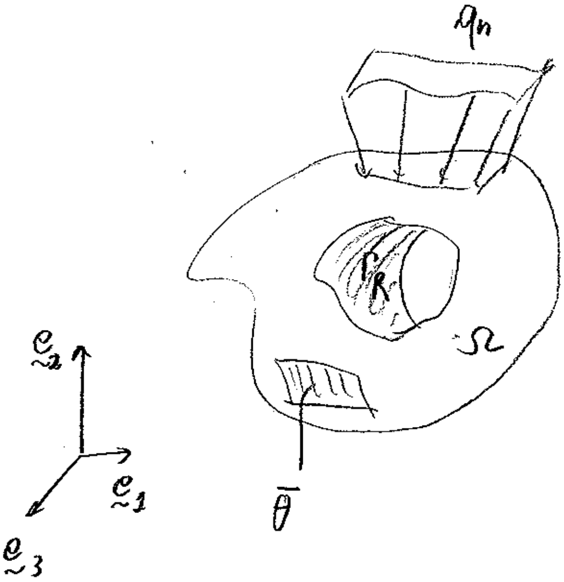
Sobre o problema do calor em cavidades

Forma forte do problema transiente

$$q_i = -K_{ij}\, v_j \qquad \text{1.a}$$

$$-q_{i,i} + G = \rho c_p \dot{\theta} \qquad \text{em } \Omega \qquad \text{1.b}$$

$$v_i = \theta_{,i} \qquad \text{1.c}$$



$$q_n + \bar{q}_n = 0 \qquad \text{em } \Gamma_{\bar{q}_n} \qquad \text{1.d}$$

$$q_n + h(\theta_a - \theta) = 0 \qquad \text{em } \Gamma_h \qquad \text{1.e}$$

$$q_n + \varepsilon\sigma(\theta_a^4 - \theta^4) = 0 \qquad \text{em } \Gamma_r \qquad \text{1.f}$$

$$q_n - \frac{\varepsilon}{\rho}(\sigma\theta^4 - R) = 0 \qquad \text{em } \Gamma_R \qquad \text{1.g}$$

$$\theta = \bar{\theta} \qquad \text{em } \Gamma_{\bar{\theta}} \qquad \text{1.h}$$

$$\theta = \bar{\theta}_0 \qquad \text{em } t = t_0 \qquad \text{1.i}$$

onde $q_n = q_i n_i$ é o fluxo segundo a normal exterior (fluxo que sai).
As condições de fronteira 1.d a 1.g estão escritas na forma

fluxo que entra = fluxo prescrito que entra $\Leftrightarrow$

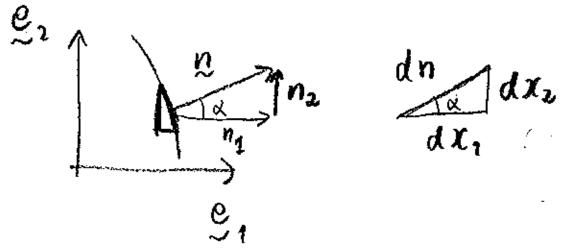
$$\Leftrightarrow \quad -q_n = \bar{q}_n \Leftrightarrow q_n + \bar{q}_n = 0$$

Note-se que $\bar{q}_n$ é o fluxo prescrito em $\Gamma_{\bar{q}_n}$, mas dirigido segundo $(-\underset{\sim}{n})$, logo a notação $\bar{q}_n$ é enganadora. (deceiving)

Assim,

$$q_n + \bar{q}_n = 0 \Leftrightarrow q_i n_i + \bar{q}_n = 0 \Leftrightarrow -K_{ij}\,\theta_{,j}\, n_i + \bar{q}_n = 0$$

Se $K_{ij} = K$, então $-K\,\theta_{,i}\, n_i + \bar{q}_n = 0 \Leftrightarrow K\,\theta_{,n} = \bar{q}_n$ que é a equação (7.3) de Bathe, pois



$$c\alpha = \frac{dx_1}{dn} = n_1$$

$$s\alpha = \frac{dx_2}{dn} = n_2$$

$$\theta_{,n} = \frac{\partial \theta}{\partial n} = \frac{\partial \theta}{\partial x_1}\frac{\partial x_1}{\partial n} + \frac{\partial \theta}{\partial x_2}\frac{\partial x_2}{\partial n} = \frac{\partial \theta}{\partial x_1} n_1 + \frac{\partial \theta}{\partial x_2} n_2 = \theta_{,i}\, n_i$$

Note-se que $\Gamma = \Gamma_{\bar{\theta}} \cup \Gamma_{q_n}$, $\Gamma_{\bar{\theta}} \cap \Gamma_{q_n} = \emptyset$ e que $\Gamma_{\bar{q}_n} \cap \Gamma_h \cap \Gamma_r$ pode ser $\neq \emptyset$.

Assim, $\Gamma_{q_n} = \Gamma_{\bar{q}_n} \cup \Gamma_h \cup \Gamma_r \cup \Gamma_R$.

Em notação direta, ter-se-á

$$\underset{\sim}{q} = -\underline{K}\,\underset{\sim}{v}$$

$$-\operatorname{div} \underset{\sim}{q} + G = \rho C_p \dot{\theta}$$

$$\underset{\sim}{v} = \underset{\sim}{\nabla}\theta$$

e ainda

$$q_n = \underset{\sim}{q} \cdot \underset{\sim}{n}$$

A condição de fronteira 1.g é devida ao fluxo de radiação interna entre as paredes das cavidades. O fluxo que sai é dado por

$$\frac{\varepsilon}{\rho}(\sigma\theta^4 - R)$$

onde ε, $\rho = 1-\varepsilon$ e R são, em geral, função do ponto considerado. R é a radiosidade, dada por,

$$R = \varepsilon\sigma\theta^4 + (1-\varepsilon)H$$

onde H é a radiação incidente.

Dado que — ao contrário do problema descrito por (1) — a equação (2A) que relaciona a radiosidade entre as diversas superfícies varia com a dimensão (d) do espaço considerada ($d = \{1, 2, 3\}$), no que se segue vai utilizar-se uma designação diferente para a fronteira de radiação em cavidades:

$$S_R^e \equiv \text{superfície } e \text{ de um domínio } \Omega \subset \mathbb{R}^3$$

$$\Gamma_R^e \equiv \text{linha } e \text{ de um domínio } \Omega \subset \mathbb{R}^2$$

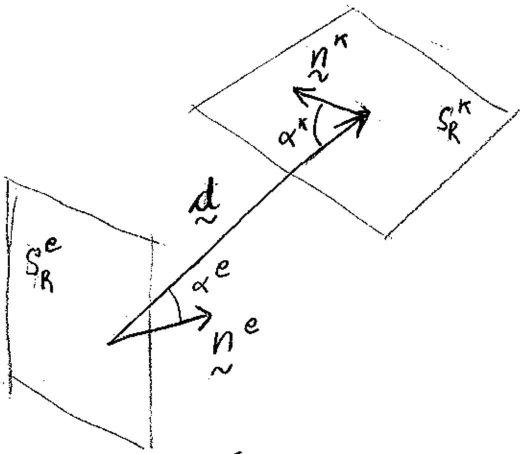
A equação que relaciona a radiosidade na superfície e com as restantes superfícies no espaço 3D será particularizada para o espaço 2D, obtendo-se a equação que relaciona a radiosidade na linha e com as restantes linhas no espaço 2D.

Note-se que as equações 1.a a 1.c são válidas independentemente da dimensão d, bastando mudar a variação dos índices, i.e., $i, j = 1 \dots d$. Tal não acontece com a "equação da radiosidade" apresentada em seguida.

