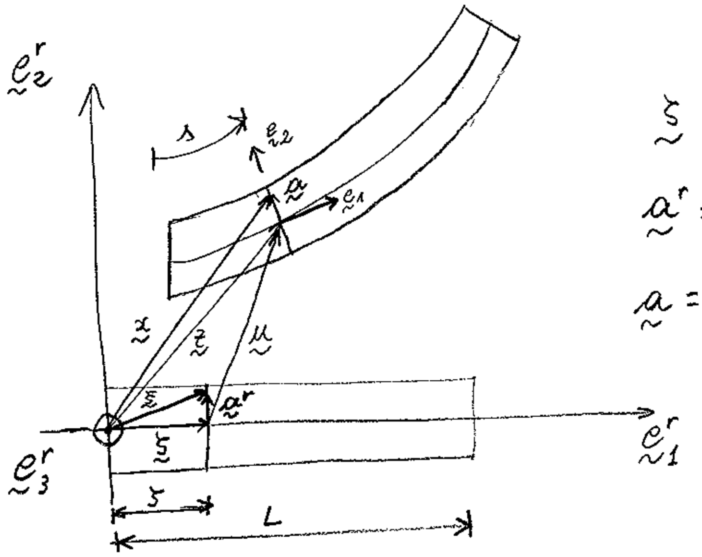
Teoria de barras geometricamente exactas no plano



$$\underset{\sim}{\zeta} = \zeta\, \underset{\sim}{e}_1^r$$

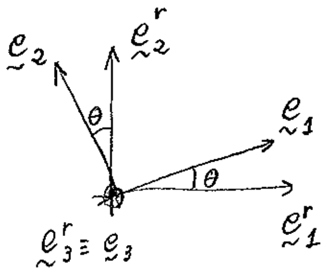
$$\underset{\sim}{a}^r = \xi\, \underset{\sim}{e}_2^r$$

$$\underset{\sim}{a} = \xi\, \underset{\sim}{e}_2$$

$\underset{\sim}{e}_1$ não coincide, em geral, com a tangente ao eixo da peça.

$\underset{\sim}{e}_2$ coincide com a posição da secção transversal.

Rotação entre eixos



$$\underset{\sim}{a} = \underline{Q}\, \underset{\sim}{a}^r \quad \text{onde} \quad \underline{Q} = \begin{bmatrix} c\theta & -s\theta & \cdot \\ +s\theta & c\theta & \cdot \\ \cdot & \cdot & 1 \end{bmatrix}$$

Geometria da barra

$$\underset{\sim}{\xi} = \underset{\sim}{\zeta} + \underset{\sim}{a}^r = \zeta\, \underset{\sim}{e}_1^r + \xi\, \underset{\sim}{e}_2^r$$

onde $0 < \zeta < L$.

ζ é o parâmetro da linha formada pelos centros de rigidez da secção transversal. Este parâmetro é o "arc-lenght", ou seja, o comprimento da linha.

Posição actual

$$\underset{\sim}{z} = \underset{\sim}{\zeta} + \underset{\sim}{u}$$

$$\underset{\sim}{x} = \underset{\sim}{z} + \underset{\sim}{a} \qquad \underset{\sim}{a} = \xi\, \underset{\sim}{e}_2 = \xi\, \underline{Q}\, \underset{\sim}{e}_2^r$$

logo

$$\underset{\sim}{x} = \underset{\sim}{\zeta} + \underset{\sim}{u} + \underset{\sim}{a} \overset{(=)}{} \begin{Bmatrix} x_1(\zeta) \\ x_2(\zeta) \\ x_3(\zeta) \end{Bmatrix} = \begin{Bmatrix} \zeta \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} u_1(\zeta) \\ u_2(\zeta) \\ 0 \end{Bmatrix} + \xi \begin{Bmatrix} -s\theta \\ c\theta \\ 0 \end{Bmatrix}$$

