$$

Forma fraca após adimensionalização

$$p = \frac{\tilde{p}}{\rho} \equiv \frac{\frac{N}{m^2}}{\frac{N s^2}{m^4}} = \frac{m^2}{s^2}$$

$$\nu = \frac{\mu}{\rho} = \frac{\frac{N s}{m^2}}{\frac{N s^2}{m^4}} \equiv \frac{m^2}{s}$$

$$\bar{\underset{\sim}{t}} = \frac{\tilde{\bar{\underset{\sim}{t}}}}{\rho} = \frac{\frac{N}{m^2}}{\frac{N s^2}{m^4}} = \frac{m^2}{s^2}$$

$$\frac{1}{\rho} \operatorname{div} \underline{T} = \frac{m^4}{N s^2} \times \frac{N}{m^3} = \frac{m}{s^2}$$

$$\int \delta \underset{\sim}{u} \cdot (\ldots) \, d\Omega \equiv \frac{m}{s} \times \frac{m}{s^2} \times m^2 = \frac{m^4}{s^3}$$

$$\int \delta p \cdot \operatorname{div} \underset{\sim}{u} \, d\Omega \equiv \frac{m^2}{s^2} \times \frac{1}{m} \, \frac{m}{s} \times m^2 = \frac{m^4}{s^3}$$

$$\nu \underset{\sim}{\nabla} \underset{\sim}{u} = \frac{m^2}{s} \, \frac{1}{m} \, \frac{m}{s} = \frac{m^2}{s^2}$$

$$\int \delta u \, (\quad) \, d\Gamma_q \equiv \frac{m}{s} \, \frac{m^2}{s^2} \, m \equiv \frac{m^4}{s^3}$$

ii) Formulação em termos de verdadeiras tensões

Subs. (3) em (2), tem-se

$$\underline{T} = -\tilde{p}\,\underline{I} + 2\mu\, \underset{\sim}{\nabla}^s \underset{\sim}{u} \qquad \text{onde } \underset{\sim}{\nabla}^s \underset{\sim}{u} = \operatorname{sym}(\underset{\sim}{\nabla} \underset{\sim}{u})$$

$$\text{e } \operatorname{sym} \underline{A} = \tfrac{1}{2}(\underline{A} + \underline{A}^T), \ \forall \underline{A}$$

e

$$\operatorname{div} \underline{T} = -\operatorname{div}(\tilde{p}\,\underline{I}) + 2\mu \operatorname{div}(\underset{\sim}{\nabla}^s \underset{\sim}{u})$$

Na eq. (11) aparecerá

$+2\mu \operatorname{div}(\underset{\sim}{\nabla}^s \underset{\sim}{u})$ em lugar de $\left(+\mu \nabla^2 \underset{\sim}{u} + \mu \underset{\sim}{\nabla}(\operatorname{div} \underset{\sim}{u})\right)$

e a função integranda do 1º termo do 1º membro de (13) será

$$\dot{\underset{\sim}{u}} + (\underset{\sim}{\nabla} \underset{\sim}{u})\underset{\sim}{u} - 2\nu \operatorname{div}(\underset{\sim}{\nabla}^s \underset{\sim}{u}) + \underset{\sim}{\nabla} p - \underset{\sim}{b} \qquad (16)$$

Em lugar de (6) ter-se-á agora $\underline{T}\,\underset{\sim}{n} = \bar{\tilde{\underset{\sim}{t}}}$, onde $\underline{T} = 2\mu\,\underset{\sim}{\varepsilon} - \tilde{p}\,\underline{I} =$
$= 2\mu\, \underset{\sim}{\nabla}^s \underset{\sim}{u} - \tilde{p}\,\underline{I}$, logo

$$(-\tilde{p}\,\underline{I} + 2\mu\, \underset{\sim}{\nabla}^s \underset{\sim}{u})\,\underset{\sim}{n} = \bar{\tilde{\underset{\sim}{t}}} \qquad (17)$$

Dividindo (17) por ρ tem-se

$$(-p\,\underline{I} + 2\nu\, \underset{\sim}{\nabla}^s \underset{\sim}{u})\,\underset{\sim}{n} = \bar{\underset{\sim}{t}} \qquad (18)$$

O termo diferente em Ω que aparece na forma fraca é

$\int_\Omega \delta \underset{\sim}{u} \cdot \operatorname{div}(\underset{\sim}{\nabla}^s \underset{\sim}{u})\, d\Omega$, que é desenvolvido em seguida.

Notando que $\operatorname{div}(\boldsymbol{\nabla}^s \boldsymbol{u}) = \operatorname{div}\left(\frac{1}{2}(u_{i,j}+u_{j,i})\,\boldsymbol{e}_i \otimes \boldsymbol{e}_j\right) = \frac{1}{2}(u_{i,j}+u_{j,i})_{,j}\,\boldsymbol{e}_i$

$$= \frac{1}{2}(u_{i,jj}\,\boldsymbol{e}_i + u_{j,ij}\,\boldsymbol{e}_i)$$

$$\int_\Omega \delta\boldsymbol{u}\cdot \operatorname{div}(\boldsymbol{\nabla}^s\boldsymbol{u})\,d\Omega = \frac{1}{2}\int_\Omega (\delta u_i\, u_{i,jj} + \delta u_i\, u_{j,ij})\,d\Omega =$$

$$= \frac{1}{2}\left(\int_\Omega (\delta u_i\, u_{i,j})_{,j}\,d\Omega - \int_\Omega \delta u_{i,j}\, u_{i,j}\,d\Omega + \int_\Omega (\delta u_i\, u_{j,i})_{,j}\,d\Omega - \int_\Omega \delta u_{i,j}\, u_{j,i}\,d\Omega\right) =$$

$$= \frac{1}{2}\left(\int_\Gamma \delta u_i\, u_{i,j}\, n_j\,d\Omega - \int_\Omega \delta u_{i,j}\, u_{i,j}\,d\Omega + \int_\Gamma \delta u_i\, u_{j,i}\, n_j\,d\Gamma - \int_\Omega \delta u_{i,j}\, u_{j,i}\,d\Omega\right) =$$

$$= \int_\Gamma \delta u_i\, \frac{1}{2}(u_{i,j}+u_{j,i})\, n_j\,d\Gamma - \int_\Omega \delta u_{i,j}\,\frac{1}{2}(u_{i,j}+u_{j,i})\,d\Omega =$$

$$= \int_\Gamma \delta\boldsymbol{u}\cdot(\boldsymbol{\nabla}^s\boldsymbol{u})\,\boldsymbol{n}\,d\Gamma - \int_\Omega \boldsymbol{\nabla}\delta\boldsymbol{u} : \boldsymbol{\nabla}^s\boldsymbol{u}\,d\Omega =$$

Mas $\boldsymbol{\nabla}\delta\boldsymbol{u} : \boldsymbol{\nabla}^s\boldsymbol{u} = (\boldsymbol{\nabla}^s\delta\boldsymbol{u} + \boldsymbol{\nabla}^{AS}\delta\boldsymbol{u}) : \boldsymbol{\nabla}^s\boldsymbol{u} = \boldsymbol{\nabla}^s\delta\boldsymbol{u} : \boldsymbol{\nabla}^s\boldsymbol{u} + \underbrace{\boldsymbol{\nabla}^{AS}\delta\boldsymbol{u} : \boldsymbol{\nabla}^s\boldsymbol{u}}_{=0}$

Logo

$$\int_\Omega \delta\boldsymbol{u}\cdot \operatorname{div}(\boldsymbol{\nabla}^s\boldsymbol{u})\,d\Omega = \int_\Gamma \delta\boldsymbol{u}\cdot(\boldsymbol{\nabla}^s\boldsymbol{u})\,\boldsymbol{n}\,d\Gamma - \int_\Omega \boldsymbol{\nabla}^s\delta\boldsymbol{u} : \boldsymbol{\nabla}^s\boldsymbol{u}\,d\Omega \qquad (19)$$

A forma fraca assume então o formato

$$\int_\Omega \delta\boldsymbol{u}\cdot\left(\dot{\boldsymbol{u}} + (\boldsymbol{\nabla}\boldsymbol{u})\,\boldsymbol{u} - 2\nu\operatorname{div}(\boldsymbol{\nabla}^s\boldsymbol{u}) + \boldsymbol{\nabla}p - \boldsymbol{b}\right)d\Omega - \int_\Omega \delta p\operatorname{div}\boldsymbol{u}\,d\Omega +$$

$$+ \int_{\Gamma_1} \delta\boldsymbol{u}\cdot\left((2\nu\,\boldsymbol{\nabla}^s\boldsymbol{u} - p\,\boldsymbol{I})\,\boldsymbol{n} - \bar{\boldsymbol{t}}\right)d\Gamma_1 = 0 \iff$$

$$\Rightarrow \int_\Omega \delta\boldsymbol{u}\cdot\dot{\boldsymbol{u}}\,d\Omega + \int_\Omega \delta\boldsymbol{u}\cdot(\boldsymbol{\nabla}\boldsymbol{u})\,\boldsymbol{u}\,d\Omega + \underline{\int_\Omega 2\nu\,\boldsymbol{\nabla}^s\delta\boldsymbol{u} : \boldsymbol{\nabla}^s\boldsymbol{u}\,d\Omega} - \int_\Omega \operatorname{div}\delta\boldsymbol{u}\,p\,d\Omega -$$

$$-\int_\Omega \delta p\,\operatorname{div}\boldsymbol{u}\,d\Omega - \underline{\cancel{\int_{\Gamma_t} 2\nu\,\delta\boldsymbol{u}\cdot(\boldsymbol{\nabla}^s\boldsymbol{u})\,\boldsymbol{n}\,d\Gamma_t}} + \cancel{\int_{\Gamma_t}\delta\boldsymbol{u}\cdot p\,\boldsymbol{n}\,d\Gamma_t} - \int_\Omega \delta\boldsymbol{u}\cdot\boldsymbol{b}\,d\Omega +$$

$$+\underline{\cancel{\int_{\Gamma_t}\delta\boldsymbol{u}\cdot 2\nu(\boldsymbol{\nabla}^s\boldsymbol{u})\,\boldsymbol{n}\,d\Gamma_t}} - \cancel{\int_{\Gamma_t}\delta\boldsymbol{u}\cdot p\,\boldsymbol{n}\,d\Gamma_t} - \int_{\Gamma_t}\delta\boldsymbol{u}\cdot\bar{\boldsymbol{t}}\,d\Gamma_t = 0 \Leftrightarrow$$

$$\Leftrightarrow \int_\Omega \delta\boldsymbol{u}\cdot\dot{\boldsymbol{u}}\,d\Omega + \int_\Omega \delta\boldsymbol{u}\cdot(\boldsymbol{\nabla}\boldsymbol{u})\,\boldsymbol{u}\,d\Omega + \underline{\int_\Omega 2\nu\,\boldsymbol{\nabla}^s\delta\boldsymbol{u} : \boldsymbol{\nabla}^s\boldsymbol{u}\,d\Omega} - \int_\Omega \operatorname{div}\delta\boldsymbol{u}\,p\,d\Omega +$$

$$+\int_\Omega \delta p\,\operatorname{div}\boldsymbol{u}\,d\Omega - \int_{\Gamma_t}\delta\boldsymbol{u}\cdot\bar{\boldsymbol{t}}\,d\Gamma_t - \int_\Omega \delta\boldsymbol{u}\cdot\boldsymbol{b}\,d\Omega = 0 \qquad (20)$$

(os termos diferentes em relação à formulação i) estão a <u>sublinhado</u>, ver equação (15)),

Note-se que nesta formulação (4) apenas é imposta ponderadamente.

A única diferença entre as formulações é o termo que envolve ν.