Divida-se a cavidade em n superfícies (não necessàriamente planas). ③
Então, a radiosidade na superfície e é dada por

$$R^e = \sigma \varepsilon^e \theta^{e^4} + \rho^e \sum_{\substack{k=1 \\ k\neq e}}^{n} \int_{S_R^k} R^k \frac{\cos\alpha^e \cos\alpha^k}{\pi d^2} \, dS_R^k \qquad (2)$$

onde $d = \|\underset{\sim}{d}\|$, α^e e α^k variam consoante os pontos considerados nas superfícies Γ_R^e e Γ_R^k. R^k varia em Γ_R^k. R^e, ε^e, ρ^e e θ^e variam em Γ_R^e.



Formas fracas de (1) e (2)

A forma fraca do equilíbrio do problema (1) é dada por

$$-\int_{\Omega} \delta\theta \left(-\operatorname{div} \underset{\sim}{q} + G - \rho c_p \dot{\theta}\right) d\Omega - \int_{\Gamma_{\bar{q}_n}} \delta\theta \left(q_n + \bar{q}_n\right) d\Gamma_{\bar{q}_n} - \qquad (3)$$

$$-\int_{\Gamma_h} \delta\theta\left(q_n + h(\theta_a - \theta)\right) d\Gamma_h - \int_{\Gamma_r} \delta\theta\left(q_n + \varepsilon\sigma(\theta_a^4 - \theta^4)\right) d\Gamma_r - \int_{\Gamma_R} \delta\theta\left(q_n - \frac{\varepsilon}{\rho}(\sigma\theta^4 - R)\right) d\Gamma_R = 0$$

$$\forall \delta\theta : \delta\theta|_{\Gamma_{\bar{\theta}}} = 0$$

Considerando que

$$\int_{\Omega} \delta\theta \operatorname{div} \underset{\sim}{q} \, d\Omega = \int_{\Omega} \delta\theta \, q_{i,i} \, d\Omega = \int_{\Omega} (\delta\theta \, q_i)_{,i} \, d\Omega - \int_{\Omega} \delta\theta_{,i} \, q_i \, d\Omega =$$

$$= \int_{\Gamma} \delta\theta q_i n_i \, d\Gamma - \int_{\Omega} \underset{\sim}{\nabla}\theta \cdot \underset{\sim}{q} \, d\Omega$$

$$= \int_{\Gamma} \delta\theta \, q_n \, d\Gamma - \int_{\Omega} \underset{\sim}{\nabla}\theta \cdot \underset{\sim}{q} \, d\Omega$$

então, a expressão (3) pode ser escrita na forma

$$-\int_{\Omega} \underset{\sim}{\nabla}\theta \cdot \underset{\sim}{q} \, d\Omega - \int_{\Omega} \delta\theta \, G \, d\Omega + \int_{\Omega} \delta\theta \, \rho c_p \dot{\theta} \, d\Omega + \int_{\Gamma_{\bar{\theta}}} \underbrace{\delta\theta}_{=0} q_n \, d\Gamma_{\bar{\theta}} + \cancel{\int_{\Gamma_{\bar{q}_n}} \delta\theta \, q_n \, d\Gamma_{\bar{q}_n}} -$$

$$+ \cancel{\int_{\Gamma_h} \delta\theta \, q_n \, d\Gamma_h} + \cancel{\int_{\Gamma_r} \delta\theta \, q_n \, d\Gamma_r} + \cancel{\int_{\Gamma_R} \delta\theta \, q_n \, d\Gamma_R} - \cancel{\int_{\Gamma_{\bar{q}_n}} \delta\theta \, q_n \, d\Gamma_{\bar{q}_n}} - \int_{\Gamma_{\bar{q}_n}} \delta\theta \, \bar{q}_n \, d\Gamma_{\bar{q}_n} +$$

$$- \cancel{\int_{\Gamma_h} \delta\theta \, q_n \, d\Gamma_h} - \int_{\Gamma_h} \delta\theta \, h(\theta_a - \theta) \, d\Gamma_h - \cancel{\int_{\Gamma_r} \delta\theta \, q_n \, d\Gamma_r} - \int_{\Gamma_r} \delta\theta \, \varepsilon\sigma(\theta_a^4 - \theta^4) \, d\Gamma_r -$$

$$- \cancel{\int_{\Gamma_R} \delta\theta \, q_n \, d\Gamma_R} + \int_{\Gamma_R} \delta\theta \, \frac{\varepsilon}{\rho}(\sigma\theta^4 - R) \, d\Gamma_R = 0 \Leftrightarrow$$

$$\Leftrightarrow \int_{\Omega} \delta\theta \, \rho c_p \dot{\theta} \, d\Omega - \int_{\Omega} \underset{\sim}{\nabla}\delta\theta \cdot \underset{\sim}{q} \, d\Omega - \int_{\Omega} \delta\theta \, G \, d\Omega - \int_{\Gamma_{\bar{q}_n}} \delta\theta \, \bar{q}_n \, d\Gamma_{\bar{q}_n} - \int_{\Gamma_h} \delta\theta \, h(\theta_a - \theta) \, d\Gamma_h -$$

$$- \int_{\Gamma_r} \delta\theta \, \varepsilon\sigma(\theta_a^4 - \theta^4) \, d\Gamma_r + \int_{\Gamma_R} \delta\theta \, \frac{\varepsilon}{\rho}(\sigma\theta^4 - R) \, d\Gamma_R = 0 \qquad (4)$$

A forma fraca da equação da radiosidade (2) é obtida através da sua ponderação por δR^e seguida de integração em Γ_R^e:

$$\int_{S_R^e} \delta R^e R^e \, dS_R^e = \int_{S_R^e} \delta R^e \sigma \varepsilon^e \theta^{e^4} dS_R^e + \int_{S_R^e} \delta R^e \rho^e \sum_{\substack{k=1 \\ k \neq e}}^{n} \int_{S_R^k} R^k \frac{\cos\alpha^e \cos\alpha^k}{\pi d^2} dS_R^k dS_R^e \quad (5)$$

Para abordar o problema plano, admita-se que a cavidade tem espessura h:

$\underset{\sim}{d} = \underset{\sim}{x}^k - \underset{\sim}{x}^e$

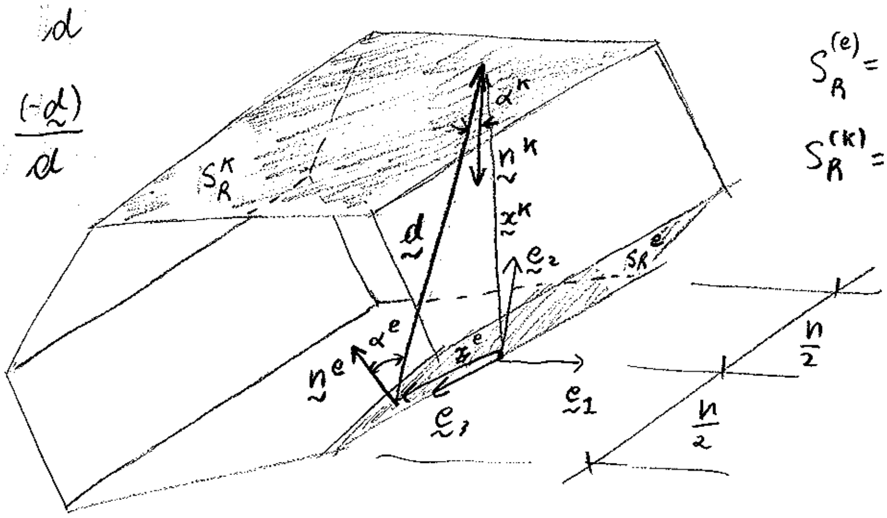
$d = \| \underset{\sim}{d} \|$

$\cos\alpha^e = \underset{\sim}{n}^e \cdot \dfrac{\underset{\sim}{d}}{d}$

$\cos\alpha^k = \underset{\sim}{n}^k \cdot \dfrac{(-\underset{\sim}{d})}{d}$

$S_R^{(e)} = \Gamma_R^{(e)} \otimes h$

$S_R^{(k)} = \Gamma_R^{(k)} \otimes h$



Assim, (5) pode escrever-se na forma

$$h \int_{\Gamma_R^e} \delta R^e R^e \, d\Gamma_R^e = h \int_{\Gamma_R^e} \delta R^e \sigma \varepsilon^e \theta^{e^4} d\Gamma_R^e + \int_{\Gamma_R^e} \int_{-\frac{h}{2}}^{\frac{h}{2}} \delta R^e \rho^e \sum_{\substack{k=1 \\ k \neq e}}^{n} \int_{\Gamma_R^k} \int_{-\frac{h}{2}}^{\frac{h}{2}} R^k \frac{\cos\alpha^e \cos\alpha^k}{\pi d^2} dx_3^k d\Gamma_R^k dx_3^e d\Gamma_R^e \quad (6)$$

Note-se que $\underset{\sim}{n}^e \cdot \underset{\sim}{e}_3 = 0$ e $\underset{\sim}{n}^k \cdot \underset{\sim}{e}_3 = 0$. Então os produtos internos

$\underset{\sim}{n}^e \cdot \underset{\sim}{d}$ e $\underset{\sim}{n}^k \cdot \underset{\sim}{d}$ também não dependerão de x_3.

Divida-se ambos os termos de (10) por h.

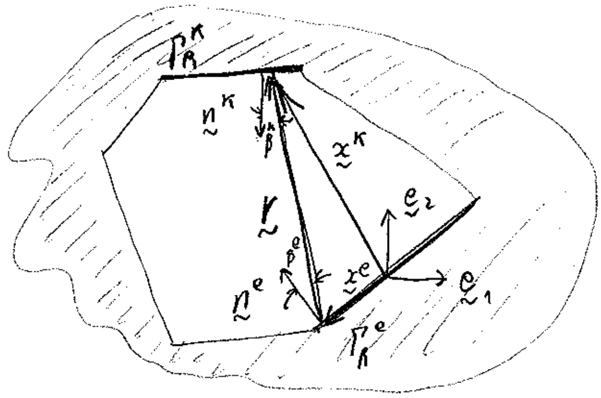
Assim, o 2º termo do 2º membro de (10) pode ser escrito na forma: (5)

$$\sum_{\substack{K=1\\K\neq e}}^{n} \int_{\Gamma_R^e}\int_{\Gamma_R^K} \delta R^e \rho^e R^K (\underset{\sim}{n}^e\cdot \underset{\sim}{d})(\underset{\sim}{n}^K\cdot \underset{\sim}{d}) \int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{\pi h \|\underset{\sim}{d}\|^2 d^2}\, dx_3^K\, dx_3^e \quad (7)$$

Seja $\underset{\sim}{d} = \underset{\sim}{x}^K - \underset{\sim}{x}^e = \underset{\sim}{r} + (x_3^K - x_3^e)\,\underset{\sim}{e}_3$, onde $\underset{\sim}{r} = (x_1^K - x_1^e)\,\underset{\sim}{e}_1 + (x_2^K - x_2^e)\,\underset{\sim}{e}_2$ é um vector no espaço bidimensional.
Então $d^2 = r^2 + (x_3^K - x_3^e)^2$, sendo $r = \|\underset{\sim}{r}\|$.

Notando que

$$\lim_{h\to\infty} \int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{1}{\pi h r^4}\, dx_3^K\, dx_3^e = \lim_{h\to\infty} \frac{1}{\pi}\, \frac{\pi - 2\,\mathrm{Atan}\left(\frac{r}{h}\right) + 2\,\mathrm{Atan}\left(\frac{h}{r}\right)}{4r^3} = \frac{1}{2r^3} \quad (8)$$



Assim, no caso plano a expressão (6) assume a forma

$$\int_{\Gamma_R^e} \delta R^e R^e\, d\Gamma_R^e = \int_{\Gamma_R^e} \delta R^e \sigma \varepsilon^e \theta^{e4}\, d\Gamma_R^e + \sum_{\substack{K=1\\K\neq e}}^{n} \int_{\Gamma_R^e}\int_{\Gamma_R^K} \frac{\delta R^e \rho^e R^K \left(\underset{\sim}{n}^e\cdot\frac{\underset{\sim}{r}}{r}\right)\left(\underset{\sim}{n}^K\cdot\frac{\underset{\sim}{r}}{r}\right)}{2r}\, d\Gamma_R^K\, d\Gamma_R^e \quad (9)$$

Esta é a forma fraca de (2) para o caso 2D, que pode ser reescrita através de

$$\int_{\Gamma_R^e} \delta R^e R^e\, d\Gamma_R^e - \int_{\Gamma_R^e} \delta R^e \sigma \varepsilon^e \theta^{e4}\, d\Gamma_R^e - \sum_{\substack{K=1\\K\neq e}}^{n} \int_{\Gamma_R^e}\int_{\Gamma_R^K} \frac{\delta R^e \rho^e R^K \cos\beta^e \cos\beta^K}{2r}\, d\Gamma_R^K\, d\Gamma_R^e = 0 \quad (10)$$

Caso δR^e, R^e, θ^e, ε^e, ρ^e sejam constantes ao longo da linha e, R^k seja constante ao longo da linha k, então a expressão (10) pode ser reescrita através de

$$\delta R^e \left(R^e \Gamma_R^e - \sigma \varepsilon^e \theta^{e4} \Gamma_R^e - \sum_{\substack{k=1 \\ k \neq e}}^{n} \rho^e R^k \int_{\Gamma_R^e} \int_{\Gamma_R^k} \frac{\cos \beta^e \cos \beta^k}{2r} \, d\Gamma_R^k \, d\Gamma_R^e \right) = 0$$

Para variações arbitrárias δR^e e dividindo o resultado pelo comprimento da linha e, $\Gamma_R^e = \int_{\Gamma_R^e} d\Gamma_R^e$, vem

$$R^e - \sigma \varepsilon^e \theta^{e4} - \sum_{\substack{k=1 \\ k \neq e}}^{n} \rho^e R^k F_{ek} = 0 \quad \text{onde} \quad F_{ek} = \frac{1}{\Gamma_R^e} \int_{\Gamma_R^e} \int_{\Gamma_R^k} \frac{\cos \beta^e \cos \beta^k}{2r} \, d\Gamma_R^k \, d\Gamma_R^e$$

é o "view factor".

Note-se que a expressão (10) - ou a similar em termos de "view factor" - aparece nas seguintes referências:

- Thermal radiation heat transfer, R. Siegel and J. R. Howell, 3ª edição, 1992, página 205.
- An isoparametric boundary solution for thermal radiation, M. A. Keavey, C. A. N. M., 4, 639-646, 1988, na página 642, equação 11.
- Discontinuous F. E. in F. D. and H. T., Li Ben, 2006, página 324, eq. (8.7).