Gradiente de deformação

$$\underline{F} = \frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{\xi}} = \frac{\partial x_i}{\partial \xi_j}\, \underset{\sim}{e}^r_i \otimes \underset{\sim}{e}^r_j = \frac{\partial \underset{\sim}{x}}{\partial \xi_i} \otimes \underset{\sim}{e}^r_i = \frac{\partial \underset{\sim}{x}}{\partial \zeta} \otimes \underset{\sim}{e}^r_1 + \frac{\partial \underset{\sim}{x}}{\partial \xi} \otimes \underset{\sim}{e}^r_2$$

$$\frac{\partial \underset{\sim}{x}}{\partial \zeta} = \frac{\partial}{\partial \zeta}\left(\underset{\sim}{\zeta} + \underset{\sim}{u} + \underset{\sim}{a}\right) = \underset{\sim}{e}^r_1 + \underset{\sim}{u}' + \frac{\partial}{\partial \zeta}\left(\underline{Q}\,\underset{\sim}{a}^r\right) = \underbrace{\underset{\sim}{e}^r_1 + \underset{\sim}{u}'}_{=\underset{\sim}{z}'} + \underbrace{\underline{Q}'\,\underline{Q}^T}_{\underline{K}}\overbrace{\underset{\sim}{a}}^{\underset{\sim}{a}^r}$$

Nota: $(\ )' = \dfrac{d}{d\zeta}(\ )$

$$\underline{Q}' = \frac{\partial}{\partial \zeta}\left(\underline{Q}(\theta(\zeta))\right) = \frac{\partial \underline{Q}(\theta)}{\partial \theta}\,\frac{\partial \theta}{\partial \zeta} = \begin{bmatrix} -s\theta & -c\theta & \cdot \\ +c\theta & -s\theta & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}\theta'$$

$$\underline{K} = \underline{Q}'\,\underline{Q}^T = \theta' \begin{bmatrix} -s\theta & -c\theta & \cdot \\ +c\theta & -s\theta & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}\begin{bmatrix} c\theta & +s\theta & \cdot \\ -s\theta & c\theta & \cdot \\ \cdot & \cdot & 1 \end{bmatrix} = \begin{bmatrix} \cdot & -1 & \cdot \\ +1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}\theta'$$

$$\underset{\sim}{\kappa} = \operatorname{axial}(\underline{K}) = +\theta'\,\underset{\sim}{e}^r_3$$

$$\frac{\partial \underset{\sim}{x}}{\partial \xi} = \frac{\partial}{\partial \xi}\left(\underset{\sim}{\zeta} + \underset{\sim}{u} + \underset{\sim}{a}\right) = \frac{\partial \underset{\sim}{a}}{\partial \xi} = \frac{\partial}{\partial \xi}\left(\xi\, \underset{\sim}{e}_2\right) = \underset{\sim}{e}_2$$

Assim,

$$\begin{aligned}
\underline{F} &= (\underbrace{\underset{\sim}{e}^r_1 + \underset{\sim}{u}'}_{=\underset{\sim}{z}'} + \underset{\sim}{\kappa} \times \underset{\sim}{a}) \otimes \underset{\sim}{e}^r_1 + \underset{\sim}{e}_2 \otimes \underset{\sim}{e}^r_2 \\
&= \underline{Q}\,\underset{\sim}{e}^r_2 \otimes \underset{\sim}{e}^r_2 + (\underset{\sim}{z}' - \underset{\sim}{e}_1 + \underset{\sim}{e}_1 + \underset{\sim}{\kappa} \times \underset{\sim}{a}) \otimes \underset{\sim}{e}^r_1 \\
&= \underline{Q}\,(\underset{\sim}{e}^r_1 \otimes \underset{\sim}{e}^r_1 + \underset{\sim}{e}^r_2 \otimes \underset{\sim}{e}^r_2) + (\underbrace{\underset{\sim}{z}' - \underset{\sim}{e}_1}_{=\underset{\sim}{\eta}} + \underset{\sim}{\kappa} \times \underset{\sim}{a}) \otimes \underset{\sim}{e}^r_1 \\
&= \underline{Q}\,(\underset{\sim}{e}^r_\alpha \otimes \underset{\sim}{e}^r_\alpha) + \underbrace{(\underset{\sim}{\eta} + \underset{\sim}{\kappa} \times \underset{\sim}{a})}_{=\underset{\sim}{\gamma}} \otimes \underset{\sim}{e}^r_1
\end{aligned}$$