Discretização no tempo: o método de Newmark

$$t^{n+1} = t^n + \Delta t$$

$$\dot{\underset{\sim}{u}}^{n+1} = \frac{1}{\gamma}\,\frac{\underset{\sim}{u}^{n+1} - \underset{\sim}{u}^{n}}{\Delta t} - \frac{1-\gamma}{\gamma}\,\dot{\underset{\sim}{u}}^{n} \qquad (21)$$

Discretização no espaço: MEF

$$\underset{\sim}{u} = \underline{\Psi}^u \hat{\underset{\sim}{u}}$$

$$\delta \underset{\sim}{u} = \underline{\Psi}^u \delta \hat{\underset{\sim}{u}} \qquad \delta \Delta \underset{\sim}{u} = \Delta \delta \underset{\sim}{u} = \underset{\sim}{0}$$

$$\dot{\underset{\sim}{u}} = \underline{\Psi}^u \dot{\hat{\underset{\sim}{u}}} \qquad (22)$$

$$\Delta \underset{\sim}{u} = \underline{\Psi}^u \Delta \hat{\underset{\sim}{u}}$$

$$p = \underline{\Psi}^p \hat{\underset{\sim}{p}}$$

$$\delta p = \underline{\Psi}^p \delta \hat{\underset{\sim}{p}} \qquad \delta \Delta p = \Delta \delta p = 0$$

$$\Delta p = \underline{\Psi}^p \Delta \hat{\underset{\sim}{p}} \qquad (23)$$

$$\boxed{\text{Termo} \quad \int_\Omega \delta \underset{\sim}{u} \cdot \dot{\underset{\sim}{u}}\, d\Omega}$$

Este termo envolve basicamente a avaliação de

$$\int_\Omega \delta \underset{\sim}{u} \cdot \dot{\underset{\sim}{u}}\, d\Omega = \int_\Omega \delta \underset{\sim}{u}^T \dot{\underset{\sim}{u}}\, d\Omega = \delta \hat{\underset{\sim}{u}}^T \int_\Omega \underline{\Psi}^{uT} \underline{\Psi}^u\, d\Omega\, \dot{\hat{\underset{\sim}{u}}} = \delta \hat{\underset{\sim}{u}}^T \underline{M}\, \dot{\hat{\underset{\sim}{u}}}$$

$$\Delta \int_\Omega \delta \underset{\sim}{u} \cdot \dot{\underset{\sim}{u}}\, d\Omega = \int_\Omega \delta \underset{\sim}{u} \cdot \Delta \dot{\underset{\sim}{u}}\, d\Omega = \delta \hat{\underset{\sim}{u}}^T \int_\Omega \underline{\Psi}^{uT} \underline{\Psi}^u\, d\Omega\, \Delta \dot{\hat{\underset{\sim}{u}}} = \delta \hat{\underset{\sim}{u}}^T \underline{M}\, \Delta \dot{\hat{\underset{\sim}{u}}} \qquad (24)$$

onde $\underline{M}$ é a matriz de massa.

$$\boxed{\text{Termo} \quad \int_\Omega \delta \underset{\sim}{u} \cdot (\underset{\sim}{\nabla} \underset{\sim}{u}) \underset{\sim}{u} \, d\Omega}$$

$$\int_\Omega \delta \underset{\sim}{u} \cdot (\underset{\sim}{\nabla} \underset{\sim}{u}) \underset{\sim}{u} \, d\Omega = \int_\Omega \delta \underset{\sim}{u} \cdot (\underset{\sim}{u}_{,i} \otimes \underset{\sim}{e}_i) \underset{\sim}{u} \, d\Omega = \int_\Omega \delta \underset{\sim}{u}^T (\underset{\sim}{u}_{,i} \underset{\sim}{e}_i^T) \underset{\sim}{u} \, d\Omega =$$

$$= \delta \underset{\sim}{\hat{u}}^T \int_\Omega \underline{\psi}^{uT} (\underline{\psi}^u_{,i} \underset{\sim}{\hat{u}} \underset{\sim}{e}_i^T) \underline{\psi}^u \, d\Omega \, \underset{\sim}{\hat{u}} = \delta \underset{\sim}{\hat{u}}^T \underline{C}_c \underset{\sim}{\hat{u}} \quad \text{(Versão H. Campelo)} \quad (25)$$

Uma forma alternativa de escrever este termo seria:

$$\int_\Omega \delta \underset{\sim}{u} \cdot (\underset{\sim}{\nabla} \underset{\sim}{u}) \underset{\sim}{u} \, d\Omega = \int_\Omega \delta \underset{\sim}{u} \cdot (\underset{\sim}{u}_{,i} \otimes \underset{\sim}{e}_i) u \, d\Omega = \int_\Omega \delta \underset{\sim}{u} \cdot \underset{\sim}{u}_{,i} (\underset{\sim}{e}_i \cdot \underset{\sim}{u}) \, d\Omega =$$

$$= \int_\Omega \delta \underset{\sim}{u}^T (\underset{\sim}{e}_i \cdot \underset{\sim}{u}) \underset{\sim}{u}_{,i} \, d\Omega = \delta \underset{\sim}{\hat{u}}^T \int_\Omega \underline{\psi}^{uT} (\underset{\sim}{e}_i^T \underline{\psi}^u \underset{\sim}{\hat{u}}) \underline{\psi}^u_{,i} \, d\Omega \, \underset{\sim}{\hat{u}} =$$

$$= \delta \underset{\sim}{\hat{u}}^T \underline{C}_H \underset{\sim}{\hat{u}} \quad \text{(Versão Huerta)} \quad (26)$$

A linearização da 1ª versão conduz a:

$$\Delta \int_\Omega \delta \underset{\sim}{u} \cdot (\underset{\sim}{\nabla} \underset{\sim}{u}) \underset{\sim}{u} \, d\Omega = \Delta \int_\Omega \delta \underset{\sim}{u} \cdot (\underset{\sim}{u}_{,i} \otimes \underset{\sim}{e}_i) \underset{\sim}{u} \, d\Omega =$$

$$= \int_\Omega [\delta \underset{\sim}{u} \cdot (\Delta \underset{\sim}{u}_{,i} \otimes \underset{\sim}{e}_i) \underset{\sim}{u} + \delta \underset{\sim}{u} \cdot (\underset{\sim}{u}_{,i} \otimes \underset{\sim}{e}_i) \Delta \underset{\sim}{u}] \, d\Omega =$$

$$= \int_\Omega [\delta \underset{\sim}{u}^T (\underset{\sim}{e}_i^T \underset{\sim}{u}) \Delta \underset{\sim}{u}_{,i} + \delta \underset{\sim}{u} \cdot (\underset{\sim}{u}_{,i} \otimes \underset{\sim}{e}_i) \Delta \underset{\sim}{u}] \, d\Omega =$$

$$= \delta \underset{\sim}{\hat{u}}^T \int_\Omega \left[\underline{\psi}^{uT} (\underset{\sim}{e}_i^T (\underline{\psi}^u \underset{\sim}{\hat{u}})) \underline{\psi}^u_{,i} + \underline{\psi}^{uT} (\underline{\psi}^u_{,i} \underset{\sim}{\hat{u}} \, \underset{\sim}{e}_i^T) \underline{\psi}^u \right] d\Omega \, \Delta \underset{\sim}{\hat{u}} = \quad (27)$$

$$= \delta \underset{\sim}{\hat{u}}^T (\underline{C}_H + \underline{C}_c) \Delta \underset{\sim}{\hat{u}}$$

A linearização da 2ª versão conduz a:

$$\Delta \int_\Omega \delta \underset{\sim}{u} \cdot (\underset{\sim}{\nabla}\, \underset{\sim}{u}) \underset{\sim}{u} \, d\Omega = \Delta \int_\Omega \delta \underset{\sim}{u} \cdot \underset{\sim}{u}_{,i} (\underset{\sim}{e}_i \cdot \underset{\sim}{u}) \, d\Omega =$$

$$= \int_\Omega \left[\delta \underset{\sim}{u} \cdot \Delta \underset{\sim}{u}_{,i} (\underset{\sim}{e}_i \cdot \underset{\sim}{u}) + \delta \underset{\sim}{u} \cdot \underset{\sim}{u}_{,i} (\underset{\sim}{e}_i \cdot \Delta \underset{\sim}{u}) \right] d\Omega =$$

$$= \int_\Omega \left[\delta \underset{\sim}{u} \cdot (\underset{\sim}{e}_i \cdot \underset{\sim}{u}) \Delta \underset{\sim}{u}_{,i} + \delta \underset{\sim}{u} \cdot (\underset{\sim}{u}_{,i} \otimes \underset{\sim}{e}_i) \Delta \underset{\sim}{u} \right] d\Omega =$$

$$= \delta \hat{\underset{\sim}{u}}^T \int_\Omega \left[\underline{\Psi}^{uT} (\underset{\sim}{e}_i^T \underline{\Psi}^u \underset{\sim}{u}) \underline{\Psi}^u_{,i} + \underline{\Psi}^{uT} \left(\underline{\Psi}^u_{,i} \hat{\underset{\sim}{u}} \, \underset{\sim}{e}_i^T \right) \underline{\Psi}^u \right] d\Omega \, \Delta \hat{\underset{\sim}{u}} = \quad (28)$$

$$= \delta \hat{\underset{\sim}{u}}^T (\underline{C}_H + \underline{C}_c) \Delta \hat{\underset{\sim}{u}}$$

Observe-se que esta matriz coincide com a anterior. Note-se também que as matrizes $\underline{C}$ de ambas as versões são diferentes, mas a sua contribuição para o vector resíduo, i. e., $\underline{C}\hat{\underset{\sim}{u}}$, é exactamente igual. Além disso, a matriz tangente é dada pela soma das matrizes $\underline{C}$ de cada uma das versões. Assim,

$$\underline{C}_c \hat{\underset{\sim}{u}} = \underline{C}_H \hat{\underset{\sim}{u}} \quad \text{e} \quad \Delta \int_\Omega \delta \underset{\sim}{u} \cdot (\underset{\sim}{\nabla} \underset{\sim}{u}) \underset{\sim}{u} \, d\Omega = \delta \hat{\underset{\sim}{u}}^T (\underline{C}_H + \underline{C}_c) \Delta \hat{\underset{\sim}{u}}$$

$\underline{C}_H$ e $\underline{C}_c$ são matrizes convectivas.

$$\boxed{\text{Termo} \int_\Omega \underset{\sim}{\nabla} \delta \underset{\sim}{u} : \nu \underset{\sim}{\nabla} \underset{\sim}{u} \, d\Omega}$$

$$\int_\Omega \underset{\sim}{\nabla} \delta \underset{\sim}{u} : \nu \underset{\sim}{\nabla} \underset{\sim}{u} \, d\Omega = \int_\Omega (\delta \underset{\sim}{u}_{,i} \otimes \underset{\sim}{e}_i) : (\nu \underset{\sim}{u}_{,j} \otimes \underset{\sim}{e}_j) \, d\Omega =$$

$$= \int_\Omega (\delta u_{i,j} \underset{\sim}{e}_i \otimes \underset{\sim}{e}_j) : (\nu u_{k,l} \underset{\sim}{e}_k \otimes \underset{\sim}{e}_l) \, d\Omega = \int_\Omega \delta u_{i,j} \nu u_{i,j} \, d\Omega =$$

$$= \int_\Omega (\delta \underset{\sim}{u}_{,j} \cdot \underset{\sim}{e}_i) \nu (\underset{\sim}{u}_{,j} \cdot \underset{\sim}{e}_i) \, d\Omega = \int_\Omega (\delta \underset{\sim}{u}_{,j}^T \underset{\sim}{e}_i) \nu (\underset{\sim}{e}_i^T \underset{\sim}{u}_{,j}) \, d\Omega =$$

$$= \delta \hat{\underset{\sim}{u}}^T \int_\Omega (\underline{\Psi}^{uT}_{,j} \underset{\sim}{e}_i) \nu (\underset{\sim}{e}_i^T \underline{\Psi}^u_{,j}) \, d\Omega \, \hat{\underset{\sim}{u}} = \delta \hat{\underset{\sim}{u}}^T \underline{K} \hat{\underset{\sim}{u}} \quad (29)$$

$$\Delta \int_\Omega \underset{\sim}{\nabla} \delta \underset{\sim}{u} : \nu \underset{\sim}{\nabla} \underset{\sim}{u} \, d\Omega = \delta \hat{\underset{\sim}{u}}^T \underline{K} \Delta \hat{\underset{\sim}{u}}$$ onde $\underline{K}$ é a matriz viscosa (30)

Simplificação da matriz viscosa

No caso do termo $\int_\Omega \underset{\sim}{\nabla}\, \delta \underset{\sim}{u} : \nu\, \underset{\sim}{\nabla}\, \underset{\sim}{u}\, d\Omega$ tem-se

$$\int_\Omega \underset{\sim}{\nabla}\, \delta \underset{\sim}{u} : \nu\, \underset{\sim}{\nabla}\, \underset{\sim}{u}\, d\Omega = \int_\Omega \nu\, \delta \underset{\sim}{u}_{,j} \cdot \underset{\sim}{e}_i \left(\underset{\sim}{e}_i \cdot \underset{\sim}{u}_{,j} \right) d\Omega = \int_\Omega \nu\, \delta \underset{\sim}{u}_{,j} \cdot \underbrace{\left(\underset{\sim}{e}_i \otimes \underset{\sim}{e}_i \right)}_{= \underline{I}} \underset{\sim}{u}_{,j}\, d\Omega =$$

$$= \int_\Omega \nu\, \delta \underset{\sim}{u}_{,j} \cdot \delta \underset{\sim}{u}_{,j}\, d\Omega = \delta \hat{\underset{\sim}{u}}^T \int_\Omega \nu\, \underline{\Psi}^{u^T}_{,j}\, \underline{\Psi}^{u}_{,j}\, d\Omega\, \hat{\underset{\sim}{u}} =$$

$$= \delta \hat{\underset{\sim}{u}}^T \int_\Omega \nu \left(\underline{\Psi}^{u^T}_{,1}\, \underline{\Psi}^{u}_{,1} + \underline{\Psi}^{u^T}_{,2}\, \underline{\Psi}^{u}_{,2} + \underline{\Psi}^{u^T}_{,3}\, \underline{\Psi}^{u}_{,3} \right) d\Omega\, \hat{\underset{\sim}{u}} =$$

$$= \delta \hat{\underset{\sim}{u}}^T \int_\Omega \nu \begin{bmatrix} \underline{\Psi}^{u^T}_{,1} & \underline{\Psi}^{u^T}_{,2} & \underline{\Psi}^{u^T}_{,3} \end{bmatrix} \begin{bmatrix} \underline{\Psi}^{u}_{,1} \\ \underline{\Psi}^{u}_{,2} \\ \underline{\Psi}^{u}_{,3} \end{bmatrix} d\Omega\, \hat{\underset{\sim}{u}} = \delta \hat{\underset{\sim}{u}}^T \int_\Omega \underline{B}^T \nu\, \underline{B}\, d\Omega\, \hat{\underset{\sim}{u}} = \delta \hat{\underset{\sim}{u}}^T\, \underline{K}\, \hat{\underset{\sim}{u}}$$

onde $\underset{(6 \times 3n)}{\underline{B}} = \begin{bmatrix} \underline{\Psi}^{u}_{,1} \\ \underline{\Psi}^{u}_{,2} \\ \underline{\Psi}^{u}_{,3} \end{bmatrix}$

Esta matriz coincide com a apresentada por Campello na pag. 20 ou a fórmula (6.23) de Huerta.