Discretização Espacial

$$\theta = \underline{\Psi}_\theta \, \hat{\underset{\sim}{\theta}} \qquad R^e = \underline{\Psi}_R \, \hat{\underset{\sim}{R}}^e$$

$$\delta\theta = \underline{\Psi}_\theta \, \delta\hat{\underset{\sim}{\theta}} \qquad R^\kappa = \underline{\Psi}_R \, \hat{\underset{\sim}{R}}^\kappa \qquad (11)$$

$$\Delta\theta = \underline{\Psi}_\theta \, \Delta\hat{\underset{\sim}{\theta}}$$

$$\Delta\delta\theta = 0$$

$$\dot{\theta} = \underline{\Psi}_\theta \, \dot{\hat{\underset{\sim}{\theta}}}$$

No que se segue considera-se que $\underset{\sim}{q} = -\underline{D} \, \underset{\sim}{\nabla} \theta$ onde $\underline{D} = \underline{D}(\theta)$.
Considera-se ainda que h, ε, c_p e ρ são dependentes de θ.

Linearização da forma fraca (4) (para o caso transiente):

$$\Delta\left(-\int_\Omega \tilde{\nabla}\delta\tilde{\theta}\cdot\tilde{q}\,d\Omega\right) = \Delta\left(-\int_\Omega \tilde{\nabla}\delta\theta\cdot(-\underline{D})\,\tilde{\nabla}\theta\,d\Omega\right) = \int_\Omega \tilde{\nabla}\delta\theta\cdot\underline{D}\,\tilde{\nabla}\Delta\theta\,d\Omega +$$

Notar que $\Delta\underline{D} = \frac{\partial\underline{D}}{\partial\theta}\Delta\theta$

$$+\int_\Omega \tilde{\nabla}\delta\theta\cdot\Delta\underline{D}\,\tilde{\nabla}\theta\,d\Omega = \delta\hat{\tilde{\theta}}^T\int_\Omega\left\{(\tilde{\nabla}\underline{\Psi}_\theta)^T\,\underline{D}\,(\tilde{\nabla}\underline{\Psi}_\theta) + \underset{(n\times 2)}{(\tilde{\nabla}\underline{\Psi}_\theta)^T}\,\underset{(2\times 2)}{\frac{\partial\underline{D}}{\partial\theta}}\,\underset{(2\times n)}{(\tilde{\nabla}\underline{\Psi}_\theta)}\,\underset{(n\times 1)}{\hat{\tilde{\theta}}}\,\underset{(1\times n)}{\underline{\Psi}_\theta}\right\}d\Omega\,\underset{(n\times 1)}{\Delta\hat{\tilde{\theta}}}$$

(12.a)

$$\Delta\int_\Omega \delta\theta\,\rho\,c_p\,\dot{\theta}\,d\Omega = \int_\Omega\left(\delta\theta\,\Delta\rho\,c_p\,\dot{\theta} + \delta\theta\,\rho\,\Delta c_p\,\dot{\theta} + \delta\theta\,\rho\,c_p\,\Delta\dot{\theta}\right)d\Omega =$$

$$= \int_\Omega\left\{\delta\theta\left(\frac{\partial\rho}{\partial\theta}c_p\dot{\theta} + \rho\frac{\partial c_p}{\partial\theta}\dot{\theta}\right)\Delta\theta + \delta\theta\,\rho\,c_p\,\Delta\dot{\theta}\right\}d\Omega =$$

$$= \delta\hat{\tilde{\theta}}^T\int_\Omega \underline{\Psi}_\theta^T\left(\frac{\partial\rho}{\partial\theta}c_p\,\underline{\Psi}_\theta\,\dot{\hat{\tilde{\theta}}} + \rho\frac{\partial c_p}{\partial\theta}\underline{\Psi}_\theta\,\dot{\hat{\tilde{\theta}}}\right)\underline{\Psi}_\theta\,d\Omega\,\Delta\hat{\tilde{\theta}} + \delta\hat{\tilde{\theta}}^T\int_\Omega \underline{\Psi}_\theta^T\rho\,c_p\,\underline{\Psi}_\theta\,d\Omega\,\Delta\dot{\hat{\tilde{\theta}}} \quad (12.b)$$

$$\Delta\int_\Omega \delta\theta\,G\,d\Omega = 0 \quad (12.c)$$

$$\Delta\int_{\Gamma_{\bar{q}_n}} \delta\theta\,\bar{q}_n\,d\Gamma_{\bar{q}_n} = 0 \quad (12.d)$$

$$\Delta\int_{\Gamma_h}\delta\theta\,h(\theta_a-\theta)\,d\Gamma_h = \int_{\Gamma_h}\delta\theta\,h(-\Delta\theta)\,d\Gamma_h + \int_{\Gamma_h}\delta\theta\,\Delta h\,(\theta_a-\theta)\,d\Gamma_h =$$

$\Delta h = \frac{\partial h}{\partial\theta}\Delta\theta$

$$= \delta\hat{\tilde{\theta}}^T\int_{\Gamma_h}\left\{\underline{\Psi}_\theta^T(-h)\,\underline{\Psi}_\theta + \underline{\Psi}_\theta^T\frac{\partial h}{\partial\theta}\underline{\Psi}_\theta(\hat{\tilde{\theta}}_a - \hat{\tilde{\theta}})\,\underline{\Psi}_\theta\right\}d\Gamma_h\,\Delta\hat{\tilde{\theta}} \quad (12.e)$$

$$\Delta \int_{\Gamma_r} \delta\theta\, \varepsilon\sigma\, (\theta_a^4 - \theta^4)\, d\Gamma_r = \int_{\Gamma_r} \Big(\delta\theta\, \Delta\varepsilon\, \sigma\, (\theta_a^4 - \theta^4) + \delta\theta\, \varepsilon\sigma\, (-4\theta^3 \Delta\theta) \Big) d\Gamma_r = ⑨$$

$$\Big\{ \Delta\varepsilon = \frac{\partial\varepsilon}{\partial\theta}\Delta\theta$$

$$= \int_{\Gamma_r} \Big(\delta\theta\, \frac{\partial\varepsilon}{\partial\theta}\Delta\theta\, \sigma\, (\theta_a^4 - \theta^4) - 4\,\delta\theta\, \varepsilon\, \sigma\, \theta^3 \Delta\theta \Big) d\Gamma_r =$$

$$= \delta\underset{\sim}{\hat{\theta}}^T \int_{\Gamma_r} \Big\{ \underline{\psi}_\theta^T \frac{\partial\varepsilon}{\partial\theta} \sigma \Big[(\underline{\psi}_\theta\, \underset{\sim}{\hat{\theta}}_a)^4 - (\underline{\psi}_\theta\, \underset{\sim}{\hat{\theta}})^4 \Big] \underline{\psi}_\theta - \underline{\psi}_\theta^T (4\varepsilon\sigma) (\underline{\psi}_\theta\, \underset{\sim}{\hat{\theta}})^3\, \underline{\psi}_\theta \Big\} d\Gamma_r\, \Delta\underset{\sim}{\hat{\theta}} \qquad (12.f)$$