Em componentes,

$$[\underline{F}] = \begin{bmatrix} c\theta & -s\theta & \cdot \\ s\theta & c\theta & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} + \begin{bmatrix} 1 + u'_1 - c\theta & \cdot & \cdot \\ + u'_2 - s\theta & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} + \xi\theta' \begin{bmatrix} -c\theta & \cdot & \cdot \\ -s\theta & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

$$\underbrace{}_{\underline{Q}\,(\boldsymbol{e}^r_\alpha \otimes \boldsymbol{e}^r_\alpha)} \qquad \underbrace{}_{\boldsymbol{\eta} \otimes \boldsymbol{e}^r_1} \qquad \underbrace{}_{(\boldsymbol{K} \times \boldsymbol{a}) \otimes \boldsymbol{e}^r_1}$$

$$\boldsymbol{\eta} = \boldsymbol{z}' - \boldsymbol{e}_1 = \boldsymbol{e}^r_1 - \boldsymbol{u}' - \boldsymbol{e}_1 = \boldsymbol{e}^r_1 - \boldsymbol{u}' - \underline{Q}\,\boldsymbol{e}^r_1$$

$$\boldsymbol{K} \times \boldsymbol{a} = \underline{K}\,\underline{Q}\,\boldsymbol{a}^r = \underline{K}\,\underline{Q}\,\boldsymbol{e}^r_2\,\xi = \theta' \begin{bmatrix} \cdot & -1 & \cdot \\ 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{Bmatrix} -s\theta \\ c\theta \\ \cdot \end{Bmatrix} \xi = \xi\theta' \begin{Bmatrix} -c\theta \\ -s\theta \\ \cdot \end{Bmatrix}$$

$$\underline{F}^r = \underline{Q}^T \underline{F} = \underbrace{\underline{Q}^T \underline{Q}}_{=\underline{I}} (\boldsymbol{e}^r_\alpha \otimes \boldsymbol{e}^r_\alpha) + \underbrace{\Big(\underbrace{\underline{Q}^T \boldsymbol{\eta}}_{=\boldsymbol{\eta}^r} + \big(\underbrace{\underline{Q}^T \boldsymbol{K}}_{=\boldsymbol{K}^r}\big) \times \big(\underbrace{\underline{Q}^T \boldsymbol{a}}_{\boldsymbol{a}^r}\big)\Big)}_{\boldsymbol{\gamma}^r} \otimes \boldsymbol{e}^r_1 =$$

$$= \boldsymbol{e}^r_\alpha \otimes \boldsymbol{e}^r_\alpha + \boldsymbol{\gamma}^r \otimes \boldsymbol{e}^r_1$$

$$\boldsymbol{\eta}^r = \underline{Q}^T \boldsymbol{z}' - \boldsymbol{e}^r_1 = \underline{Q}^T(\boldsymbol{e}^r_1 + \boldsymbol{u}') - \boldsymbol{e}^r_1 = \begin{Bmatrix} c\theta + c\theta u'_1 + s\theta u'_2 - 1 \\ -s\theta - s\theta u'_1 + c\theta u'_2 \\ 0 \end{Bmatrix}$$

$$\boldsymbol{K}^r = \underline{Q}^T \boldsymbol{K} = \underline{Q}^T \theta' \boldsymbol{e}^r_3 = \theta' \boldsymbol{e}^r_3 = \begin{Bmatrix} 0 \\ 0 \\ \theta' \end{Bmatrix}$$

Em componentes,

$$[\underline{F}^r] = \begin{bmatrix} 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} + \begin{bmatrix} c\theta + c\theta u'_1 + s\theta u'_2 - 1 & \cdot & \cdot \\ -s\theta - s\theta u'_1 + c\theta u'_2 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} + \xi\theta' \begin{bmatrix} -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