Nota: uma forma alternativa de simplificar este termo é usar

$$\int_\Omega \underset{\sim}{\nabla}\, \delta \underset{\sim}{u} : \nu\, \underset{\sim}{\nabla}\, \underset{\sim}{u}\, d\Omega = \int_\Omega \nu \left(\delta \underset{\sim}{u}_{,j} \cdot \underset{\sim}{e}_i \right) \left(\underset{\sim}{u}_{,j}\, \underset{\sim}{e}_i \right) d\Omega =$$

$$= \int_\Omega \nu\, \delta u_{i,j}\, u_{i,j}\, d\Omega = \int_\Omega \nu\, \delta \underset{\sim}{u}_{,j}\, \underset{\sim}{u}_{,j}\, d\Omega$$

$$\boxed{\text{Termo } \int_\Omega \nabla^s \delta \boldsymbol{u} : (2\nu) \nabla^s \boldsymbol{u} \, d\Omega}$$

$$\int_\Omega \nabla^s \delta \boldsymbol{u} : (2\nu) \nabla^s \boldsymbol{u} \, d\Omega = \int_\Omega 2\nu \, \frac{\delta u_{i,j} + \delta u_{j,i}}{2} \; \frac{u_{i,j} + u_{j,i}}{2} \, d\Omega =$$

$$= \frac{1}{4} \int_\Omega 2\nu \left[\delta u_{i,j} u_{i,j} + \delta u_{i,j} u_{j,i} + \delta u_{j,i} u_{i,j} + \delta u_{j,i} u_{j,i} \right] d\Omega =$$

$$= \int_\Omega \nu \left(\delta u_{i,j} u_{i,j} + \delta u_{i,j} u_{j,i} \right) d\Omega =$$

$$= \delta \hat{\boldsymbol{u}}^T \int_\Omega \nu \left[\left(\underline{\Psi}^{uT}_{,j} \boldsymbol{e}_i \right) \left(\boldsymbol{e}_i^T \underline{\Psi}^{u}_{,j} \right) + \left(\underline{\Psi}^{uT}_{,j} \boldsymbol{e}_i \right) \left(\boldsymbol{e}_j^T \underline{\Psi}^{u}_{,i} \right) \right] d\Omega \, \hat{\boldsymbol{u}} =$$

$$= \delta \hat{\boldsymbol{u}}^T \underline{K} \, \hat{\boldsymbol{u}} \qquad (31)$$

$$\Delta \int_\Omega \nabla^s \delta \boldsymbol{u} : \nabla^s \boldsymbol{u} \, d\Omega = \delta \hat{\boldsymbol{u}}^T \underline{K} \, \Delta \hat{\boldsymbol{u}} \qquad (32)$$

$$\boxed{\text{Termo } -\int_\Omega \operatorname{div} \delta \boldsymbol{u} \, p \, d\Omega}$$

$$-\int_\Omega \operatorname{div} \delta \boldsymbol{u} \, p \, d\Omega = -\int_\Omega \delta u_{i,i} \, p \, d\Omega = -\int_\Omega \left(\delta \boldsymbol{u}_{,i} \cdot \boldsymbol{e}_i \right) p \, d\Omega = -\int_\Omega \left(\delta \boldsymbol{u}_{,i}^T \boldsymbol{e}_i \right) p \, d\Omega =$$

$$= \delta \hat{\boldsymbol{u}}^T \left(-\int_\Omega \left(\underline{\Psi}^{uT}_{,i} \boldsymbol{e}_i \right) \underline{\Psi}^{p} \, d\Omega \right) \hat{\boldsymbol{p}} = \delta \hat{\boldsymbol{u}}^T \underline{G} \, \hat{\boldsymbol{p}} \qquad (33)$$

$$\Delta \left(-\int_\Omega \operatorname{div} \delta \boldsymbol{u} \, p \, d\Omega \right) = \delta \hat{\boldsymbol{u}}^T \underline{G} \, \Delta \hat{\boldsymbol{p}} \qquad (34)$$

Simplificação da matriz viscosa

No caso do termo $\int_\Omega \underset{\sim}{\nabla}^s \delta \underset{\sim}{u} : (2\nu)\, \underset{\sim}{\nabla}^s \underset{\sim}{u}\, d\Omega$ tem-se

$$\int_\Omega \underset{\sim}{\nabla}^s \delta \underset{\sim}{u} : (2\nu)\, \underset{\sim}{\nabla}^s \underset{\sim}{u}\, d\Omega = \int_\Omega \nu \left(\delta u_{i,j}\, u_{i,j} + \delta u_{i,j}\, u_{j,i} \right) d\Omega =$$

$$= \int_\Omega \nu \left(\delta \underset{\sim}{u}_{,j} \cdot (\underset{\sim}{e}_i \otimes \underset{\sim}{e}_i)\, \underset{\sim}{u}_{,j} + \delta \underset{\sim}{u}_{,j} \cdot (\underset{\sim}{e}_i \otimes \underset{\sim}{e}_j)\, \underset{\sim}{u}_{,i} \right) d\Omega =$$

$$= \delta \hat{\underset{\sim}{u}}^T \int_\Omega \nu \Big(2 \left(\underline{\psi}^{uT}_{,1}\, \underset{\sim}{e}_1 \underset{\sim}{e}_1^T\, \underline{\psi}^u_{,1} + \underline{\psi}^{uT}_{,2}\, \underset{\sim}{e}_2 \underset{\sim}{e}_2^T\, \underline{\psi}^u_{,2} \right) + \underline{\psi}^{uT}_{,1}\, \underset{\sim}{e}_2 \underset{\sim}{e}_2^T\, \underline{\psi}^u_{,1} + \underline{\psi}^{uT}_{,2}\, \underset{\sim}{e}_1 \underset{\sim}{e}_1^T\, \underline{\psi}^u_{,2} +$$

$$+ \underline{\psi}^{uT}_{,2}\, \underset{\sim}{e}_1 \underset{\sim}{e}_2^T\, \underline{\psi}^u_{,1} + \underline{\psi}^{uT}_{,1}\, \underset{\sim}{e}_2 \underset{\sim}{e}_1^T\, \underline{\psi}^u_{,2} \Big) d\Omega\, \hat{\underset{\sim}{u}} =$$

$$= \delta \hat{\underset{\sim}{u}}^T \int_\Omega \nu \Big(2 \left(\underline{\psi}^{uT}_{,1}\, \underset{\sim}{e}_1 \underset{\sim}{e}_1^T\, \underline{\psi}^u_{,1} + \underline{\psi}^{uT}_{,2}\, \underset{\sim}{e}_2 \underset{\sim}{e}_2^T\, \underline{\psi}^u_{,2} \right) + \left(\underline{\psi}^{uT}_{,1}\, \underset{\sim}{e}_2 + \underline{\psi}^{uT}_{,2}\, \underset{\sim}{e}_1 \right) \underset{\sim}{e}_2^T\, \underline{\psi}^u_{,1} +$$

$$+ \left(\underline{\psi}^{uT}_{,2}\, \underset{\sim}{e}_1 + \underline{\psi}^{uT}_{,1}\, \underset{\sim}{e}_2 \right) \underset{\sim}{e}_1^T\, \underline{\psi}^u_{,2} \Big) d\Omega\, \hat{\underset{\sim}{u}} =$$

$$= \delta \hat{\underset{\sim}{u}}^T \int_\Omega \nu \Big(2 \left(\underline{\psi}^{uT}_{,1}\, \underset{\sim}{e}_1 \underset{\sim}{e}_1^T\, \underline{\psi}^u_{,1} + \underline{\psi}^{uT}_{,2}\, \underset{\sim}{e}_2 \underset{\sim}{e}_2^T\, \underline{\psi}^u_{,2} \right) +$$

$$+ \left(\underline{\psi}^{uT}_{,1}\, \underset{\sim}{e}_2 + \underline{\psi}^{uT}_{,2}\, \underset{\sim}{e}_1 \right) \left(\underset{\sim}{e}_1^T\, \underline{\psi}^u_{,2} + \underset{\sim}{e}_2^T\, \underline{\psi}^u_{,1} \right) \Big) d\Omega\, \hat{\underset{\sim}{u}} =$$

$$= \delta \hat{\underset{\sim}{u}}^T \int_\Omega \nu \left[\underline{\psi}^{uT}_{,1}\, \underset{\sim}{e}_1 \;\vdots\; \underline{\psi}^{uT}_{,2}\, \underset{\sim}{e}_2 \;\vdots\; \underline{\psi}^{uT}_{,1}\, \underset{\sim}{e}_2 + \underline{\psi}^{uT}_{,2}\, \underset{\sim}{e}_1 \right]^T \begin{bmatrix} 2 & \cdot & \cdot \\ \cdot & 2 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix} \begin{bmatrix} \underset{\sim}{e}_1^T\, \underline{\psi}^u_{,1} \\ \underset{\sim}{e}_2^T\, \underline{\psi}^u_{,2} \\ \underset{\sim}{e}_1^T\, \underline{\psi}^u_{,2} + \underset{\sim}{e}_2^T\, \underline{\psi}^u_{,1} \end{bmatrix} d\Omega\, \hat{\underset{\sim}{u}}$$

$$= \delta \hat{\underset{\sim}{u}}^T \int_\Omega \underline{B}_u^T\, \underline{C}_\nu\, \underline{B}_u\, d\Omega \quad \text{onde}$$

$$\underline{C}_\nu = \nu \begin{bmatrix} 2 & \cdot & \cdot \\ \cdot & 2 & \cdot \\ \cdot & \cdot & 1 \end{bmatrix}$$

e

$$\begin{array}{c} \begin{array}{cccccc} \text{nó 1} & & \text{nó 2} & & \text{nó 3} & \\ u_1 & u_2 & u_1 & u_2 & u_1 & u_2 \end{array} \\ \underline{B}_u = \left[\begin{array}{cc|cc|cc|c} \psi^u_{1,1} & \cdot & \psi^u_{2,1} & \cdot & \psi^u_{3,1} & \cdot & \\ \cdot & \psi^u_{1,2} & \cdot & \psi^u_{2,2} & \cdot & \psi^u_{3,2} & \cdots \\ \psi^u_{1,2} & \psi^u_{1,1} & \psi^u_{2,2} & \psi^u_{2,1} & \psi^u_{3,2} & \psi^u_{3,1} & \end{array} \right] \end{array}$$

Notar que i e j variam sempre de 1 a 3. Por simplicidade, aqui apenas se obteve a forma final para o caso 2D.

$$\boxed{\text{Termo } -\int_\Omega \delta p \; \mathrm{div}\, \underset{\sim}{u} \, d\Omega}$$

$$-\int_\Omega \delta p \; \mathrm{div}\, \underset{\sim}{u} \, d\Omega = -\int_\Omega \delta p \left(\underset{\sim}{e}_i \cdot \underset{\sim}{u}_{,i} \right) d\Omega = \delta \hat{\underline{p}}^T \left(-\int_\Omega \underline{\psi}^{pT} \left(\underset{\sim}{e}_i^T \, \underline{\psi}^{u}_{,i} \right) d\Omega \right) \hat{\underset{\sim}{u}} =$$

$$= \delta \hat{\underline{p}}^T \, \underline{G}^T \, \hat{\underset{\sim}{u}} \qquad (35)$$

$$\Delta \left(-\int_\Omega \delta p \; \mathrm{div}\, \underset{\sim}{u} \, d\Omega \right) = \delta \hat{p}^T \, \underline{G}^T \Delta \hat{\underset{\sim}{u}} \qquad (36)$$

$$\boxed{\text{Termo } \int_\Omega \delta \underset{\sim}{u} \cdot \bar{\underset{\sim}{b}} \, d\Omega}$$

$$\int_\Omega \delta \underset{\sim}{u} \cdot \bar{\underset{\sim}{b}} \, d\Omega = \int_\Omega \delta \underset{\sim}{u}^T \, \bar{\underset{\sim}{b}} \, d\Omega = \delta \hat{\underset{\sim}{u}}^T \int_\Omega \underline{\psi}^{uT} \, \bar{\underset{\sim}{b}} \, d\Omega = \delta \hat{\underset{\sim}{u}}^T \underline{f}^\Omega$$

$$\Delta \int_\Omega \delta \underset{\sim}{u} \cdot \hat{\underset{\sim}{b}} \, d\Omega = 0$$ (assume-se que $\underset{\sim}{b}$ não é função de $\underset{\sim}{u}$ e p)

$$\boxed{\text{Termo } \int_{\Gamma_t} \delta \underset{\sim}{u} \cdot \bar{\underset{\sim}{t}} \, d\Gamma_t}$$

$$\int_{\Gamma_t} \delta \underset{\sim}{u} \cdot \bar{\underset{\sim}{t}} \, d\Gamma_t = \int_{\Gamma_t} \delta \underset{\sim}{u}^T \, \bar{\underset{\sim}{t}} \, d\Gamma_t = \delta \hat{\underset{\sim}{u}}^T \int_{\Gamma_t} \underline{\psi}^{uT} \, \bar{\underset{\sim}{t}} \, d\Gamma_t = \delta \hat{\underset{\sim}{u}}^T \underline{f}^\Gamma$$

$$\Delta \int_{\Gamma_t} \delta \underset{\sim}{u} \cdot \bar{\underset{\sim}{t}} \, d\Gamma_t = 0$$ (assume-se que $\bar{\underset{\sim}{t}}$ não é função de $\underset{\sim}{u}$ e p)

Inclusão do termo de estabilização da convecção

Às formas fracas expressas através de (15) e (20) é adicionado o termo:

$$\tau_{SUPG} \int_{\Omega} (\nabla \delta \mathbf{u})\, \mathbf{u} \cdot \mathbf{r}^{LM}\, d\Omega \qquad (37)$$

onde o resíduo da equação do momento linear (Linear Momentum), dividida por ρ, é dado por

$$\mathbf{r}^{LM} = \dot{\mathbf{u}} + (\nabla \mathbf{u})\,\mathbf{u} - \nu \nabla^2 \mathbf{u} + \nabla p - \mathbf{b} \qquad (38)$$

e

$$\tau_{SUPG} = \left(\frac{1}{\tau_{S1}^r} + \frac{1}{\tau_{S2}^r} + \frac{1}{\tau_{S3}^r} \right)^{-\frac{1}{r}} \qquad (39)$$

$$\tau_{S1} = \frac{\|\underline{C}\|}{\|\underline{C}_\tau\|} \; ; \quad \tau_{S2} = \frac{\Delta t}{2} \frac{\|\underline{C}\|}{\|\underline{M}_\tau\|} \; ; \quad \tau_{S3} = \tau_{S1} R_e \; , \quad R_e = \frac{\|\mathbf{u}\|^2}{\nu} \frac{\|\underline{C}\|}{\|\underline{C}_\tau\|}$$

$$\|\cdot\| = \sqrt{tr[(\cdot)(\cdot)^T]} \; , \quad r = 2$$

A origem da matriz $\underline{C}$ é $\int_{\Omega} \delta \mathbf{u} \cdot (\nabla \mathbf{u})\,\mathbf{u}\, d\Omega$ e foi definida de duas formas alternativas na página 9.