No caso do Γ_R há que especificar o elemento em análise. Ir-se-á considerar os elementos (e) e (k)

$$\Delta \Big(\int_{\Gamma_R^{(e)}} \delta\theta \frac{\varepsilon}{1-\varepsilon} (\sigma\theta^4 - \hat{R}^{(e)})\, d\Gamma_R^{(e)} \Big) = \int_{\Gamma_R^{(e)}} \Big\{ \delta\theta\, \Delta\Big(\frac{\varepsilon}{1-\varepsilon}\Big) (\sigma\theta^4 - \hat{R}^{(e)}) + \delta\theta \Big(\frac{\varepsilon}{1-\varepsilon}\Big) (\sigma 4\theta^3 \Delta\theta - \Delta\hat{R}^{(e)}) \Big\} d\Gamma_R^{(e)}$$

$$= \int_{\Gamma_R^{(e)}} \Big\{ \delta\theta \frac{1}{(\varepsilon-1)^2} \Delta\theta (\sigma\theta^4 - \hat{R}^{(e)}) + \delta\theta \Big(\frac{\varepsilon}{1-\varepsilon}\Big) (\sigma 4\theta^3 \Delta\theta - \Delta\hat{R}^{(e)}) \Big\} d\Gamma_R^{(e)}$$

$$\Big\{ \Delta\Big(\frac{\varepsilon}{1-\varepsilon}\Big) = \frac{\partial}{\partial\theta}\Big(\frac{\varepsilon}{1-\varepsilon}\Big)\Delta\theta = \frac{1}{(\varepsilon-1)^2}\Delta\theta \;\; ?$$

$$= \delta\underset{\sim}{\hat{\theta}}^T \int_{\Gamma_R^{(e)}} \Big\{ \underline{\psi}_\theta^T \frac{1}{(\varepsilon-1)^2} \Big(\sigma (\underline{\psi}_\theta\, \underset{\sim}{\hat{\theta}})^4 - \underline{\psi}_R\, \underset{\sim}{\hat{R}}^{(e)} \Big) \underline{\psi}_\theta + \underline{\psi}_\theta^T \Big(\frac{\varepsilon}{1-\varepsilon}\Big) 4\sigma (\underline{\psi}_\theta\, \underset{\sim}{\hat{\theta}})^3\, \underline{\psi}_\theta \Big\} d\Gamma_R^{(e)} \Delta\underset{\sim}{\hat{\theta}} -$$

$$- \delta\underset{\sim}{\hat{\theta}}^T \int_{\Gamma_R^{(e)}} \underline{\psi}_\theta^T \Big(\frac{\varepsilon}{1-\varepsilon}\Big) \underline{\psi}_R\, d\Gamma_R^{(e)} \Delta\underset{\sim}{\hat{R}}^{(e)} \qquad (12.g)$$

Linearização da forma fraca (10), relativa à "eq. da radiosidade"

$$\Delta \int_{\Gamma_R^e} \delta R^e R^e d\Gamma_R^e = \int_{\Gamma_R^e} \delta R^e \Delta R^e d\Gamma_R^e = \delta \hat{\underset{\sim}{R}}^{eT} \int_{\Gamma_R^e} \underline{\Psi}_R^T \underline{\Psi}_R \, d\Gamma_R^e \, \Delta \hat{\underset{\sim}{R}}^e \qquad (13.a)$$

$$\Delta \int_{\Gamma_R^e} \delta R^e \sigma \varepsilon^e \theta^{e4} d\Gamma_R^e = \int_{\Gamma_R^e} \delta R^e \left(\sigma \Delta\varepsilon^e \theta^{e4} + \sigma \varepsilon^e 4 \theta^{e3} \Delta\theta^e\right) d\Gamma_R^e = \left| \Delta\varepsilon^e = \frac{\partial \varepsilon^e}{\partial \theta^e}\Delta\theta^e \right.$$

$$= \int_{\Gamma_R^e} \delta R^e \left(\sigma \frac{\partial \varepsilon^e}{\partial \theta} \theta^{e4} + \sigma \varepsilon^e 4 \theta^{e3}\right) \Delta\theta^e d\Gamma_R^e =$$

$$= \delta \hat{\underset{\sim}{R}}^{eT} \int_{\Gamma_R^e} \underline{\Psi}_R^T \left(\sigma \frac{\partial \varepsilon^e}{\partial \theta} \left(\underline{\Psi}_\theta \hat{\underset{\sim}{\theta}}^e\right)^4 + \sigma \varepsilon^e \left(\underline{\Psi}_\theta \hat{\underset{\sim}{\theta}}^e\right)^3\right) \underline{\Psi}_\theta \, d\Gamma_R^e \, \Delta \hat{\underset{\sim}{\theta}}^e \qquad (13.b)$$

A variação do termo

$$\Delta \left(\sum_{\substack{k=1 \\ k\neq e}}^{n} \int_{\Gamma_R^e} \int_{\Gamma_R^k} \frac{\delta R^e \rho^e R^k \cos\beta^e \cos\beta^k}{2r} d\Gamma_R^k d\Gamma_R^e \right) = (*) \qquad (13.c)$$

pode ser obtida a partir da variação da função integranda, i. e.,

$$\Delta \left(\frac{\delta R^e \rho^e R^k \cos\beta^e \cos\beta^k}{2r} \right) = \frac{\delta R^e \cos\beta^e \cos\beta^k}{2r} \left(\Delta\rho^e R^k + \rho^e \Delta R^k \right) = \left| \begin{array}{l} \rho^e = 1 - \varepsilon^e \text{ e} \\ \Delta\rho^e = -\Delta\varepsilon^e \\ \quad = -\frac{\partial \varepsilon^e}{\partial \theta^e}\Delta\theta^e \end{array} \right.$$

$$= \frac{\delta R^e \cos\beta^e \cos\beta^k}{2r} \left(-\frac{\partial \varepsilon^e}{\partial \theta^e} \Delta\theta^e R^k + (1-\varepsilon^e) \Delta R^k \right)$$

Assim, (13.c) assume a forma

$$(*) = \delta \hat{\underset{\sim}{R}}^{eT} \left\{ \sum_{\substack{k=1 \\ k\neq e}}^{n} \int_{\Gamma_R^e} \int_{\Gamma_R^k} \underline{\Psi}_R^T \left(-\frac{\partial \varepsilon^e}{\partial \theta^e} \right) \left(\underline{\Psi}_R \hat{\underset{\sim}{R}}^k\right) \frac{\cos\beta^e \cos\beta^k}{2r} \underline{\Psi}_\theta \, d\Gamma_R^k d\Gamma_R^e \Delta \hat{\underset{\sim}{\theta}}^e + \right.$$

$$\left. + \sum_{\substack{k=1 \\ k\neq e}}^{n} \int_{\Gamma_R^e} \int_{\Gamma_R^k} \underline{\Psi}_R^T (1-\varepsilon^e) \frac{\cos\beta^e \cos\beta^k}{2r} \underline{\Psi}_R \, d\Gamma_R^k d\Gamma_R^e \Delta \hat{\underset{\sim}{R}}^k \right\} \qquad (13.d)$$

Em resumo, o problema a resolver é dado por

$$\boldsymbol{r} = 0 \text{, onde } \boldsymbol{r} = \begin{Bmatrix} \boldsymbol{r}_\theta \\ \boldsymbol{r}_R \end{Bmatrix}. \qquad (14)$$