Assim,

$$\boldsymbol{\varepsilon}^r = \begin{Bmatrix} \varepsilon \\ \gamma \\ \chi \end{Bmatrix} = \begin{Bmatrix} (1 + u'_1)\,c\theta + u'_2\,s\theta - 1 \\ u'_2\,c\theta - (1 + u'_1)\,s\theta \\ \theta' \end{Bmatrix}$$

(medidas objectivas de deformação que coincidem com as de Reissner e de Wriggers)

O trabalho virtual interior é dado por:

$$\delta w^{int} = \int_0^L \underline{P} : \delta \underline{F} \, d\zeta = \int_0^L \underset{\sim}{\sigma}^r \cdot \delta \underset{\sim}{\varepsilon}^r \, d\zeta$$

onde $\underset{\sim}{\sigma}^r = \begin{Bmatrix} N \\ V \\ M \end{Bmatrix}$ e $\delta \underset{\sim}{\varepsilon}^r = \begin{Bmatrix} \delta \varepsilon \\ \delta \gamma \\ \delta \chi \end{Bmatrix}$ } Todas estas grandezas são materias, as únicas que realmente interessam.

As variações das deformações generalizadas são dadas por:

$$\delta(\varepsilon) = \delta\Big((1+u_1')\,c\theta + u_2'\,s\theta - 1\Big) = \delta u_1'\,c\theta + (1+u_1')(-s\theta)\,\delta\theta + \delta u_2'\,s\theta + u_2'\,c\theta\,\delta\theta$$

$$\delta(\gamma) = \delta\big(u_2'\,c\theta - (1+u_1')\,s\theta\big) = \delta u_2'\,c\theta + u_2'(-s\theta)\,\delta\theta - \delta u_1'\,s\theta - (1+u_1')\,c\theta\,\delta\theta$$

$$\delta(\chi) = \delta(\theta') = \delta\theta'$$

ou seja,

$$\underset{(3\times1)}{\delta \underset{\sim}{\varepsilon}^r} = \underset{(3\times4)}{\underline{\Psi}}\ \underset{(4\times3)}{\underline{\Delta}}\ \underset{(3\times1)}{\delta \underset{\sim}{d}} =$$

$$= \begin{bmatrix} c\theta & s\theta & \cdot & -(1+u_1')s\theta + u_2'c\theta \\ -s\theta & c\theta & \cdot & -(1+u_1')c\theta - u_2's\theta \\ \cdot & \cdot & 1 & \cdot \end{bmatrix} \underbrace{\begin{bmatrix} \partial/\partial\zeta & \cdot & \cdot \\ \cdot & \partial/\partial\zeta & \cdot \\ \cdot & \cdot & \partial/\partial\zeta \\ \cdot & \cdot & 1 \end{bmatrix} \begin{Bmatrix} \delta u_1 \\ \delta u_2 \\ \delta\theta \end{Bmatrix}}_{= \begin{Bmatrix} \delta u_1' \\ \delta u_2' \\ \delta\theta' \\ \delta\theta \end{Bmatrix}}$$

Assim,

$$\delta w^{int} = \int_0^L \delta \underset{\sim}{\varepsilon}^r \cdot \underset{\sim}{\sigma}^r(\underset{\sim}{\varepsilon}^r)\, d\zeta = \int_0^L (\underline{\Delta}\,\delta \underset{\sim}{d})^T\, \underline{\Psi}^T\, \underset{\sim}{\sigma}^r(\underset{\sim}{\varepsilon}^r)\, d\zeta$$

A matriz de rigidez tangente é dada por

$$\Delta\delta W^{int} = \int_0^L \Delta\underset{\sim}{\sigma}^r \cdot \delta\underset{\sim}{\varepsilon}^r \, d\xi + \int_0^L \underset{\sim}{\sigma}^r \cdot \Delta\delta\underset{\sim}{\varepsilon}^r \, d\xi$$