A origem da matriz $\underline{C}_\tau$ é $\int_{\Omega} (\nabla \delta \mathbf{u})\,\mathbf{u} \cdot (\nabla \mathbf{u})\,\mathbf{u}\, d\Omega$ e pode ser obtida a partir de:

$$\int_{\Omega} (\nabla \delta \mathbf{u})\,\mathbf{u} \cdot (\nabla \mathbf{u})\,\mathbf{u}\, d\Omega = \int_{\Omega} (\delta \mathbf{u}_{,i} \otimes \mathbf{e}_i)\,\mathbf{u} \cdot (\mathbf{u}_{,j} \otimes \mathbf{e}_j)\,\mathbf{u}\, d\Omega =$$

$$= \int_\Omega \delta \mathbf{u}_{,i} (\mathbf{u}\cdot \mathbf{e}_i) \cdot \mathbf{u}_{,j}(\mathbf{u}\cdot\mathbf{e}_j)\, d\Omega =$$

$$= \int_\Omega \delta\mathbf{u}_{,i}^T (\mathbf{u}^T \mathbf{e}_i)\, \mathbf{u}_{,j}(\mathbf{u}^T\mathbf{e}_j)\, d\Omega =$$

$$= \delta\hat{\mathbf{u}}^T \underbrace{\int_\Omega \underline{\Psi}^{u\,T}_{,i}\left((\underline{\Psi}^u\hat{\mathbf{u}})^T\mathbf{e}_i\right)\left((\underline{\Psi}^u\hat{\mathbf{u}})^T\mathbf{e}_j\right)\underline{\Psi}^u_{,j}\, d\Omega}_{\underline{C}_\tau}\,\hat{\mathbf{u}} = \delta\hat{\mathbf{u}}^T\underline{C}_\tau\hat{\mathbf{u}} \qquad (40)$$

A origem de $\underline{M}_\tau$ é $\int_\Omega (\nabla\delta\mathbf{u})\mathbf{u}\cdot\dot{\mathbf{u}}\, d\Omega$ e pode ser obtida a partir de:

$$\int_\Omega (\nabla\delta\mathbf{u})\mathbf{u}\cdot\dot{\mathbf{u}}\, d\Omega = \int_\Omega (\delta\mathbf{u}_{,i}\otimes\mathbf{e}_i)\mathbf{u}\cdot\dot{\mathbf{u}}\, d\Omega = \int_\Omega \delta\mathbf{u}_{,i}(\mathbf{u}\cdot\mathbf{e}_i)\cdot\dot{\mathbf{u}}\, d\Omega =$$

$$= \int_\Omega \delta\mathbf{u}_{,i}^T(\mathbf{u}^T\mathbf{e}_i)\,\dot{\mathbf{u}}\, d\Omega = \delta\hat{\mathbf{u}}^T \underbrace{\int_\Omega \underline{\Psi}^{u\,T}_{,i}\left((\underline{\Psi}^u\hat{\mathbf{u}})^T\mathbf{e}_i\right)\underline{\Psi}^u\, d\Omega}_{\underline{M}_\tau}\,\dot{\hat{\mathbf{u}}} = \delta\hat{\mathbf{u}}^T\underline{M}_\tau\dot{\hat{\mathbf{u}}} \qquad (41)$$

Esta Matriz é simétrica da matriz $\underline{C}_\tau$ obtida para a "versão Huerta", ver remark 12 no paper de Tezduyar e Osawa.

Em alternativa, substituindo $(\nabla\delta\mathbf{u})\mathbf{u}$ por $(\nabla\mathbf{u})\delta\mathbf{u}$, pode escrever-se

$$\int_\Omega (\nabla\mathbf{u})\delta\mathbf{u}\cdot\dot{\mathbf{u}}\, d\Omega = \int_\Omega (\mathbf{u}_{,i}\otimes\mathbf{e}_i)\delta\mathbf{u}\cdot\dot{\mathbf{u}}\, d\Omega = \int_\Omega \left[(\mathbf{u}_{,i}\,\mathbf{e}_i^T)\delta\mathbf{u}\right]^T\dot{\mathbf{u}}\, d\Omega =$$

$$= \int_\Omega \delta\mathbf{u}^T(\mathbf{u}_{,i}\,\mathbf{e}_i^T)^T\,\dot{\mathbf{u}}\, d\Omega = \int_\Omega \delta\mathbf{u}^T(\mathbf{e}_i\,\mathbf{u}_{,i}^T)\,\dot{\mathbf{u}}\, d\Omega =$$

$$= \delta\hat{\mathbf{u}}^T\int_\Omega \underline{\Psi}^{u\,T}\left(\mathbf{e}_i(\underline{\Psi}^u_{,i}\hat{\mathbf{u}})^T\right)\underline{\Psi}^u\, d\Omega\,\dot{\hat{\mathbf{u}}} = \delta\hat{\mathbf{u}}^T\underline{M}_\tau\dot{\hat{\mathbf{u}}}, \qquad (42)$$

obtendo-se a matriz simétrica de $\underline{C}$ da "versão Campelo".

A avaliação de $\underset{\sim}{r}^{LM}$ pode ser efectuada através de

$$\underset{\sim}{r}^{LM} = \dot{\underset{\sim}{u}} + (\underset{\sim}{\nabla}\underset{\sim}{u})\underset{\sim}{u} - \nu\,\nabla^2\underset{\sim}{u} + \underset{\sim}{\nabla}p - \underset{\sim}{b} =$$

$$= \dot{\underset{\sim}{u}} + (\underset{\sim}{u}_{,i}\otimes\underset{\sim}{e}_i)\underset{\sim}{u} - \nu\,\underset{\sim}{u}_{,ii} + p_{,i}\underset{\sim}{e}_i - \underset{\sim}{b} =$$

$$= \dot{\underset{\sim}{u}} + \underset{\sim}{u}_{,i}(\underset{\sim}{e}_i\cdot\underset{\sim}{u}) - \nu\,\underset{\sim}{u}_{,ii} + p_{,i}\underset{\sim}{e}_i - \underset{\sim}{b} = \qquad (43)$$

$$= \underline{\Psi}^u\dot{\hat{\underset{\sim}{u}}} + \underline{\Psi}^u_{,i}\hat{\underset{\sim}{u}}(\underset{\sim}{e}_i^T\underline{\Psi}^u\hat{\underset{\sim}{u}}) - \nu\,\underline{\Psi}^u_{,ii}\hat{\underset{\sim}{u}} + \underset{\sim}{e}_i\underline{\Psi}^p_{,i}\hat{\underset{\sim}{p}} - \underset{\sim}{b}$$

Note-se que o termo $\underset{\sim}{u}_{,ii}$ envolve segundas derivadas de $\underset{\sim}{u}$ em relação aos eixos globais.

Assim, depois de avaliado $\underset{\sim}{r}^{LM}$, tem-se

$$\tau_{SUPG}\int_\Omega(\underset{\sim}{\nabla}\delta\underset{\sim}{u})\underset{\sim}{u}\cdot\underset{\sim}{r}^{LM}\,d\Omega = \tau_{SUPG}\int_\Omega(\delta\underset{\sim}{u}_{,i}\otimes\underset{\sim}{e}_i)\underset{\sim}{u}\cdot\underset{\sim}{r}^{LM}\,d\Omega =$$

$$= \tau_{SUPG}\int_\Omega\delta\underset{\sim}{u}_{,i}(\underset{\sim}{e}_i\cdot\underset{\sim}{u})\cdot\underset{\sim}{r}^{LM}\,d\Omega = \tau_{SUPG}\int_\Omega\delta\underset{\sim}{u}_{,i}^T(\underset{\sim}{e}_i^T\underset{\sim}{u})\underset{\sim}{r}^{LM}\,d\Omega =$$

$$= \tau_{SUPG}\,\delta\hat{\underset{\sim}{u}}^T\int_\Omega\underline{\Psi}^{uT}_{,i}(\underset{\sim}{e}_i^T\underline{\Psi}^u\hat{\underset{\sim}{u}})\underset{\sim}{r}^{LM}\,d\Omega = \delta\hat{\underset{\sim}{u}}^T\tau_{SUPG}\underset{\sim}{r}^{SUPG} \qquad (44)$$

Assumindo que τ_{SUPG} é constante, a linearização do termo associado é dada por

$$\Delta\left(\tau_{SUPG}\int_\Omega(\underset{\sim}{\nabla}\delta\underset{\sim}{u})\underset{\sim}{u}\cdot\underset{\sim}{r}^{LM}\,d\Omega\right) = \tau_{SUPG}\int_\Omega\left[\Delta\left((\underset{\sim}{\nabla}\delta\underset{\sim}{u})\underset{\sim}{u}\right)\cdot\underset{\sim}{r}^{LM} + (\underset{\sim}{\nabla}\delta\underset{\sim}{u})\underset{\sim}{u}\cdot\Delta\underset{\sim}{r}^{LM}\right]d\Omega =$$

$$= \tau_{SUPG}\int_\Omega\left[(\underset{\sim}{\nabla}\delta\underset{\sim}{u})\Delta\underset{\sim}{u}\cdot\underset{\sim}{r}^{LM} + (\underset{\sim}{\nabla}\delta\underset{\sim}{u})\underset{\sim}{u}\cdot\Delta\underset{\sim}{r}^{LM}\right]d\Omega \qquad (45)$$

Mas,

$$\underset{\sim}{\Delta r}^{LM} = \Delta\left(\dot{\underset{\sim}{u}} + (\underset{\sim}{\nabla}\underset{\sim}{u})\underset{\sim}{u} - \nu\,\nabla^2\underset{\sim}{u} + \underset{\sim}{\nabla} p - \underset{\sim}{b}\right) =$$

