HYBRID STRESS ELEMENTS FOR CONTINUUM DAMAGE ANALYSIS

J.A. TEIXEIRA DE FREITAS AND L.M.S. CASTRO
Departamento de Engenharia Civil e Arquitectura
Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisboa, Portugal

ABSTRACT: The stress model of the hybrid finite element formulation is formulated for the damage analysis of quasi-static structural problems. Simple local and non-local continuum damage models are used to support the development of the alternative hybrid-mixed, hybrid and hybrid-Trefftz finite element formulations. They are derived directly from the relevant fundamental structural conditions, namely the equilibrium and compatibility conditions and the constitutive relations. The finite element solving system for the finite step incremental analysis is encoded as a recursive sequence of symmetric parametric linear complementarity problems (SPLCP), which can be solved using a direct extension of the restricted basis linear programming algorithm.

KEY WORDS: Hybrid stress elements, continuum damage.

INTRODUCTION
The stress model of the hybrid-mixed formulation and the displacement model of the hybrid and hybrid-Trefftz formulations of the finite element method [1] have been recently applied to the analysis of concrete structures using isotropic elastic continuum damage models [2].

The limitations that are reported there on the performance of both the stress model and the Trefftz formulation motivated a reassessment based on the alternative description of the constitutive relations presented here, which derives directly from the application of the hybrid to the elastoplastic analysis of structures using both local and gradient-dependent constitutive relations [3,4].

A (finite) incremental formulation is used here, which is based on a (internal) time series expansion to exploit the fact that the loading programme is quasi-static. Moreover, small strains and small displacements are assumed, as the nonlinearity of the response is constrained to the material properties. Although different damage models are assessed in Ref. [1], the reassessment addressed here contemplates only a local and a gradient-dependent isotropic elastic model.

The report is organized in seven sections. The time discretization procedure is recalled first and used to establish the kinematically linear description of the local domain and boundary equilibrium and compatibility conditions. The local damage model of the constitutive relation
is stated next and extended to the gradient-dependent variant. The report closes with the presentation of the stress model of the alternative hybrid-mixed, hybrid and hybrid-Trefftz finite element formulations, and a brief comment on their numerical implementation.

## TIME DISCRETIZATION

Let \( x \) and \( t \) represent the space and (internal) time frames used to describe the response of the structure. Assume that the structural variables, say variable \( v \), are written in finite incremental form and that their increments are expanded in a Taylor time series:

\[
\Delta v(x,t) = \sum_{n=0}^{\infty} \frac{\Delta t^n}{n!} v^{(n)}(x,t)
\]

Treating similarly the general transformation (3) governing the response of the structure and equating next the same order terms in the series expansion (1), this transformation generates the infinite sequence, \( n = 1, 2, 3, \ldots \), of asymptotic approximations described by equation (4):

\[
L_n(x,t,v) v(x,t) = s(x,t,v) \quad \text{(3)}
\]

\[
L_n(x,t,v) v^{(n)}(x,t) = s^{(n)}(x,t,v) + R^{(n)}(x,t,v,v^{(n)}) \quad \text{(4)}
\]

The sequence of systems (4) is linear when the transformation operator \( L_n \) and the stipulation vector \( s^{(n)} \) depend on the state variables at the instant the time increment is implemented, and recursive when the \( n \)-th order residual vector \( R_n \) depends on variables of order lower than the \( n \)-th:

\[
R^{(n)} = R^{(n)}(x,t,v,v^{(0)}, \ldots, v^{(n-1)}) \quad \text{(5)}
\]

When transformation (3) is linear, the transformation operator is invariant, \( L_n = L \), and the \( n \)-th order residual vector is the null vector, \( R^{(n)} = 0 \). It is noted that the first-order residual vector is also the null vector, \( \hat{R} = 0 \) when the transformation is non-linear.