Para o caso de um material elástico linear, $\underset{\sim}{\sigma}^r(\underset{\sim}{\varepsilon}^r) = \underline{D}\,\underset{\sim}{\varepsilon}^r$ e

$$\Delta\underset{\sim}{\sigma}^r \cdot \delta\underset{\sim}{\varepsilon}^r = (\underline{D}\Delta\underset{\sim}{\varepsilon}^r) \cdot \delta\underset{\sim}{\varepsilon}^r \qquad \text{onde} \quad \underline{D} = \frac{\partial\underset{\sim}{\sigma}^r}{\partial\underset{\sim}{\varepsilon}^r}$$

logo

$$\int_0^L \Delta\underset{\sim}{\sigma}^r \cdot \delta\underset{\sim}{\varepsilon}^r \, d\xi = \int_0^L (\underline{\Delta}\,\delta\underset{\sim}{d})^T \underline{\Psi}^T \underline{D}\, \underline{\Psi}\, (\underline{\Delta}\,\Delta\underset{\sim}{d})\, d\xi$$

$$\underset{\sim}{\sigma}^r \cdot \Delta\delta\underset{\sim}{\varepsilon}^r = N\,\Delta\delta\varepsilon + V\,\Delta\delta\gamma + M\,\Delta\delta\chi$$

onde

$$\Delta\delta\varepsilon = \Delta\left(\delta u_1' C\theta + (1+u_1')(-S\theta)\delta\theta + \delta u_2' S\theta + u_2' C\theta\,\delta\theta\right) =$$

$$= \delta u_1'(-S\theta)\Delta\theta + (1+u_1')(-C\theta)\delta\theta\Delta\theta + \Delta u_1'(-S\theta)\delta\theta + \delta u_2' C\theta\Delta\theta +$$

$$+ \Delta u_2' C\theta\,\delta\theta + u_2'(-S\theta)\delta\theta\Delta\theta$$

$$\Delta\delta\gamma = \Delta\left(\delta u_2' C\theta + u_2'(-S\theta)\delta\theta - \delta u_1' S\theta - (1+u_1')C\theta\,\delta\theta\right) =$$

$$= \delta u_2'(-S\theta)\Delta\theta + \Delta u_2'(-S\theta)\delta\theta + u_2'(-C\theta)\delta\theta\Delta\theta - \delta u_1' C\theta\Delta\theta - \Delta u_1' C\theta\delta\theta -$$

$$- (1+u_1')(-S\theta)\delta\theta\Delta\theta$$

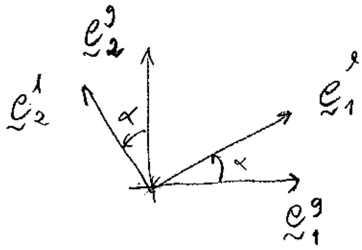
$\Delta\delta\kappa = \Delta(\delta\theta') = 0$ (neste caso o termo $\underset{\sim}{\sigma}^r \cdot \Delta\delta\underset{\sim}{\varepsilon}^r$ não depende de M).

Assim,

$$\underset{\sim}{\sigma}^r \cdot \Delta \delta \underset{\sim}{\varepsilon}^r = (\Delta\delta \underset{\sim}{d}) \cdot \underline{G}(\Delta\, \Delta \underset{\sim}{d}) =$$

$$= \begin{Bmatrix} \delta u'_1 \\ \delta u'_2 \\ \delta \theta' \\ \delta \theta \end{Bmatrix} \cdot \begin{bmatrix} \cdot & \cdot & \cdot & -NS\theta - VC\theta \\ \cdot & \cdot & \cdot & NC\theta - VS\theta \\ \cdot & \cdot & \cdot & \cdot \\ -NS\theta - VC\theta & NC\theta - VS\theta & \cdot & (1+u'_1)(VS\theta - NC\theta) - u'_2(NS\theta + VC\theta) \end{bmatrix} \begin{Bmatrix} \Delta u'_1 \\ \Delta u'_2 \\ \Delta \theta' \\ \Delta \theta \end{Bmatrix}$$

Rotação de vectores entre o referencial local e global



$$\begin{Bmatrix} u_1 \\ u_2 \\ \theta \end{Bmatrix}^l = \begin{bmatrix} C\alpha & S\alpha & \cdot \\ -S\alpha & C\alpha & \cdot \\ \cdot & \cdot & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \theta \end{Bmatrix}^g$$