(assume-se que $\Delta \underset{\sim}{b} = 0$)

$$= \Delta\dot{\underset{\sim}{u}} + (\underset{\sim}{\nabla}\Delta\underset{\sim}{u})\underset{\sim}{u} + (\underset{\sim}{\nabla}\underset{\sim}{u})\Delta\underset{\sim}{u} - \nu\,\nabla^2\Delta\underset{\sim}{u} + \underset{\sim}{\nabla}\Delta p - \Delta\underset{\sim}{b} =$$

$$= \Delta\dot{\underset{\sim}{u}} + \Delta\underset{\sim}{u}_{,i}(\underset{\sim}{e}_i\cdot\underset{\sim}{u}) + (\underset{\sim}{u}_{,i}\otimes\underset{\sim}{e}_i)\Delta\underset{\sim}{u} - \nu\,\Delta\underset{\sim}{u}_{,ii} + \Delta p_{,i}\,\underset{\sim}{e}_i \tag{46}$$

e

$$(\underset{\sim}{\nabla}\delta\underset{\sim}{u})\Delta\underset{\sim}{u}\cdot\underset{\sim}{r}^{LM} = (\delta\underset{\sim}{u}_{,i}\otimes\underset{\sim}{e}_i)\Delta\underset{\sim}{u}\cdot\underset{\sim}{r}^{LM} = \delta\underset{\sim}{u}_{,i}(\Delta\underset{\sim}{u}\cdot\underset{\sim}{e}_i)\cdot\underset{\sim}{r}^{LM} =$$

$$= \delta\underset{\sim}{u}_{,i}\cdot\underset{\sim}{r}^{LM}(\Delta\underset{\sim}{u}\cdot\underset{\sim}{e}_i) = \delta\underset{\sim}{u}_{,i}\cdot(\underset{\sim}{r}^{LM}\otimes\underset{\sim}{e}_i)\Delta\underset{\sim}{u} \tag{47}$$

$$(\underset{\sim}{\nabla}\delta\underset{\sim}{u})\underset{\sim}{u}\cdot\Delta\underset{\sim}{r}^{LM} = \delta\underset{\sim}{u}_{,i}(\underset{\sim}{u}\cdot\underset{\sim}{e}_i)\cdot\Delta\underset{\sim}{r}^{LM} \tag{48}$$

logo, a expressão (45) assume a forma $\Delta\left(\tau_{SUPG}\int_\Omega(\underset{\sim}{\nabla}\delta\underset{\sim}{u})\underset{\sim}{u}\cdot\underset{\sim}{r}^{LM}\,d\Omega\right) =$

$$= \tau_{SUPG}\int_\Omega\left[\delta\underset{\sim}{u}_{,i}\cdot(\underset{\sim}{r}^{LM}\otimes\underset{\sim}{e}_i)\Delta\underset{\sim}{u} + \delta\underset{\sim}{u}_{,i}(\underset{\sim}{u}\cdot\underset{\sim}{e}_i)\cdot\Delta\underset{\sim}{r}^{LM}\right]d\Omega =$$

$$= \tau_{SUPG}\int_\Omega\Big[\delta\underset{\sim}{u}_{,i}\cdot(\underset{\sim}{r}^{LM}\otimes\underset{\sim}{e}_i)\Delta\underset{\sim}{u} + \delta\underset{\sim}{u}_{,i}(\underset{\sim}{u}\cdot\underset{\sim}{e}_i)\cdot\Big(\Delta\dot{\underset{\sim}{u}} + \Delta\underset{\sim}{u}_{,j}(\underset{\sim}{e}_j\cdot\underset{\sim}{u}) + (\underset{\sim}{u}_{,j}\otimes\underset{\sim}{e}_j)\Delta\underset{\sim}{u} -$$
$$- \nu\,\Delta\underset{\sim}{u}_{,jj} + \Delta p_{,j}\,\underset{\sim}{e}_j\Big)\Big]d\Omega =$$

$$= \tau_{SUPG}\Big\{\delta\hat{\underset{\sim}{u}}^T\int_\Omega \underline{\Psi}^{u\,T}_{,i}((\underline{\Psi}^u\hat{\underset{\sim}{u}})^T\underset{\sim}{e}_i)\,\underline{\Psi}^u\,d\Omega\,\Delta\dot{\hat{\underset{\sim}{u}}} + \delta\hat{\underset{\sim}{u}}^T\int_\Omega\Big[\underline{\Psi}^{u\,T}_{,i}(\underset{\sim}{r}^{LM}\,\underset{\sim}{e}_i^T)\,\underline{\Psi}^u +$$
$$+ \underline{\Psi}^{u\,T}_{,i}((\underline{\Psi}^u\hat{\underset{\sim}{u}})^T\underset{\sim}{e}_i)\Big[\left(\underset{\sim}{e}_j^T(\underline{\Psi}^u\hat{\underset{\sim}{u}})\right)\underline{\Psi}^u_{,j} + (\underline{\Psi}^u_{,j}\hat{\underset{\sim}{u}})\,\underset{\sim}{e}_j^T\,\underline{\Psi}^u - \nu\,\underline{\Psi}^u_{,jj}\Big]\Big]d\Omega\,\Delta\hat{\underset{\sim}{u}} +$$
$$+ \delta\hat{\underset{\sim}{u}}^T\int_\Omega\underline{\Psi}^{u\,T}_{,i}((\underline{\Psi}^u\hat{\underset{\sim}{u}})^T\underset{\sim}{e}_i)\,\underbrace{(\underset{\sim}{e}_j\,\underline{\Psi}^p_{,j})}_{\text{(}\ne\text{ de Campelo.)}}\,d\Omega\,\Delta\hat{\underset{\sim}{p}}\Big\} \tag{49}$$

$$= \tau_{SUPG} \left\{ \delta \hat{\underset{\sim}{u}}^T \underline{M}^{SUPG} \Delta \dot{\hat{\underset{\sim}{u}}} + \delta \hat{\underset{\sim}{u}}^T \underline{C}^{SUPG} \Delta \hat{\underset{\sim}{u}} + \delta \hat{\underset{\sim}{u}}^T \underline{G}^{SUPG} \Delta \hat{\underset{\sim}{p}} \right\}$$ (17)

onde

$$\underline{M}^{SUPG} = \int_\Omega \underline{\Psi}^{uT}_{,i} \left((\underline{\Psi}^u \hat{\underset{\sim}{u}})^T \underset{\sim}{e}_i \right) \underline{\Psi}^u \, d\Omega$$ (que coincide com $\underline{C}^T_M$)

$$\underline{C}^{SUPG} = \int_\Omega \left[\underline{\Psi}^{uT}_{,i} \left(\underset{\sim}{r}^{LM} \underset{\sim}{e}^T_i \right) \underline{\Psi}^u + \underline{\Psi}^{uT}_{,i} \left((\underline{\Psi}^u \hat{\underset{\sim}{u}})^T \underset{\sim}{e}_i \right) \left[\left(\underset{\sim}{e}^T_j (\underline{\Psi}^u \hat{\underset{\sim}{u}}) \right) \underline{\Psi}^u_{,j} + \right.\right.$$

$$\left.\left. + \left(\underline{\Psi}^u_{,j} \hat{\underset{\sim}{u}} \right) \underset{\sim}{e}^T_j \underline{\Psi}^u - \nu \underline{\Psi}^u_{,jj} \right]\right] d\Omega$$

$$\underline{G}^{SUPG} = \int_\Omega \underline{\Psi}^{uT}_{,i} \left((\underline{\Psi}^u \hat{\underset{\sim}{u}})^T \underset{\sim}{e}_i \right) \left(\underset{\sim}{e}_j \underline{\Psi}^p_{,j} \right) d\Omega$$

Forma final em termos de pseudo-tensão

Forma fraca resultante de (15) ⊕ (37):

$$\int_\Omega \delta\boldsymbol{u}\cdot\dot{\boldsymbol{u}}\,d\Omega+\int_\Omega \delta\boldsymbol{u}\cdot(\nabla\boldsymbol{u})\boldsymbol{u}\,d\Omega+\int_\Omega \nabla\delta\boldsymbol{u}:\nu\nabla\boldsymbol{u}\,d\Omega-\int_\Omega \mathrm{div}\,\delta\boldsymbol{u}\,p\,d\Omega-$$

$$-\int_\Omega \delta p\,\mathrm{div}\,\boldsymbol{u}\,d\Omega-\int_{\Gamma_t}\delta\boldsymbol{u}\cdot\bar{\boldsymbol{t}}\,d\Gamma_t-\int_\Omega \delta\boldsymbol{u}\cdot\bar{\boldsymbol{b}}\,d\Omega+\tau_{SUPG}\int_\Omega(\nabla\delta\boldsymbol{u})\boldsymbol{u}\cdot\boldsymbol{r}^{LM}\,d\Omega=0 \qquad (50)$$

e $\boldsymbol{r}^{LM}$ é dado por (38).

Após discretização espacial, tem-se

$$\delta\hat{\boldsymbol{u}}^T\left\{\underline{M}\,\dot{\hat{\boldsymbol{u}}}+(\underline{C}+\underline{K})\,\hat{\boldsymbol{u}}+\underline{G}\,\hat{\boldsymbol{p}}-\boldsymbol{f}^\Gamma-\boldsymbol{f}^\Omega+\tau^{SUPG}\boldsymbol{r}^{SUPG}\right\}+\delta\hat{\boldsymbol{p}}^T\,\underline{G}^T\hat{\boldsymbol{u}}=0 \qquad (51)$$

Para valores arbitrários de $\delta\hat{\boldsymbol{u}}$ e $\delta\hat{\boldsymbol{p}}$,

$$\begin{bmatrix}\underline{M} & \underline{0}\\ \underline{0} & \underline{0}\end{bmatrix}\begin{Bmatrix}\dot{\hat{\boldsymbol{u}}}\\ \dot{\hat{\boldsymbol{p}}}\end{Bmatrix}+\begin{bmatrix}\underline{C}(\hat{\boldsymbol{u}})+\underline{K} & \underline{G}\\ \underline{G}^T & \underline{0}\end{bmatrix}\begin{Bmatrix}\hat{\boldsymbol{u}}\\ \hat{\boldsymbol{p}}\end{Bmatrix}+\begin{Bmatrix}\tau^{SUPG}\boldsymbol{r}^{SUPG}(\hat{\boldsymbol{u}},\hat{\boldsymbol{p}})-\boldsymbol{f}^\Omega-\boldsymbol{f}^\Gamma\\ \boldsymbol{0}\end{Bmatrix}=\boldsymbol{0} \qquad (52)$$

onde

$$\underline{M}=\int_\Omega \underline{\Psi}^{uT}\underline{\Psi}^u\,d\Omega \qquad (53.a)$$

$$\underline{C}=\underline{C}_C(\boldsymbol{u})=\int_\Omega \underline{\Psi}^{uT}(\underline{\Psi}^u_{,i}\,\hat{\boldsymbol{u}}\,\boldsymbol{e}_i^T)\underline{\Psi}^u\,d\Omega \text{ ou } \underline{C}=\underline{C}_H(\boldsymbol{u})=\int_\Omega \underline{\Psi}^{uT}(\boldsymbol{e}_i^T\underline{\Psi}^u\hat{\boldsymbol{u}})\underline{\Psi}^u_{,i}\,d\Omega \qquad (53.b)$$

$$\underline{K}=\int_\Omega(\underline{\Psi}^{uT}_{,j}\boldsymbol{e}_i)\nu(\boldsymbol{e}_i^T\underline{\Psi}^u_{,j})\,d\Omega \qquad (53.c)$$

$$\underline{G}=-\int_\Omega(\underline{\Psi}^{uT}_{,i}\boldsymbol{e}_i)\underline{\Psi}^p\,d\Omega \qquad (53.d)$$

$$\underset{\sim}{r}^{SUPG}(\hat{\underset{\sim}{u}},\hat{\underset{\sim}{p}}) = \int_{\Omega} \underline{\Psi}_{,j}^{uT}\,(\underset{\sim}{e}_j^T\,\underline{\Psi}^u\,\hat{\underset{\sim}{u}})\Big[\underline{\Psi}^u\,\dot{\hat{\underset{\sim}{u}}} + (\underline{\Psi}^u_{,i}\,\hat{\underset{\sim}{u}})\,(\underset{\sim}{e}_i^T\,\underline{\Psi}^u\hat{\underset{\sim}{u}}) - \nu\,\underline{\Psi}^u_{,ii}\,\hat{\underset{\sim}{u}} +$$

$$+\ \underset{\sim}{e}_i\,\underline{\Psi}^p_{,i}\,\hat{\underset{\sim}{p}} - \underset{\sim}{b}\Big]\,d\Omega \qquad \text{(onde foi usada (43))} \qquad (53,e)$$

$$\underset{\sim}{f}^{\Omega} = \int_{\Omega} \underline{\Psi}^{uT}\,\bar{\underset{\sim}{b}}\,d\Omega \qquad (53,f)$$

$$\underset{\sim}{f}^{\Gamma} = \int_{\Gamma_t} \underline{\Psi}^{uT}\,\bar{\underset{\sim}{t}}\,d\Gamma_t \qquad (53.g)$$

A linearização do 1º membro de (50) conduz a:

$$\delta\hat{\underset{\sim}{u}}^T\Big\{(\underline{M}+\tau^{SUPG}\underline{M}^{SUPG})\,\Delta\dot{\hat{\underset{\sim}{u}}} + (\underline{K}+\underline{C}_c+\underline{C}_H+\tau^{SUPG}\underline{C}^{SUPG})\,\Delta\hat{\underset{\sim}{u}} + (\underline{G}+\tau^{SUPG}\underline{G}^{SUPG})\,\Delta\hat{\underset{\sim}{p}}\Big\} +$$

$$+\ \delta\hat{\underset{\sim}{p}}^T\,\underline{G}^T\,\hat{\underset{\sim}{u}} \qquad (54)$$

Assim, o operador tangente é dado por

$$\begin{Bmatrix}\delta\hat{\underset{\sim}{u}}\\ \delta\hat{p}\end{Bmatrix}^T\left(\begin{bmatrix}\underline{M}+\tau^{SUPG}\underline{M}^{SUPG} & \underline{0}\\ \underline{0} & \underline{0}\end{bmatrix}\begin{Bmatrix}\Delta\dot{\hat{\underset{\sim}{u}}}\\ \Delta\dot{\hat{\underset{\sim}{p}}}\end{Bmatrix} + \begin{bmatrix}\underline{K}+\underline{C}_c+\underline{C}_H+\tau^{SUPG}\underline{C}^{SUPG} & \underline{G}+\tau^{SUPG}\underline{G}^{SUPG}\\ \underline{G}^T & \underline{0}\end{bmatrix}\begin{Bmatrix}\Delta\hat{\underset{\sim}{u}}\\ \Delta\hat{p}\end{Bmatrix}\right) \qquad (55)$$

onde

$$\underline{M}^{SUPG} = \int_{\Omega} \underline{\Psi}^{uT}_{,i}\,((\underline{\Psi}^u\,\hat{\underset{\sim}{u}})^T\,\underset{\sim}{e}_i)\,\underline{\Psi}^u\,d\Omega = \underline{C}_H^T$$

$$\underline{C}^{SUPG} = \int_{\Omega}\Big[\underline{\Psi}^{uT}_{,i}\,(\underset{\sim}{r}^{cM}\,\underset{\sim}{e}_i^T)\,\underline{\Psi}^u + \underline{\Psi}^{uT}_{,i}\,((\underline{\Psi}^u\,\hat{\underset{\sim}{u}})^T\underset{\sim}{e}_i)\Big[(\underset{\sim}{e}_j^T(\underline{\Psi}^u\,\hat{\underset{\sim}{u}})\,\underline{\Psi}^u_{,j} +$$

$$+\ (\underline{\Psi}^u_{,j}\,\hat{\underset{\sim}{u}})\,\underset{\sim}{e}_j^T\,\underline{\Psi}^u - \nu\,\underline{\Psi}^u_{,jj}\Big]\Big]\,d\Omega$$

$$\underline{G}^{SUPG} = \int_{\Omega} \underline{\Psi}^{uT}_{,i}\,((\underline{\Psi}^{uT}\hat{\underset{\sim}{u}})^T\underset{\sim}{e}_i)\,(\underset{\sim}{e}_j\,\underline{\Psi}^p_{,j})\,d\Omega$$

O termo $\tau_{SUPG}\,\underset{\sim}{r}^{SUPG}$ presente em (52) encerra matrizes que multiplicam por $\dot{\hat{\underset{\sim}{u}}}$, $\hat{\underset{\sim}{u}}$ e $\hat{\underset{\sim}{p}}$, para além de termos constantes.