$\boldsymbol{r}_\theta$ é obtido substituindo (11) em (4) para $\forall\, \delta\theta^e : \delta\theta^e\big|_{\Gamma_{\bar\theta}} = 0$, obtendo-se para o elemento genérico (e),

$$\boldsymbol{r}_\theta^{(e)} = \int_{\Omega^e} \underline{\Psi}_\theta^T \rho c_p \underline{\Psi}_\theta \, d\Omega^e \dot{\hat{\boldsymbol{\theta}}} + \int_{\Omega^e} (\nabla \underline{\Psi}_\theta)^T \underline{D} (\nabla \underline{\Psi}_\theta) \, d\Omega^e \hat{\boldsymbol{\theta}}^e - \int_{\Omega^e} \underline{\Psi}_\theta^T \underline{\Psi}_\theta \, d\Omega^e \hat{\boldsymbol{g}} -$$

$$- \int_{\Gamma^e_{\bar q_n}} \underline{\Psi}_\theta^T \underline{\Psi}_\theta \, d\Gamma^e_{\bar q_n} \hat{\bar{\boldsymbol{q}}}_n - \int_{\Gamma_h^e} \underline{\Psi}_\theta^T h \underline{\Psi}_\theta \, d\Gamma_h^e (\hat{\boldsymbol{\theta}}_a - \hat{\boldsymbol{\theta}}) - \int_{\Gamma_r^e} \underline{\Psi}_\theta^T \varepsilon \sigma \left[(\underline{\Psi}_\theta \hat{\boldsymbol{\theta}}_a)^4 - (\underline{\Psi}_\theta \hat{\boldsymbol{\theta}})^4 \right] d\Gamma_r^e +$$

$$+ \int_{\Gamma_R^e} \underline{\Psi}_\theta^T \frac{\varepsilon}{\rho} \left[\sigma (\underline{\Psi}_\theta \hat{\boldsymbol{\theta}})^4 - \underline{\Psi}_R \hat{\boldsymbol{R}}^e \right] d\Gamma_R^e \qquad (15)$$

$\boldsymbol{r}_R^{(e)}$ é obtido substituindo (11) em (10) para $\forall\, \delta R^e$, obtendo-se para o elemento genérico (e),

$$\boldsymbol{r}_R^{(e)} = \int_{\Gamma_R^e} \underline{\Psi}_R^T \underline{\Psi}_R \, d\Gamma_R^e \hat{\boldsymbol{R}}^e - \int_{\Gamma_R^e} \underline{\Psi}_R^T \sigma \varepsilon^e (\underline{\Psi}_\theta \hat{\boldsymbol{\theta}}^e)^4 d\Gamma_R^e - \sum_{\substack{k=1 \\ k\neq e}}^{n} \int_{\Gamma_R^e} \int_{\Gamma_R^k} \frac{\underline{\Psi}_R^T (1-\varepsilon^e) \underline{\Psi}_R \cos\beta^e \cos\beta^k}{2r} d\Gamma_R^k d\Gamma_R^e \hat{\boldsymbol{R}}^k \qquad (16)$$

A matriz tangente é obtida a partir das linearizações (12) e (13). Estas podem ser escritas - para elementos de radiosidade genéricos e e k - na forma

$$\begin{Bmatrix} \delta \hat{\boldsymbol{\theta}}^e \\ \delta \hat{\boldsymbol{R}}^e \end{Bmatrix}^T \left(\begin{bmatrix} \dfrac{\partial \boldsymbol{r}_\theta^e}{\partial \dot{\hat{\boldsymbol{\theta}}}^e} & \underline{0} & \underline{0} \\ \underline{0} & \underline{0} & \underline{0} \end{bmatrix} \begin{Bmatrix} \Delta \dot{\hat{\boldsymbol{\theta}}}^e \\ \Delta \dot{\hat{\boldsymbol{R}}}^e \\ \Delta \dot{\hat{\boldsymbol{R}}}^k \end{Bmatrix} + \begin{bmatrix} \dfrac{\partial \boldsymbol{r}_\theta^e}{\partial \hat{\boldsymbol{\theta}}^e} & \dfrac{\partial \boldsymbol{r}_\theta^e}{\partial \hat{\boldsymbol{R}}^e} & \underline{0} \\ \dfrac{\partial \boldsymbol{r}_R^e}{\partial \hat{\boldsymbol{\theta}}^e} & \dfrac{\partial \boldsymbol{r}_R^e}{\partial \hat{\boldsymbol{R}}^e} & \dfrac{\partial \boldsymbol{r}_R^e}{\partial \hat{\boldsymbol{R}}^k} \end{bmatrix} \begin{Bmatrix} \Delta \hat{\boldsymbol{\theta}}^e \\ \Delta \hat{\boldsymbol{R}}^e \\ \Delta \hat{\boldsymbol{R}}^k \end{Bmatrix} \right) \qquad (17)$$

Os termos presentes em (17) são dados por

$$\frac{\partial \underset{\sim}{r}_\theta^{(e)}}{\partial \underset{\sim}{\dot{\hat{\theta}}}^{(e)}} = \int_\Omega \underline{\psi}_\theta^T \rho c_p \underline{\psi}_\theta \, d\Omega \qquad (18.a)$$

$$\frac{\partial \underset{\sim}{r}_\theta^{(e)}}{\partial \underset{\sim}{\hat{\theta}}^{(e)}} = \int_\Omega \underline{\psi}_\theta^T \left(\frac{\partial \rho}{\partial \theta} c_p \left(\underline{\psi}_\theta \underset{\sim}{\dot{\hat{\theta}}}\right) + \rho \frac{\partial c_p}{\partial \theta} \left(\underline{\psi}_\theta \underset{\sim}{\dot{\hat{\theta}}}\right)\right) \underline{\psi}_\theta \, d\Omega +$$

$$+ \int_\Omega \left(\left(\underset{\sim}{\nabla} \underline{\psi}_\theta\right)^T \underline{D} \left(\underset{\sim}{\nabla} \underline{\psi}_\theta\right) + \underline{\left(\underset{\sim}{\nabla} \underline{\psi}_\theta\right)^T \frac{\partial \underline{D}}{\partial \theta} \left(\underset{\sim}{\nabla} \underline{\psi}_\theta\right) \underset{\sim}{\hat{\theta}} \, \underline{\psi}_\theta} \right) d\Omega +$$

Nota: o termo sublinhado é não-simétrico

$$+ \int_{\Gamma_h} \underline{\psi}_\theta^T \left(-h + \frac{\partial h}{\partial \theta} \underline{\psi}_\theta \left(\underset{\sim}{\hat{\theta}}_a - \underset{\sim}{\hat{\theta}}\right)\right) \underline{\psi}_\theta \, d\Gamma_h +$$

$$+ \int_{\Gamma_r} \underline{\psi}_\theta^T \left\{ \frac{\partial \varepsilon}{\partial \theta} \sigma \left[\left(\underline{\psi}_\theta \underset{\sim}{\hat{\theta}}_a\right)^4 - \left(\underline{\psi}_\theta \underset{\sim}{\hat{\theta}}\right)^4 \right] - 4 \varepsilon \sigma \left(\underline{\psi}_\theta \underset{\sim}{\hat{\theta}}\right)^3 \right\} \underline{\psi}_\theta \, d\Gamma_r +$$

$$+ \int_{\Gamma_R^{(e)}} \underline{\psi}_\theta^T \left\{ \frac{1}{(1-\varepsilon)^2} \left[\sigma \left(\underline{\psi}_\theta \underset{\sim}{\hat{\theta}}\right)^4 - \underline{\psi}_R \underset{\sim}{\hat{R}}^{(e)} \right] + \frac{\varepsilon^e}{1-\varepsilon^e} 4 \sigma \left(\underline{\psi}_\theta \underset{\sim}{\hat{\theta}}\right)^3 \right\} \underline{\psi}_\theta \, d\Gamma_R^{(e)} \qquad (18.b)$$

$$\frac{\partial \underset{\sim}{r}_\theta^{(e)}}{\partial \underset{\sim}{\hat{R}}^{(e)}} = - \int_{\Gamma_R^{(e)}} \underline{\psi}_\theta^T \frac{\varepsilon}{1-\varepsilon} \underline{\psi}_R \, d\Gamma_R^{(e)} \qquad (18,c)$$