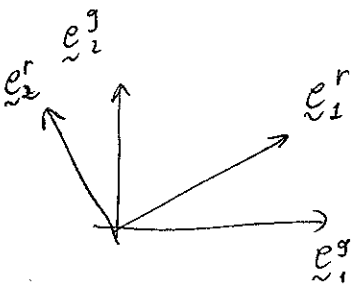
$$\underset{\sim}{u}^l_I = \overline{\underline{T}}_I \, \underset{\sim}{u}^g_I$$

$$\underset{\sim}{R}^g_I = \overline{\underline{T}}_I^T \underset{\sim}{R}^l_I$$

$$\underline{K}^g_I = \overline{\underline{T}}_I^T \, \underline{K}^l_I \, \overline{\underline{T}}_I$$

É mais simples avaliar

$$\begin{Bmatrix} u_1 \\ u_2 \\ \theta \end{Bmatrix}^g = \begin{bmatrix} \overbrace{C\alpha}^{e_1^r} & \overbrace{-S\alpha}^{e_2^r} & \cdot \\ S\alpha & C\alpha & \cdot \\ \cdot & \cdot & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \theta \end{Bmatrix}^l$$



$\underset{\sim}{e}^r_1$ é o vector tangente unitário segundo o eixo do elemento na posição indeformada.

O trabalho virtual exterior é dado por

$$\delta W^{ext} = \int_0^L \delta \underset{\sim}{d}^T (\lambda \underset{\sim}{f}) \, d\xi$$

Discretização

Sendo $\underset{\sim}{d} = \underline{N}\,\underset{\sim}{\tilde{d}}$, vem $\delta \underset{\sim}{d} = \underline{N}\,\delta \underset{\sim}{\tilde{d}}$ e $\Delta \underset{\sim}{d} = \underline{N}\,\Delta \underset{\sim}{\tilde{d}}$.

Assim,

$$\delta w^{int} = \int_0^L (\underline{\Delta}\,\delta \underset{\sim}{d})^T \underline{\Psi}^T \underset{\sim}{\sigma}^r(\underset{\sim}{\varepsilon}^r)\, d\xi = \delta \underset{\sim}{\tilde{d}}^T \underbrace{\int_0^L (\underline{\Delta}\,\underline{N})^T \underline{\Psi}^T \underset{\sim}{\sigma}^r(\underset{\sim}{\varepsilon}^r)\, d\xi}_{\underset{\sim}{R}^{int}} = \delta \underset{\sim}{\tilde{d}}^T \underset{\sim}{R}^{int}$$

O resíduo é dado por $\underset{\sim}{R} = \underset{\sim}{R}^{int} + \underset{\sim}{R}^{ext}$

$$\underset{\sim}{R}^{int} = \int_0^L (\underline{\Delta}\,\underline{N})^T \underline{\Psi}^T \underset{\sim}{\sigma}^r(\underset{\sim}{\varepsilon}^r)\, d\xi \quad ; \quad \underset{\sim}{R}^{ext} = \lambda \int_0^L \underline{N}^T \underset{\sim}{f}\, d\xi \quad \text{e} \quad \underset{\sim}{f} = \begin{Bmatrix} \bar{p}_1 \\ \bar{p}_2 \\ \bar{m} \end{Bmatrix}$$

A parcela geométrica de $\underline{K}$ é dada por

$$\delta w^{int} = \dots + \int_0^L (\underline{\Delta}\,\delta \underset{\sim}{d})^T \underline{G} (\underline{\Delta}\,\Delta \underset{\sim}{d})\, d\xi = \delta \underset{\sim}{\tilde{d}}^T \underbrace{\int_0^L (\underline{\Delta}\,\underline{N})^T \underline{G} (\underline{\Delta}\,\underline{N})\, d\xi}_{\underline{K}_g} \Delta \underset{\sim}{\tilde{d}} =$$

$$= \delta \underset{\sim}{\tilde{d}}^T \underline{K}_g \Delta \underset{\sim}{\tilde{d}}$$

onde $\underline{K}_g = \int_0^L (\underline{\Delta}\,\underline{N})^T \underline{G} (\underline{\Delta}\,\underline{N})\, d\xi$