Assim, o resíduo pode ser escrito de forma a salientar estas contribuições:

$$\left[\begin{array}{c|c} \underline{M}+\tau^{SUPG}\underline{M}^{SUPG} & \underline{0} \\ \hline \underline{0} & \underline{0} \end{array}\right]\left\{\begin{array}{c} \dot{\hat{\underset{\sim}{u}}} \\ \dot{\hat{\underset{\sim}{p}}} \end{array}\right\}+\left[\begin{array}{c|c} \underline{C}(\hat{\underset{\sim}{u}})+\underline{K}+\tau^{SUPG}\underline{H}^{SUPG} & \underline{G}+\tau^{SUPG}\underline{C}^{SUPG} \\ \hline \underline{G}^{T} & \underline{0} \end{array}\right]\left\{\begin{array}{c} \hat{\underset{\sim}{u}} \\ \hat{\underset{\sim}{p}} \end{array}\right\}+$$

$$+\left\{\begin{array}{c} -\underset{\sim}{f}^{\Omega}-\underset{\sim}{f}^{\Gamma}+\tau^{SUPG}\underset{\sim}{f}^{SUPG\Omega} \\ \hline \underset{\sim}{0} \end{array}\right\}=\underset{\sim}{0}$$

onde

$$\underline{H}^{SUPG}=\int_{\Omega}\underline{\Psi}^{uT}_{,j}\left(\underset{\sim}{e}_j^T\,\underline{\Psi}^u\,\hat{\underset{\sim}{u}}\right)\left[\left(\underline{\Psi}^u_{,i}\,\hat{\underset{\sim}{u}}\right)\left(\underset{\sim}{e}_i^T\,\underline{\Psi}^u\right)-\nu\,\underline{\Psi}^u_{,ii}\right]d\Omega$$

$$\underset{\sim}{f}^{\Omega SUPG}=-\int_{\Omega}\underline{\Psi}^{uT}_{,j}\left(\underset{\sim}{e}_j^T\,\underline{\Psi}^u\,\hat{\underset{\sim}{u}}\right)\underset{\sim}{b}\,d\Omega$$

Forma Final em termos de verdadeiras tensões

Na forma fraca (50) o termo $\int_\Omega \underset{\sim}{\nabla} \delta \underset{\sim}{u} : \nu \underset{\sim}{\nabla} \underset{\sim}{u} \, d\Omega$ é substituído por $\int_\Omega \underset{\sim}{\nabla}^s \delta \underset{\sim}{u} : 2\nu \underset{\sim}{\nabla}^s \underset{\sim}{u} \, d\Omega$.

Em (51),(52),(54) e (55) em lugar de se usar $\underline{K}$ dado por (29) usa-se $\underline{\kappa}$ dado por (31).

Forma final após discretização temporal

Substituindo (21) em (52) vem, para o instante $(n+1)$

$$\underline{M}\left(\frac{1}{\gamma\Delta t}\left(\hat{\underset{\sim}{u}}^{n+1} - \hat{\underset{\sim}{u}}^{n}\right) - \frac{1-\gamma}{\gamma}\dot{\hat{\underset{\sim}{u}}}^{n}\right) + \left(\underline{C}(\hat{\underset{\sim}{u}}^{n+1}) + \underline{K}\right)\hat{\underset{\sim}{u}}^{n+1} + \underline{G}\,\hat{\underset{\sim}{p}}^{n+1} +$$

$$+ \tau^{SUPG} \underset{\sim}{r}^{SUPG}(\hat{\underset{\sim}{u}}^{n}, \hat{\underset{\sim}{p}}^{n}) - \underset{\sim}{f}^{\Omega\, n+1} - \underset{\sim}{f}^{\Gamma\, n+1} = \underset{\sim}{0} \iff$$

$$\left(\frac{1}{\gamma\Delta t}\underline{M} + \underline{C}(\hat{\underset{\sim}{u}}^{n+1}) + \underline{K}\right)\hat{\underset{\sim}{u}}^{n+1} + \underline{G}\,\hat{\underset{\sim}{p}}^{n+1} - \underset{\sim}{f}^{\Omega\, n+1} - \underset{\sim}{f}^{\Gamma\, n+1} - \frac{1-\gamma}{\gamma}\underline{M}\dot{\hat{\underset{\sim}{u}}}^{n} + \tau^{SUPG}\underset{\sim}{r}^{SUPG}(\hat{\underset{\sim}{u}}^{n}, \hat{\underset{\sim}{p}}^{n}) -$$
$$- \frac{1}{\gamma\Delta t}\underline{M}\hat{\underset{\sim}{u}}^{n} = 0 \qquad (56)$$

Assumiu-se que $\underset{\sim}{r}^{SUPG}$ é avaliado usando $\hat{\underset{\sim}{u}}$ e $\hat{\underset{\sim}{p}}$ do passo anterior, n.

Usou-se ainda a discretização temporal para cada grau de liberdade

$$\dot{\underset{\sim}{u}}^{n+1} = \frac{1}{\gamma}\frac{\underset{\sim}{u}^{n+1} - \underset{\sim}{u}^{n}}{\Delta t} - \frac{1-\gamma}{\gamma}\dot{\underset{\sim}{u}}^{n} \iff \dot{\hat{\underset{\sim}{u}}}^{n+1} = \frac{1}{\gamma}\frac{\hat{\underset{\sim}{u}}^{n+1} - \hat{\underset{\sim}{u}}^{n}}{\Delta t} - \frac{1-\gamma}{\gamma}\dot{\hat{\underset{\sim}{u}}}^{n} \qquad (57)$$

A forma final do vector resíduo é, então,

$$\left[\begin{array}{c|c} \frac{1}{\gamma\Delta t}\underline{M} + \underline{C}(\mathbf{u}^{n+1}) + \underline{K} & \underline{G} \\ \hline \underline{G}^T & \underline{0} \end{array}\right]\left\{\begin{array}{c} \hat{\mathbf{u}}^{n+1} \\ \hat{\mathbf{p}}^{n+1} \end{array}\right\} + \left\{\begin{array}{c} -\mathbf{f}^{\Omega n+1} - \mathbf{f}^{\Gamma n+1} - \frac{1-\gamma}{\gamma}\underline{M}\dot{\hat{\mathbf{u}}}^n + \tau^{SUPG}\mathbf{r}^{SUPG}(\hat{\mathbf{u}}^n, \hat{\mathbf{p}}^n) \\ - \frac{1}{\gamma\Delta t}\underline{M}\hat{\mathbf{u}}^n \\ \mathbf{0} \end{array}\right\} = \mathbf{0} \quad (58)$$

Note-se que por se tratar de um vector resíduo, deve ser escrito na forma $\mathbf{r} = \mathbf{0}$, tal como em (58) e não na forma (3.28) ou (3.37) de Campelo, pois não se trata de "um sistema de equações algébricas".

A partir de (55) obtém-se a matriz tangente associada a (58).

Neste caso,

$$\Delta\dot{\hat{\mathbf{u}}} = \Delta\left(\frac{1}{\gamma}\frac{\hat{\mathbf{u}}^{n+1} - \hat{\mathbf{u}}^n}{\Delta t} - \frac{1-\gamma}{\gamma}\dot{\hat{\mathbf{u}}}^n\right) = \frac{\Delta\hat{\mathbf{u}}}{\gamma\Delta t}$$

, pois as quantidades associadas ao passo n são constantes. Assim, a matriz tangente é dada por

$$\left[\begin{array}{c|c} \frac{1}{\gamma\Delta t}\left(\underline{M} + \tau^{SUPG}\underline{M}^{SUPG}\right) + \underline{C}_c + \underline{C}_H + \tau^{SUPG}\underline{C}^{SUPG} + \underline{K} & \underline{G} + \tau^{SUPG}\underline{G}^{SUPG} \\ \hline \underline{G}^T & \underline{0} \end{array}\right] \quad (59)$$

Nas expressões (56), (58) e (59) $\underline{K}$ pode ser obtido a partir de (29) ou (31) consoante se trate de pseudo-tensões ou verdadeiras tensões.

O problema acoplado transferência de calor/escoamento de fluidos

A inclusão da hipótese de Boussinesq nas equações de Navier-Stokes

Neste caso, é acrescentado o termo

$$\underset{\sim}{f}_B = -\beta \underset{\sim}{g} (\theta - \theta_0) \qquad (60)$$

onde β é o coeficiente de expansão térmica, θ_0 é uma temperatura de referência e $\underset{\sim}{g} = -g \underset{\sim}{e}_2$ é o vector da aceleração da gravidade.

$\beta \equiv \left[\frac{1}{{}^\circ C}\right] \qquad \theta \equiv [{}^\circ C] \qquad \underset{\sim}{g} \equiv \left[\frac{m}{s^2}\right] \qquad \underset{\sim}{f}_B \equiv \left[\frac{m}{s^2}\right]$

Assim, tem-se $\dot{\underset{\sim}{u}} + (\underset{\sim}{\nabla} \underset{\sim}{u}) \underset{\sim}{u} - \nu \underset{\sim}{\nabla}^2 \underset{\sim}{u} + \underset{\sim}{\nabla} p - \underset{\sim}{b} - \underset{\sim}{f}_B = \underset{\sim}{0}$.

Às expressões da forma fraca (15) e (20) é acrescentado o termo:

$$-\int_\Omega \delta \underset{\sim}{u} \cdot \underset{\sim}{f}_B \, d\Omega$$

cuja discretização, assumindo $\theta = \underline{\Psi}^\theta \underset{\sim}{\hat{\theta}}$, conduz a

$$-\int_\Omega \delta \underset{\sim}{u} \cdot \underset{\sim}{f}_B \, d\Omega = -\delta \underset{\sim}{\hat{u}}^T \int_\Omega \underline{\Psi}^{u^T} (-\beta) \underset{\sim}{g} (\underline{\Psi}^\theta \underset{\sim}{\hat{\theta}} - \underset{\sim}{\theta}_0) \, d\Omega =$$

$$= \delta \underset{\sim}{\hat{u}}^T \left\{ \int_\Omega \underline{\Psi}^{u^T} \beta \underset{\sim}{g} \underline{\Psi}^\theta \, d\Omega \, \underset{\sim}{\hat{\theta}} - \int_\Omega \underline{\Psi}^{u^T} \beta \underset{\sim}{g} \, \theta_0 \, d\Omega \right\}$$

Expressando θ_0 na forma $\theta_0 = \underline{\Psi}^\theta \underset{\sim}{\hat{\theta}}_0$, então este termo escreve-se na forma

$$\delta \underset{\sim}{\hat{u}}^T \left\{ \int_\Omega \underline{\Psi}^{u^T} \beta \underset{\sim}{g} \underline{\Psi}^\theta \, d\Omega \, \underset{\sim}{\hat{\theta}} - \int_\Omega \underline{\Psi}^{u^T} \beta \underset{\sim}{g} \underline{\Psi}^\theta \underset{\sim}{\hat{\theta}}_0 \, d\Omega \right\} \qquad (61)$$

A sua linearização é dada por:

$$\Delta\left(-\int_{\Omega} \delta \underset{\sim}{u} \cdot \underset{\sim}{f}_{B}\, d\Omega\right) = -\int_{\Omega} \delta \underset{\sim}{u} \cdot \Delta\left(-\beta \underset{\sim}{g} (\theta - \theta_0)\right) d\Omega = \int_{\Omega} \delta \underset{\sim}{u} \cdot \beta \underset{\sim}{g} \Delta \theta \, d\Omega =$$

$$= \delta \underset{\sim}{\hat{u}}^{T} \int_{\Omega} \underline{\Psi}^{u^{T}} \beta \underset{\sim}{g} \underline{\Psi}^{\theta} \, d\Omega \Delta \underset{\sim}{\hat{\theta}} = \delta \underset{\sim}{\hat{u}}^{T} \underline{B} \, \Delta \underset{\sim}{\hat{\theta}}$$