$$\frac{\partial \underset{\sim}{r}_R^{(e)}}{\partial \underset{\sim}{\hat{\theta}}^{(e)}} = \int_{\Gamma_R^{(e)}} \underline{\psi}_R^T \left[\sigma \frac{\partial \varepsilon^e}{\partial \theta} \left(\underline{\psi}_\theta \underset{\sim}{\hat{\theta}}^{(e)}\right)^4 - \sigma \varepsilon^{(e)} \left(\underline{\psi}_\theta \underset{\sim}{\hat{\theta}}^{(e)}\right)^3 \right] \underline{\psi}_\theta \, d\Gamma_R^{(e)} +$$

$$+ \int_{\Gamma_R^e} \int_{\Gamma} \underline{\psi}_R^T \left(- \frac{\partial \varepsilon^e}{\partial \theta^e} \right) \left(\underline{\psi}_R \underset{\sim}{\hat{R}}^k \right) \frac{\cos \beta^e \cos \beta^k}{2r} \underline{\psi}_\theta \, d\Gamma_R^k \, d\Gamma_R^e \qquad (18.d)$$

$$\frac{\partial \underset{\sim}{r}_R^{(e)}}{\partial \underset{\sim}{\hat{R}}^{(e)}} = \int_{\Gamma_R^e} \underline{\Psi}_R^T \, \underline{\Psi}_R \, d\Gamma_R^e \qquad (18.e)$$

$$\frac{\partial \underset{\sim}{r}_R^{(e)}}{\partial \underset{\sim}{\hat{R}}^{(k)}} = \int_{\Gamma_R^e} \int_{\Gamma_R^k} \underline{\Psi}_R^T \, (1-\varepsilon^e) \, \frac{\cos\beta^e \cos\beta^k}{2r} \, \underline{\Psi}_R \, d\Gamma_R^k \, d\Gamma_R^e \qquad (18.f)$$

Uma alternativa mais precisa seria escrever (17) na forma

$$\begin{Bmatrix} \delta \hat{\underset{\sim}{\theta}}^{(e)} \\ \delta \hat{\underset{\sim}{R}}^{(e)} \end{Bmatrix}^T \left(\begin{bmatrix} \dfrac{\partial \underset{\sim}{r}_\theta^{(e)}}{\partial \dot{\hat{\underset{\sim}{\theta}}}^{(e)}} & \underline{0} & \dots & \underline{0} \\ \underline{0} & \underline{0} & & \underline{0} \end{bmatrix} \begin{Bmatrix} \Delta \dot{\hat{\underset{\sim}{\theta}}}^{(e)} \\ \Delta \dot{\hat{\underset{\sim}{R}}}^{(1)} \\ \vdots \\ \Delta \dot{\hat{\underset{\sim}{R}}}^{(n)} \end{Bmatrix} + \right.$$

$$\left. + \begin{bmatrix} \dfrac{\partial \underset{\sim}{r}_\theta^{(e)}}{\partial \hat{\underset{\sim}{\theta}}^{(e)}} & \dfrac{\partial \underset{\sim}{r}_\theta^{(e)}}{\partial \hat{\underset{\sim}{R}}^{(1)}} & \dots & \dfrac{\partial \underset{\sim}{r}_\theta^{(e)}}{\partial \hat{\underset{\sim}{R}}^{(n)}} \\ \dfrac{\partial \underset{\sim}{r}_R^{(e)}}{\partial \hat{\underset{\sim}{\theta}}^{(e)}} & \dfrac{\partial \underset{\sim}{r}_R^{(e)}}{\partial \hat{\underset{\sim}{R}}^{(1)}} & & \dfrac{\partial \underset{\sim}{r}_R^{(e)}}{\partial \hat{\underset{\sim}{R}}^{(n)}} \end{bmatrix} \begin{Bmatrix} \Delta \hat{\underset{\sim}{\theta}}^{(e)} \\ \Delta \hat{\underset{\sim}{R}}^{(1)} \\ \vdots \\ \Delta \hat{\underset{\sim}{R}}^{(n)} \end{Bmatrix} \right) \qquad (19)$$

Neste caso, as derivadas em ordem a $\dot{\hat{\underset{\sim}{\theta}}}^{(e)}$ e $\hat{\underset{\sim}{\theta}}^{(e)}$ são as expressas por (18.a), (18.b) e (18.d).

As derivadas em ordem a $\hat{\underset{\sim}{R}}^{(k)}$ para $k = \{1 \dots n\} \setminus \{e\}$ são dadas por

$$\frac{\partial \underset{\sim}{r}_\theta^{(e)}}{\partial \hat{\underset{\sim}{R}}^{(k)}} = \underline{0}$$

$$\frac{\partial \underset{\sim}{r}_R^{(e)}}{\partial \hat{\underset{\sim}{R}}^{(k)}} = \int_{\Gamma_R^e} \int_{\Gamma_R^k} \underline{\psi}_R^T (1 - \varepsilon^e) \frac{\cos\beta^e \cos\beta^k}{2r} \underline{\psi}_R \, d\Gamma_R^k \, d\Gamma_R^e \qquad \text{(igual a 18.f)}$$

As derivadas em ordem a $\hat{\underset{\sim}{R}}^{(e)}$ são dadas por (18.c) e (18.e).

Discretização temporal

Seja conhecida a solução no instante t^n, dada por $\hat{\underset{\sim}{\theta}}^n$, $\dot{\hat{\underset{\sim}{\theta}}}^n$ e $\hat{\underset{\sim}{R}}^n$, e a solicitação em t^{n+1} expressa através de $\hat{\underset{\sim}{g}}^{n+1}$, $\hat{\underset{\sim}{q}}_n^{n+1}$, $\hat{\underset{\sim}{\theta}}_a^{n+1}$. No instante t^{n+1} assume-se que

$$t^{n+1} = t^n + \Delta t$$

$$\dot{\hat{\underset{\sim}{\theta}}}^{n+1} = \frac{1}{\gamma} \frac{\hat{\underset{\sim}{\theta}}^{n+1} - \hat{\underset{\sim}{\theta}}^n}{\Delta t} - \frac{1-\gamma}{\gamma} \dot{\hat{\underset{\sim}{\theta}}}^n \qquad (20)$$

Substituindo (20) em (15), no instante t^{n+1}, tem-se para $\underset{\sim}{r}_\theta^{(e)}$:

$$\underset{\sim}{r}_\theta^{(e)(n+1)} = \int_{\Omega^e} \underline{\Psi}_\theta^T \rho^{n+1} c_p^{n+1} \underline{\Psi}_\theta \, d\Omega^e \left(\frac{1}{\gamma} \frac{\hat{\underset{\sim}{\theta}}^{n+1} - \hat{\underset{\sim}{\theta}}^n}{\Delta t} - \frac{1-\gamma}{\gamma} \dot{\hat{\underset{\sim}{\theta}}}^n \right) +$$

$$+ \int_{\Omega^e} (\underset{\sim}{\nabla} \underline{\Psi}_\theta)^T \underline{D}^{n+1} (\underset{\sim}{\nabla} \underline{\Psi}_\theta) \, d\Omega^e \, \hat{\underset{\sim}{\theta}}^{n+1} - \int_{\Omega^e} \underline{\Psi}_\theta^T \underline{\Psi}_\theta \, d\Omega^e \, \hat{\underset{\sim}{g}}^{n+1} -$$