A parcela material de $\underline{K}$ é dada por

$$\delta w^{int} = \dots + \int_0^L (\underline{\Delta}\,\delta \underset{\sim}{d})^T \underline{\Psi}\,\underline{D}\,\underline{\Psi} (\underline{\Delta}\,\Delta \underset{\sim}{d})\, d\xi =$$

$$= \delta \underset{\sim}{\tilde{d}}^T \underbrace{\int_0^L (\underline{\Delta}\,\underline{N})^T \underline{\Psi}\,\underline{D}\,\underline{\Psi} (\underline{\Delta}\,\underline{N})\, d\xi}_{\underline{K}_m} \Delta \underset{\sim}{\tilde{d}}$$

onde $\underline{K}_m = \int_0^L (\underline{\Delta}\,\underline{N})^T \underline{\Psi}^T \underline{D}\,\underline{\Psi} (\underline{\Delta}\,\underline{N})\, d\xi$

Assim, a matriz tangente pode assumir a forma:

$$\underline{K} = \int_0^L (\Delta \underline{N})^T (\underline{\Upsilon}^T \underline{D}\, \underline{\Upsilon} + \underline{G})(\Delta \underline{N})\, ds$$

Método de Newton

No caso de não haver deslocamentos impostos e não se pretender avaliar as reacções de apoio, tem-se:

$\tilde{d} = 0$, (deslocamento total)

$\Delta \tilde{d} = 0$ (" incremental) $\}$ (é desnecessário)

$\delta \tilde{d} = 0$ (" iterativo) $\}$

Avalia $\underline{F}$ (vector de forças nodais equivalentes final)

For istep = 1 : nstep

$\lambda = \frac{istep}{nstep}$ (parâmetro de carga no incremento istep)

Avalia $\tilde{R} = \tilde{R}^{int}(\tilde{d}) + \tilde{R}^{ext}(\lambda)$

$\Delta \tilde{d} = 0$

while $\|\tilde{R}\| > tol$

Avalia $\underline{K}(\tilde{d} + \Delta\tilde{d})$; $\tilde{R}(\tilde{d} + \Delta \tilde{d})$

$\delta \tilde{d} = \underline{K}^{-1}(\tilde{R})$

$\Delta \tilde{d} = \Delta \tilde{d} + \delta \tilde{d}$

End while

$\tilde{d} = \tilde{d} + \Delta \tilde{d}$

END FOR

Neste procedimento foi omitido o subscrito f, pois apenas é necessário avaliar os g.l. livres. Este índice aplicar-se-ia a $\tilde{d}$, $\Delta\tilde{d}$, $\delta\tilde{d}$, $\underline{F}$, $\tilde{R}$, $\tilde{R}^{int}$, $\tilde{R}^{ext}$ e $\underline{K}$.

No caso de se pretender avaliar as reacções de apoio e/ou existirem deslocamentos impostos, tem-se:

$$\tilde{\underset{\sim}{d}}_f = \underset{\sim}{0}$$

$$\underset{\sim}{T}$$

Avalia $\underset{\sim}{F} = \begin{Bmatrix} \underset{\sim}{F}_f \\ \underset{\sim}{F}_r \end{Bmatrix}$

For istep = 1 : nstep

$\lambda = \dfrac{istep}{nstep}$

$\Delta \tilde{\underset{\sim}{d}}_f = 0$;

Avalia $\underline{K}(\tilde{\underset{\sim}{d}} + \Delta \tilde{\underset{\sim}{d}}, \underset{\sim}{T} + \Delta \underset{\sim}{T})$; $\underset{\sim}{R}(\tilde{\underset{\sim}{d}} + \Delta \tilde{\underset{\sim}{d}}, \underset{\sim}{T} + \Delta \underset{\sim}{T}, \lambda)$

While $\| \underset{\sim}{R}_f \| > tol$

$\delta \tilde{\underset{\sim}{d}}_f = \underline{K}_{ff}^{-1} \left(-\left(\underset{\sim}{F}_f - \underline{K}_{fr} \tilde{\underset{\sim}{d}}_r \right) \right)$