Equação da energia no fluido

$$\rho_0 c_p (\dot{\theta} + \underset{\sim}{u} \cdot \underset{\sim}{\nabla} \theta) + \mathrm{div}\, \underset{\sim}{q} - G = 0 \qquad \text{em } \Omega \qquad (62.a)$$

$$\theta = \bar{\theta} \qquad \text{em } \Gamma_\theta \qquad (62.b)$$

$$q_n + \bar{q}_n = 0 \qquad \text{em } \Gamma_q \qquad (62.c)$$

$$q_n + h(\theta_a - \theta) = 0 \qquad \text{em } \Gamma_h \qquad (62.d)$$

$$q_n + \varepsilon\sigma(\theta_a^4 - \theta^4) = 0 \qquad \text{em } \Gamma_r \qquad (62.e)$$

onde $q_n = \underset{\sim}{q} \cdot \underset{\sim}{n}$, $\quad \underset{\sim}{q} = -\underline{D}\, \underset{\sim}{v}$ e $\underset{\sim}{v} = \underset{\sim}{\nabla} \theta$

Forma fraca das equações de equilíbrio

$$\int_\Omega \delta\theta \left(\rho_0 c_p (\dot{\theta} + \underset{\sim}{u} \cdot \underset{\sim}{\nabla} \theta) + \mathrm{div}\, \underset{\sim}{q} - G \right) d\Omega - \int_{\Gamma_q} \delta\theta (q_n + \bar{q}_n)\, d\Gamma_q -$$

$$- \int_{\Gamma_h} \delta\theta (q_n + h(\theta_a - \theta))\, d\Gamma_h - \int_{\Gamma_r} \delta\theta (q_n + \varepsilon\sigma(\theta_a^4 - \theta^4))\, d\Gamma_r = 0 \qquad (63)$$

Notando que

$$\int_\Omega \delta\theta\, \mathrm{div}\, \underset{\sim}{q}\, d\Omega = \int_\Omega \delta\theta\, q_{i,i}\, d\Omega = \int_\Gamma \delta\theta\, q_i n_i\, d\Gamma - \int_\Omega \delta\theta_{,i}\, q_i\, d\Omega =$$

$$= \int_\Gamma \delta\theta\, q_n\, d\Gamma - \int_\Omega \underset{\sim}{\nabla} \delta\theta \cdot \underset{\sim}{q}\, d\Omega$$

a expressão (63) pode ser escrita na forma

$$\int_\Omega \delta\theta\, \rho_0 c_p \dot{\theta}\, d\Omega + \int_\Omega \delta\theta\, \rho_0 c_p (\underset{\sim}{u} \cdot \underset{\sim}{\nabla}\theta)\, d\Omega - \int_\Omega \underset{\sim}{\nabla}\delta\theta \cdot \underset{\sim}{q}\, d\Omega + \cancel{\int_{\Gamma_q} \delta\theta\, q_n\, d\Gamma_q} +$$

$$+ \cancel{\int_{\Gamma_h} \delta\theta\, q_n\, d\Gamma_h} + \cancel{\int_{\Gamma_r} \delta\theta\, q_n\, d\Gamma_r} - \int_\Omega \delta\theta\, G\, d\Omega - \cancel{\int_{\Gamma_q} \delta\theta\, q_n\, d\Gamma_q} - \int_{\Gamma_q} \delta\theta\, \bar{q}_n\, d\Gamma_q -$$

$$- \cancel{\int_{\Gamma_h} \delta\theta\, q_n\, d\Gamma_h} - \int_{\Gamma_h} \delta\theta\, h(\theta_a - \theta)\, d\Gamma_h - \cancel{\int_{\Gamma_r} \delta\theta\, q_n\, d\Gamma_r} - \int_{\Gamma_r} \delta\theta\, \varepsilon\sigma(\theta_a^4 - \theta^4)\, d\Gamma_r = 0 \;(=)$$

$$\underset{\sim}{q} = -\underline{D}(\underset{\sim}{\nabla}\theta)$$

$$(=) \int_\Omega \delta\theta\, \rho_0 c_p \dot{\theta}\, d\Omega + \int_\Omega \delta\theta\, \rho_0 c_p (\underset{\sim}{u} \cdot \underset{\sim}{\nabla}\theta)\, d\Omega + \int_\Omega \underset{\sim}{\nabla}\delta\theta \cdot (+\underline{D}\, \underset{\sim}{\nabla}\theta) - \int_\Omega \delta\theta\, G\, d\Omega -$$

$$- \int_{\Gamma_q} \delta\theta\, \bar{q}_n\, d\Gamma_q - \int_{\Gamma_h} \delta\theta\, h\,(\theta_a - \theta)\, d\Gamma_h - \int_{\Gamma_r} \delta\theta\, \varepsilon\sigma(\theta_a^4 - \theta^4)\, d\Gamma_r = 0 \qquad (64)$$

As discretizações espacial e temporal são semelhantes às apresentadas em (22) e (21), respectivamente, bastando substituir $\underset{\sim}{u}$ por θ.

$$\boxed{\int_\Omega \delta\theta\, \rho_0 c_p\, \dot{\theta}\, d\Omega}$$

$$\int_\Omega \delta\theta\, \rho_0 c_p\, \dot{\theta}\, d\Omega = \delta\hat{\underset{\sim}{\theta}}^T \int_\Omega \underline{\Psi}^{\theta T} \rho_0 c_p\, \underline{\Psi}^\theta\, d\Omega\, \dot{\hat{\underset{\sim}{\theta}}} = \delta\hat{\underset{\sim}{\theta}}^T\, \underline{N}\, \dot{\hat{\underset{\sim}{\theta}}} \qquad (65)$$

$$\Delta \int_\Omega \delta\theta\, \rho_0 c_p\, \dot{\theta}\, d\Omega = \delta\hat{\underset{\sim}{\theta}}^T\, \underline{N}\, \Delta\dot{\hat{\underset{\sim}{\theta}}} \qquad (66)$$

$$\boxed{\int_\Omega \delta\theta\, \rho_0 c_p (\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\theta)\, d\Omega}$$

$$\int_\Omega \delta\theta\, \rho_0 c_p (\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\theta)\, d\Omega = \int_\Omega \delta\theta\, \rho_0 c_p (\underset{\sim}{u}\cdot\underset{\sim}{e}_i)\theta_{,i}\, d\Omega =$$

$$= \delta\hat{\underset{\sim}{\theta}}^T \int_\Omega \underline{\Psi}^{\theta T} \rho_0 c_p\, \underset{\sim}{e}_i^T (\underline{\Psi}^u \hat{\underset{\sim}{u}})\, \underline{\Psi}^\theta_{,i}\, d\Omega\, \hat{\underset{\sim}{\theta}} = \delta\hat{\underset{\sim}{\theta}}^T\, \underline{D}_\theta\, \hat{\underset{\sim}{\theta}} \qquad (67)$$

$$\Delta \int_\Omega \delta\theta\, \rho_0 c_p (\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\theta)\, d\Omega = \int_\Omega \delta\theta\, \rho_0 c_p (\Delta\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\theta)\, d\Omega + \int_\Omega \delta\theta\, \rho_0 c_p (\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\Delta\theta)\, d\Omega =$$

$$= \delta\hat{\underset{\sim}{\theta}}^T \int_\Omega \underline{\Psi}^{\theta T} \rho_0 c_p \left(\underline{\Psi}^\theta_{,i}\, \hat{\underset{\sim}{\theta}}\right)\left(\underset{\sim}{e}_i^T\, \underline{\Psi}^u\right) d\Omega\, \Delta\hat{\underset{\sim}{u}} + \delta\hat{\underset{\sim}{\theta}}^T \int_\Omega \underline{\Psi}^{\theta T} \rho_0 c_p\, \underset{\sim}{e}_i^T (\underline{\Psi}^u \hat{\underset{\sim}{u}})\, \underline{\Psi}^\theta_{,i}\, d\Omega\, \Delta\hat{\underset{\sim}{\theta}}$$

$$= \delta\hat{\underset{\sim}{\theta}}^T\, \underline{E}\, \Delta\hat{\underset{\sim}{u}} + \delta\hat{\underset{\sim}{\theta}}^T\, \underline{D}_\theta\, \Delta\hat{\underset{\sim}{\theta}} \qquad (68)$$

$$\boxed{\int_{\Omega} \nabla \delta \theta \cdot \underline{D} \nabla \theta \, d\Omega}$$

Nota: $\nabla \delta\theta \cdot \underline{D} \nabla \theta = (\delta\theta_{,i} e_i) \cdot (D_{kl}\, e_k \otimes e_l)(e_j \theta_{,j}) =$
$= (\delta\theta_{,i} e_i)\cdot(D_{kl}\, e_k (e_l \cdot e_j)\theta_{,j}) = (\delta\theta_{,i} e_i)\cdot(D_{kj}\, e_k \theta_{,j}) =$
$= \delta\theta_{,j} D_{kj} \theta_{,j} (e_i \cdot e_k) = \delta\theta_{,i} D_{ij} \theta_{,j}$

$$\int_{\Omega} \nabla \delta \theta \cdot \underline{D} \nabla \theta \, d\Omega = \delta \hat{\theta}^T \int_{\Omega} \underline{\psi}^{\theta T}_{,i} D_{ij} \underline{\psi}^{\theta}_{,i} \, d\Omega \, \hat{\theta} = \delta \hat{\theta}^T \underline{L} \, \hat{\theta} \tag{69}$$

$$\Delta \int_{\Omega} \nabla \delta \theta \cdot \underline{D} \nabla \theta \, d\Omega = \delta \hat{\theta}^T \int_{\Omega} \underline{\psi}^{\theta T}_{,i} D_{ij} \underline{\psi}_{,i} \, d\Omega \, \Delta\hat{\theta} = \delta \hat{\theta}^T \underline{L} \, \Delta \hat{\theta} \tag{70}$$

$$\boxed{-\int_{\Omega} \delta \theta \, G \, d\Omega}$$

$$-\int_{\Omega} \delta\theta \, G \, d\Omega = \delta \hat{\theta}^T \int_{\Omega} \underline{\psi}^{\theta T}(-G)\, d\Omega = \delta \hat{\theta}^T \left(-\int_{\Omega} \underline{\psi}^{\theta T} \underline{\psi}^{\theta} \, d\Omega \, \hat{G} \right) = \delta \hat{\theta}^T f^{\Omega} \tag{71}$$

$$\Delta \int_{\Omega} \delta \theta \, G \, d\Omega = 0 \tag{72}$$

$$\boxed{-\int_{\Gamma_q} \delta \theta \, \bar{q}_n \, d\Gamma_q}$$

$$-\int_{\Gamma_q} \delta\theta \, \bar{q}_n \, d\Gamma_q = \delta \hat{\theta}^T \int_{\Gamma_q} \underline{\psi}^{\theta T}(-\bar{q}_n)\, d\Omega = \delta \hat{\theta}^T \left(-\int_{\Gamma_q} \underline{\psi}^{\theta T} \underline{\psi}^{\theta} \, d\Gamma_q \, \bar{q}_n \right) = \delta \hat{\theta}^T f^{\Gamma_q} \tag{73}$$

$$\Delta \int_{\Omega} \delta \theta \, \bar{q}_n \, d\Gamma_q = 0 \tag{74}$$

$$\boxed{-\int_{\Gamma_h} \delta\theta\, h\,(\theta_a - \theta)\, d\Gamma_h}$$

$$-\int_{\Gamma_h} \delta\theta\, h\,(\theta_a - \theta)\, d\Gamma_h = -\delta \hat{\underset{\sim}{\theta}}^T \int_{\Gamma_h} \underline{\Psi}^{\theta T} h\,(\underline{\Psi}^\theta \hat{\underset{\sim}{\theta}}_a - \underline{\Psi}^\theta \hat{\underset{\sim}{\theta}})\, d\Gamma_h =$$

$$= \delta \hat{\underset{\sim}{\theta}}^T \left(-\int_{\Gamma_h} \underline{\Psi}^{\theta T} h\, \underline{\Psi}^\theta\, d\Gamma_h\, \hat{\underset{\sim}{\theta}}_a \right) + \delta \hat{\underset{\sim}{\theta}}^T \int_{\Gamma_h} \underline{\Psi}^{\theta T} h\, \underline{\Psi}^\theta\, d\Gamma_h\, \hat{\underset{\sim}{\theta}} =$$

$$= \delta \hat{\underset{\sim}{\theta}}^T \underset{\sim}{f}^{\Gamma_h} + \delta \hat{\underset{\sim}{\theta}}^T \underline{K}_h \hat{\underset{\sim}{\theta}} \qquad (75)$$

$$\Delta\left(-\int_{\Gamma_h} \delta\theta\, h\,(\theta_a - \theta)\, d\Gamma_h \right) = \int_{\Gamma_h} \delta\theta\, h\, \Delta\theta\, d\Gamma_h = \delta \hat{\underset{\sim}{\theta}}^T \int_{\Gamma} \underline{\Psi}^{\theta T} h\, \underline{\Psi}^\theta\, d\Gamma_h\, \Delta \hat{\underset{\sim}{\theta}} = \delta \hat{\underset{\sim}{\theta}}^T \underline{K}_h \Delta \hat{\underset{\sim}{\theta}} \qquad (76)$$

$$\boxed{-\int_{\Gamma_r} \delta\theta\, \varepsilon\sigma\,(\theta_a^4 - \theta^4)\, d\Gamma_r}$$

$$-\int_{\Gamma_r} \delta\theta\, \varepsilon\sigma\,(\theta_a^4 - \theta^4)\, d\Gamma_r = \delta \hat{\underset{\sim}{\theta}} \int_{\Gamma_r} \underline{\Psi}^{\theta T} \varepsilon\sigma \left(-(\underline{\Psi}^\theta \hat{\underset{\sim}{\theta}}_a)^4 + (\underline{\Psi}^\theta \hat{\underset{\sim}{\theta}})^4 \right) d\Gamma_r = \delta \hat{\underset{\sim}{\theta}}^T \underset{\sim}{f}^{\Gamma_r}(\hat{\underset{\sim}{\theta}}) \qquad (77)$$

$$\Delta\left(-\int_{\Gamma_r} \delta\theta\, \varepsilon\sigma\,(\theta_a^4 - \theta^4)\, d\Gamma_r \right) = \int_{\Gamma_r} \delta\theta\, \varepsilon\sigma\,(+4\theta^3 \Delta\theta)\, d\Gamma_r = \delta \hat{\underset{\sim}{\theta}}^T \int_{\Gamma_r} \underline{\Psi}^{\theta T} \varepsilon\sigma\, 4 (\underline{\Psi}^\theta \hat{\underset{\sim}{\theta}})^3 \underline{\Psi}^\theta\, d\Gamma_r\, \Delta \hat{\underset{\sim}{\theta}} = \qquad (78)$$

$$= \delta \hat{\underset{\sim}{\theta}}^T \underline{K}_r\, \Delta \hat{\underset{\sim}{\theta}}$$