$$- \int_{\Gamma_{\bar{q}_n}^e} \underline{\Psi}_\theta^T \underline{\Psi}_\theta \, d\Gamma_{\bar{q}_n}^e \, \hat{\underset{\sim}{q}}_n^{n+1} - \int_{\Gamma_h^e} \underline{\Psi}_\theta^T h^{n+1} \underline{\Psi}_\theta \, d\Gamma_h^e \left(\hat{\underset{\sim}{\theta}}_a^{n+1} - \hat{\underset{\sim}{\theta}}^{n+1} \right) -$$

$$- \int_{\Gamma_r^e} \underline{\Psi}_\theta^T \varepsilon^{n+1} \sigma \left[\left(\underline{\Psi}_\theta \hat{\underset{\sim}{\theta}}_a^{n+1} \right)^4 - \left(\underline{\Psi}_\theta \hat{\underset{\sim}{\theta}}^{n+1} \right)^4 \right] d\Gamma_r^e +$$

$$+ \int_{\Gamma_R^e} \underline{\Psi}_\theta^T \frac{\varepsilon^{n+1}}{\rho^{n+1}} \left[\sigma \left(\underline{\Psi}_\theta \hat{\underset{\sim}{\theta}}^{n+1} \right)^4 - \underline{\Psi}_R \hat{\underset{\sim}{R}}^{(e)(n+1)} \right] d\Gamma_R^e \qquad (21)$$

Note-se que as propriedades $\underline{D}^{n+1}$, h^{n+1}, ε^{n+1}, ρ^{n+1} e a radiosidade, $\hat{\underset{\sim}{R}}^{e\,n+1}$, em geral, dependerão, do instante considerado.

O resíduo $\underset{\sim}{r}_R^{(e)}$ não depende de $\hat{\dot{\underset{\sim}{\theta}}}^{(e)}$, pelo que a sua escrita no instante t^{n+1} é trivial:

$$\underset{\sim}{r}_R^{(e)} = \int_{\Gamma_R^{(e)}} \underline{\Psi}_R^T \underline{\Psi}_R \, d\Gamma_R^{(e)} \, \hat{\underset{\sim}{R}}^{(e)(n+1)} - \int_{\Gamma_R^{(e)}} \underline{\Psi}_R^T \, \sigma \, \varepsilon^{e(n+1)} \left(\underline{\Psi}_\theta \, \hat{\underset{\sim}{\theta}}^{e(n+1)} \right) d\Gamma_R^e -$$

$$- \sum_{\substack{k=1 \\ k \neq e}}^{n} \int_{\Gamma_R^e} \int_{\Gamma_R^k} \frac{\underline{\Psi}_R^T (1 - \varepsilon^{e(n+1)}) \underline{\Psi}_R \cos\beta^e \cos\beta^k}{2r} d\Gamma_R^k \, d\Gamma_R^e \, \hat{\underset{\sim}{R}}^{k(n+1)} \qquad (22)$$

A linearização de (20) conduz a

$$\Delta\left(\hat{\dot{\underset{\sim}{\theta}}}^{n+1}\right) = \Delta\left(\frac{1}{\gamma} \frac{\hat{\underset{\sim}{\theta}}^{n+1} - \hat{\underset{\sim}{\theta}}^n}{\Delta t} - \frac{1-\gamma}{\gamma} \hat{\dot{\underset{\sim}{\theta}}}^n \right) = \frac{\Delta \hat{\underset{\sim}{\theta}}^{n+1}}{\gamma \Delta t}$$

A matriz tangente incluída em (19) assume agora a forma

$$\begin{Bmatrix} \delta \hat{\underset{\sim}{\theta}}^{(e)} \\ \delta \hat{\underset{\sim}{R}}^{(e)} \end{Bmatrix}^{(n+1)T} \left(\begin{bmatrix} \frac{1}{\gamma\Delta t} \frac{\partial \underset{\sim}{r}_\theta^e}{\partial \hat{\underset{\sim}{\theta}}^e} & \underline{0} & \cdots & \underline{0} \\ \underline{0} & \underline{0} & & \underline{0} \end{bmatrix}^{(n+1)} \begin{Bmatrix} \Delta \hat{\underset{\sim}{\theta}}^e \\ \Delta \hat{\underset{\sim}{R}}^{(1)} \\ \vdots \\ \Delta \hat{\underset{\sim}{R}}^{(n)} \end{Bmatrix}^{(n+1)} + \right.$$

$$\left. + \begin{bmatrix} \frac{\partial \underset{\sim}{r}_\theta^e}{\partial \hat{\underset{\sim}{\theta}}} & \frac{\partial \underset{\sim}{r}_\theta^e}{\partial \hat{\underset{\sim}{R}}^{(1)}} & \cdots & \frac{\partial \underset{\sim}{r}_\theta^e}{\partial \hat{\underset{\sim}{R}}^{(n)}} \\ \frac{\partial \underset{\sim}{r}_R^e}{\partial \hat{\underset{\sim}{\theta}}} & \frac{\partial \underset{\sim}{r}_R^e}{\partial \hat{\underset{\sim}{R}}^{(1)}} & & \frac{\partial \underset{\sim}{r}_R^e}{\partial \hat{\underset{\sim}{R}}^{(n)}} \end{bmatrix}^{(n+1)} \begin{Bmatrix} \Delta \hat{\underset{\sim}{\theta}}^{(e)} \\ \Delta \hat{\underset{\sim}{R}}^{(1)} \\ \vdots \\ \Delta \hat{\underset{\sim}{R}}^{(n)} \end{Bmatrix}^{(n+1)} \right) =$$

$$\begin{Bmatrix} \delta \hat{\underset{\sim}{\theta}}^{(e)} \\ \partial \hat{\underset{\sim}{R}}^{(e)} \end{Bmatrix}^{(n+1)T} \left(\begin{bmatrix} \frac{1}{\gamma\Delta t} \frac{\partial \underset{\sim}{r}^e}{\partial \hat{\underset{\sim}{\theta}}^e} + \frac{\partial \underset{\sim}{r}_\theta^e}{\partial \hat{\underset{\sim}{\theta}}^e} & \frac{\partial \underset{\sim}{r}_\theta^e}{\partial \hat{\underset{\sim}{R}}^{(1)}} & \cdots & \frac{\partial \underset{\sim}{r}_\theta^e}{\partial \hat{\underset{\sim}{R}}^{(n)}} \\ \frac{\partial \underset{\sim}{r}_R^e}{\partial \hat{\underset{\sim}{\theta}}^e} & \frac{\partial \underset{\sim}{r}_R^e}{\partial \hat{\underset{\sim}{R}}^{(1)}} & & \frac{\partial \underset{\sim}{r}_R^e}{\partial \hat{\underset{\sim}{R}}^{(n)}} \end{bmatrix}^{(n+1)} \begin{Bmatrix} \Delta \hat{\underset{\sim}{\theta}}^{(e)} \\ \Delta \hat{\underset{\sim}{R}}^{(1)} \\ \vdots \\ \Delta \hat{\underset{\sim}{R}}^{(n)} \end{Bmatrix}^{(n+1)} \right)$$

Nota: há uma ambiguidade na notação. n designa o instante t^n e o número de paredes da cavidade.

Os termos presentes na matriz tangente são dados por (18a) a (18f), bastando inserir o superescrito $(n+1)$ em ρ, c_p, $\hat{\dot{\underset{\sim}{\theta}}}$, $\hat{\underset{\sim}{\theta}}$, $\underline{\rho}$, h, ε, $\hat{\underset{\sim}{R}}^{(e)}$, $\hat{\underset{\sim}{R}}^{(n)}$ e substituir $\hat{\dot{\underset{\sim}{\theta}}}^{n+1}$ por (20).