$\Delta \tilde{\underset{\sim}{d}}_f = \Delta \tilde{\underset{\sim}{d}}_f + \delta \tilde{\underset{\sim}{d}}_f$

Avalia $\underline{K}(\tilde{\underset{\sim}{d}} + \Delta \tilde{\underset{\sim}{d}}, \underset{\sim}{T} + \Delta \underset{\sim}{T})$; $\underset{\sim}{R}(\tilde{\underset{\sim}{d}} + \Delta \tilde{\underset{\sim}{d}}, \underset{\sim}{T} + \Delta \underset{\sim}{T}, \lambda)$

END While

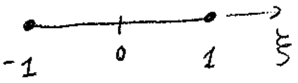
$\tilde{\underset{\sim}{d}} = \tilde{\underset{\sim}{d}} + \Delta \tilde{\underset{\sim}{d}}$

$\underset{\sim}{T} = \underset{\sim}{R}_r$

END For

Implementação numérica das integrações

As funções de aproximação são definidas no elemento mestre:



(notar o conflito de notação entre a variável $-1<\xi<1$ e $\underset{\sim}{a}^r = \xi\, \underset{\sim}{e}_2^r$)

Assim,

$$\int_0^L f(\zeta)\, d\zeta \underset{\substack{\uparrow \\ \zeta(\xi)=\sum_{i=1}^{2} N_i(\xi)\, x_i}}{=} \int_{-1}^{1} f(\zeta(\xi))\, \frac{d\zeta(\xi)}{d\xi}\, d\xi \cong \sum_{i=1}^{n_G} f(\zeta(\xi_i))\, \left.\frac{d\zeta(\xi)}{d\xi}\right|_{\xi=\xi_i} \omega_i$$

Nota: no presente caso $f(\zeta)$ é formado pelas funções $N_i(\xi)$, que já se encontram expressas em função de ξ. Assim, não se utiliza $N(\zeta(\xi))$, mas sim $N(\xi)$ directamente, pois $N(\zeta)$ não é utilizada.

Por vezes é também necessário avaliar

$$f'(\zeta) = \frac{d f(\zeta)}{d\zeta} = \frac{d f(\zeta(\xi))}{d\xi} \frac{d\xi}{d\zeta} = \frac{d f(\zeta(\xi))}{d\xi} \frac{1}{\frac{d\zeta}{d\xi}}$$

Mais uma vez, não se utiliza $\frac{dN(\zeta(\xi))}{d\xi}$, mas sim $\frac{dN(\xi)}{d\xi}$ directamente.

A teoria linear de barras de Timoshenko pode facilmente ser (6)
recuperada a partir da presente formulação. As deformações seriam
dadas por:

$$\underset{\sim}{\varepsilon}^r(\underset{\sim}{d})\Big|_{\underset{\sim}{d}=\underset{\sim}{0}} \cong \underset{\sim}{\varepsilon}^r(\underset{\sim}{0}) + \frac{\partial \underset{\sim}{\varepsilon}^r}{\partial \underset{\sim}{d}}\Big|_{\underset{\sim}{d}=\underset{\sim}{0}} \cdot \underset{\sim}{d} = \begin{Bmatrix} u_1' \\ u_2' \\ \theta' \end{Bmatrix} + \begin{bmatrix} \cdot & \cdot & u_2' \\ \cdot & \cdot & -(1+u_1') \\ \cdot & \cdot & \cdot \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \theta \end{Bmatrix} =$$

$$= \begin{Bmatrix} u_1' + \theta u_2' \\ u_2' - (1+u_1')\theta \\ \theta' \end{Bmatrix} = \begin{Bmatrix} u_1' + \underline{\underline{\theta u_2'}} \\ u_2' - \theta - \underline{\underline{u_1'\theta}} \\ \theta' \end{Bmatrix}$$

Notando que os termos a sublinhado são produtos de
deslocamentos, logo não lineares, então estes devem ser eliminados.

Assim,

$$\underset{\sim}{\varepsilon}^r_{linear} = \begin{Bmatrix} u_1' \\ u_2' - \theta \\ \theta' \end{Bmatrix}$$