Em resumo, a forma fraca (64), após discretização espacial, pode escrever-se

$$\delta\hat{\underset{\sim}{\theta}}^T\left(\underline{N}\,\dot{\hat{\underset{\sim}{\theta}}} + (\underline{D}_\theta(\underset{\sim}{u})+\underline{L}+\underline{K}_h)\,\hat{\underset{\sim}{\theta}} + \underset{\sim}{f}^{\Omega} + \underset{\sim}{f}^{\Gamma_q} + \underset{\sim}{f}^{\Gamma_h} + \underset{\sim}{f}^{\Gamma_r}(\hat{\underset{\sim}{\theta}})\right) = 0 \Leftrightarrow \delta\hat{\underset{\sim}{\theta}}^T \underset{\sim}{r}_\theta = 0 \tag{79}$$

cuja linearização é dada por

$$\delta\hat{\underset{\sim}{\theta}}^T\left(\underline{N}\Delta\dot{\hat{\underset{\sim}{\theta}}} + \underline{E}\,\Delta\hat{\underset{\sim}{u}} + (\underline{D}_\theta + \underline{L} + \underline{K}_h + \underline{K}_r)\,\Delta\hat{\underset{\sim}{\theta}}\right) \tag{80}$$

Notar a presença de:

i) termo de acoplamento com velocidade, $\underline{E}$.

ii) " da radiação, $\underline{K}_r$.

Conhecida a solução no instante n, i.e., $\hat{\underset{\sim}{\theta}}^n$ e $\dot{\hat{\underset{\sim}{\theta}}}^n$, o sistema não-linear que conduz a $\hat{\underset{\sim}{\theta}}^{n+1}$ é dado por

$$\underset{\sim}{r}(\hat{\underset{\sim}{\theta}}^{n+1}) = \underset{\sim}{0} \Leftrightarrow \underline{N}\left(\frac{1}{\gamma}\,\frac{\hat{\underset{\sim}{\theta}}^{n+1}-\hat{\underset{\sim}{\theta}}^{n}}{\Delta t} - \frac{1-\gamma}{\gamma}\,\dot{\hat{\underset{\sim}{\theta}}}^{n}\right) + (\underline{D}_\theta + \underline{L} + \underline{K}_h)\,\hat{\underset{\sim}{\theta}}^{n+1} + \underset{\sim}{f}^{\Omega} + \underset{\sim}{f}^{\Gamma_q} + \underset{\sim}{f}^{\Gamma_h} + \underset{\sim}{f}^{\Gamma_r}(\hat{\underset{\sim}{\theta}}^{n+1}) = \underset{\sim}{0} \Leftrightarrow$$

$$\Leftrightarrow \left(\frac{1}{\gamma\Delta t}\underline{N} + \underline{D}_\theta + \underline{L} + \underline{K}_h\right)\hat{\underset{\sim}{\theta}}^{n+1} + \left(\underset{\sim}{f}^{\Omega} + \underset{\sim}{f}^{\Gamma_q} + \underset{\sim}{f}^{\Gamma_h} + \underset{\sim}{f}^{\Gamma_r}(\hat{\underset{\sim}{\theta}}^{n+1}) + \underline{N}\left(-\frac{1}{\gamma\Delta t}\hat{\underset{\sim}{\theta}}^{n} + \frac{1-\gamma}{\gamma}\dot{\hat{\underset{\sim}{\theta}}}^{n}\right)\right) = \underset{\sim}{0} \tag{81}$$

Atendendo a que $\Delta\dot{\hat{\underset{\sim}{\theta}}} = \Delta\left(\frac{1}{\gamma}\frac{\hat{\underset{\sim}{\theta}}^{n+1}-\hat{\underset{\sim}{\theta}}^{n}}{\Delta t} - \frac{1-\gamma}{\gamma}\dot{\hat{\underset{\sim}{\theta}}}^{n}\right) = \frac{\Delta\hat{\underset{\sim}{\theta}}^{n+1}}{\gamma\Delta t}$, a linearização de $\underset{\sim}{r}(\hat{\underset{\sim}{\theta}}^{n+1})$ é:

$$\left(\frac{1}{\gamma\Delta t}\underline{N} + \underline{D}_\theta + \underline{L} + \underline{K}_h + \underline{K}_r\right)\Delta\hat{\underset{\sim}{\theta}}^{n+1} + \underline{E}\,\Delta\hat{\underset{\sim}{u}}^{n+1} \tag{82}$$

Estabilização da equação da energia

À forma fraca (64) é adicionado o termo

$$\tau_{SUPG\theta} \int_\Omega \underset{\sim}{u} \cdot \underset{\sim}{\nabla} \delta\theta \, r_{SUPG\theta} \, d\Omega \qquad (83)$$

onde

$$r_{SUPG\theta} = \rho_0 c_p (\dot{\theta} + \underset{\sim}{u} \cdot \underset{\sim}{\nabla}\theta) + \operatorname{div} \underset{\sim}{q} - G \qquad (84)$$

é o resíduo da equação da energia.

A discretização espacial de (84) conduz a

($\underline{K}$ isotrópico)

$$\begin{aligned} r_{SUPG\theta} &= \rho_0 c_p \left(\underline{\Psi}^\theta \dot{\hat{\underset{\sim}{\theta}}} + \underset{\sim}{e}_i^T (\underline{\Psi}^u \hat{\underset{\sim}{u}}) \underline{\Psi}^\theta_{,i} \hat{\underset{\sim}{\theta}} \right) - K_{ij,i} \underline{\Psi}^\theta_{,j} \hat{\underset{\sim}{\theta}} - K_{ij} \underline{\Psi}^\theta_{,ji} \hat{\underset{\sim}{\theta}} - \underline{\Psi}^\theta \hat{\underset{\sim}{G}} = \\ &= \rho_0 c_p \left(\underline{\Psi}^\theta \dot{\hat{\underset{\sim}{\theta}}} + \underset{\sim}{e}_i^T (\underline{\Psi}^u \hat{\underset{\sim}{u}}) \underline{\Psi}^\theta_{,i} \hat{\underset{\sim}{\theta}} \right) - K_{,i} \underline{\Psi}^\theta_{,i} \hat{\underset{\sim}{\theta}} - K \underline{\Psi}^\theta_{,ii} \hat{\underset{\sim}{\theta}} - \underline{\Psi}^\theta \hat{\underset{\sim}{G}} \end{aligned}$$

Note-se que $q_i = -K_{ij}\theta_{,j}$, logo

$$\operatorname{div} \underset{\sim}{q} = q_{i,i} = (-K_{ij}\theta_{,j})_{,i} = -K_{ij,i}\theta_{,j} - K_{ij}\theta_{,ji}$$

No caso de $\underline{K}$ ser diagonal, então

$$\operatorname{div} \underset{\sim}{q} = -K_{11,1}\theta_{,1} - K_{22,2}\theta_{,2} - K_{33,3}\theta_{,3} - K_{11}\theta_{,11} - K_{22}\theta_{,22} - K_{33}\theta_{,33}$$

No caso de $\underline{K}$ ser isotrópico, e $K_{ij} = K$, então $\operatorname{div} \underset{\sim}{q} = -\underset{\sim}{\nabla} K \cdot \underset{\sim}{\nabla}\theta - K\nabla^2\theta$

Assim, (83) pode ser avaliado através de

$$\tau_{SUPG\theta} \int_\Omega \underset{\sim}{u} \cdot \underset{\sim}{\nabla} \delta\underset{\sim}{\theta} \, r_{SUPG\theta} \, d\Omega = \delta\hat{\underset{\sim}{\theta}}^T \tau_{SUPG\theta} \int_\Omega \underline{\Psi}^{\theta T}_{,i} (\underline{\Psi}^u_{,i} \hat{\underset{\sim}{u}}) \underset{\sim}{r}_{SUPG\theta} \, d\Omega \qquad (85)$$

A linearização de (83) é dada por

$$\Delta\left(\tau_{SUPG\theta} \int_\Omega \underset{\sim}{u} \cdot \underset{\sim}{\nabla}\delta\theta \, r_{SUPG\theta} \, d\Omega \right) = \tau_{SUPG\theta} \int_\Omega \Delta\underset{\sim}{u} \cdot \underset{\sim}{\nabla}\delta\theta \, r_{SUPG\theta} \, d\Omega + \qquad (86)$$

$$+ \tau_{SUPG\theta} \int_\Omega \underset{\sim}{u} \cdot \underset{\sim}{\nabla}\delta\theta \, \Delta r_{SUPG\theta} \, d\Omega$$

Mas

$$\Delta r_{SUPG\theta} = \Delta\left(\rho_0 c_p(\dot{\theta} + \underset{\sim}{u}\cdot\underset{\sim}{\nabla}\theta) + \mathrm{div}\,\underset{\sim}{q} - G\right) =$$

(assume-se que $\Delta G = 0$ e $\underline{K}$ isotrópico)

$$= \rho_0 c_p\left(\Delta\dot{\theta} + \Delta\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\theta + \underset{\sim}{u}\cdot\underset{\sim}{\nabla}\Delta\theta\right) + \mathrm{div}\,\Delta\underset{\sim}{q} - \Delta G =$$

$$= \rho_0 c_p\left(\Delta\dot{\theta} + \Delta\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\theta + \underset{\sim}{u}\cdot\underset{\sim}{\nabla}\Delta\theta\right) - \underset{\sim}{\nabla}K\cdot\underset{\sim}{\nabla}\Delta\theta - K\nabla^2\Delta\theta \qquad (87)$$

$$= \rho_0 c_p\left(\Delta\dot{\theta} + \Delta u_{,i}\,\theta_{,i} + u_i\,\Delta\theta_{,i}\right) - K_{,i}\,\Delta\theta_{,i} - K\,\Delta\theta_{,ii}$$

e

$$\Delta\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\delta\theta\, r_{SUPG\theta} = \delta\theta_{,i}\, r_{SUPG\theta}\,\Delta u_{,i} \qquad (88)$$

$$\underset{\sim}{u}\cdot\underset{\sim}{\nabla}\delta\theta\,\Delta r_{SUPG\theta} = \delta\theta_{,i}\, u_i\,\Delta r_{SUPG\theta} \qquad (89)$$

logo, a expressão (86) assume a forma

$$\Delta\left(\tau_{SUPG\theta}\int_\Omega \underset{\sim}{u}\cdot\underset{\sim}{\nabla}\delta\theta\, r_{SUPG\theta}\, d\Omega\right) = \tau_{SUPG\theta}\Big\{\delta\underset{\sim}{\hat{\theta}}^T\int_\Omega \underline{\psi}^{\theta T}_{,j}\Big(\rho_0 c_p\big(\underline{\psi}^\theta\dot{\underset{\sim}{\hat{\theta}}} + \underset{\sim}{e}_i^T(\underline{\psi}^u\underset{\sim}{\hat{u}})\,\underline{\psi}^\theta_{,i}\underset{\sim}{\hat{\theta}}\big) - K_{,i}\,\underline{\psi}^\theta_{,i}\underset{\sim}{\hat{\theta}} - K\,\underline{\psi}^\theta_{,ii}\underset{\sim}{\hat{\theta}} - \underline{\psi}^\theta\underset{\sim}{\hat{G}}\Big)\underline{\psi}^u_{,j}\, d\Omega\,\Delta\underset{\sim}{\hat{u}} +$$

$$+ \delta\underset{\sim}{\hat{\theta}}^T\int_\Omega \underline{\psi}^{\theta T}_{,i}\left(\underset{\sim}{e}_i^T\underline{\psi}^u\underset{\sim}{\hat{u}}\right)\Big(\rho_0 c_p\big(\underline{\psi}^\theta\Delta\dot{\underset{\sim}{\hat{\theta}}} + \underline{\psi}^\theta_{,i}\underset{\sim}{\hat{\theta}}\,\underline{\psi}^u_{,i}\Delta\underset{\sim}{\hat{u}} + (\underset{\sim}{e}_i^T\underline{\psi}^u\underset{\sim}{\hat{u}})\,\underline{\psi}^\theta_{,i}\Delta\underset{\sim}{\hat{\theta}}\big) -$$

$$- K_{,i}\,\underline{\psi}^\theta_{,i}\Delta\underset{\sim}{\hat{\theta}} - K\,\underline{\psi}^\theta_{,ii}\Delta\underset{\sim}{\hat{\theta}}\Big)\, d\Omega\Big\}$$

Adicionando (85) a (79), tem-se a expressão final do resíduo: 32