The incremental solution procedure used here consists in exploring the knowledge of the state of stress and strain of the structure at a given instant, \( t \), to support the implementation of the time series and thus model the structural response during the ensuing finite time increment, \( \Delta t \), using the solving system in form (4). The state of stress and strain of the structure at instant \( t + \Delta t \) is recovered using equations (1) and (2).
It is assumed that the time series expansion (2) is convergent. The series is truncated into form (6) and, for a pre-selected precision $\tau_j$, the time increment is bounded by condition (7) on the highest order term used in the approximation of variable $v_j$:

$$\Delta v(x,t) = \sum_{n=1}^{k} v^{(n)} \frac{\Delta t^n}{n!}$$  \hspace{1cm} (6)

$$|v^{(k)}_j| \frac{\Delta t^k}{k!} \leq \tau_j$$  \hspace{1cm} (7)

EQUILIBRIUM AND COMPATIBILITY CONDITIONS

Assume that the structure under analysis is discretized into elements with domain $V$ and boundary $\Gamma$, and let the domain of a critical element, wherein damage may develop, be discretized into plastic cells with domain $V_d$ and boundary $\Gamma_d$.

After applying the time discretization procedure described above, the description that is found for the equilibrium and compatibility conditions is summarized in Table 1.

<table>
<thead>
<tr>
<th>Equilibrium</th>
<th>Compatibility</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D \sigma^{(n)} + b^{(n)} = \theta$ in $V$</td>
<td>$\varepsilon^{(n)} = D^* u^{(n)}$ in $V$</td>
</tr>
<tr>
<td>$N \sigma^{(n)} = \bar{T}^{(n)}$ on $\Gamma_N$</td>
<td>$u^{(n)} = \bar{u}^{(n)}$ on $\Gamma_D$</td>
</tr>
</tbody>
</table>

In the domain equilibrium and compatibility conditions (8) and (9), vectors $\sigma$ and $\varepsilon$ collect the independent components of the stress and strain tensors, respectively, $u$ and $b$ are the displacement and body force vectors, and the differential equilibrium and compatibility operators $D$ and $D^*$ are linear and adjoint, under the assumption of geometrically linear analysis.

In the Neumann condition (10), vector $\bar{T}$ defines the tractions (or Cauchy stresses) prescribed on the static boundary $\Gamma_t$ in equilibrium with the stress field. The boundary equilibrium matrix $N$ collects the components of the unit outward normal vector associated with the differential operators present in the domain equilibrium matrix $D$. In the Dirichlet condition (11), vector $\bar{u}$ defines the displacements prescribed on the complementary, kinematic boundary, $\Gamma_u$, of the element. Mixed boundary conditions are assumed to be accounted for in the usual notation for geometric complementarity: $\Gamma = \Gamma_N \cup \Gamma_D$, $\emptyset = \Gamma_N \cap \Gamma_D$.  

3
LOCAL CONSTITUTIVE RELATIONS

The simplest, single-parameter local damage model is used here, e.g. Ref. [5]. This model is recalled first and specialized next to specific damage conditions.

Definitions

The stress-strain relation is stated in form,

$$\sigma = (1 - \lambda) k \varepsilon \quad \text{in } V$$  \hspace{1cm} (12)$$

where $0 \leq \lambda \leq 1$ is the damage parameter, and $k$ is the local stiffness matrix of the bulk material, to yield the following expanded form (4):

$$\sigma^{(n)} = k \left[ (1 - \lambda) \varepsilon^{(n)} - \varepsilon^{(n)} \lambda^{(n)} - R^{(n)} \right] \quad \text{in } V$$  \hspace{1cm} (13)$$

$$\dot{R} = 0; \quad \ddot{R} = 2 \ddot{\varepsilon} + \dot{\varepsilon}; \quad \dddot{R} = 3(\dddot{\varepsilon} + \dddot{\varepsilon}) \ldots$$

Similarly to the modelling of elastoplastic responses, it is convenient to separate the deformation vector into ‘elastic’ and ‘damage’ components,

$$\varepsilon = \varepsilon^{e} + \varepsilon^{d} \quad \text{in } V$$  \hspace{1cm} (14)$$

to yield the following expressions for the ‘elastic’ constitutive relation,

$$\sigma^{(n)} = k \varepsilon^{(n)} \quad \text{in } V$$  \hspace{1cm} (15)$$

and for the ‘damage’ component of the deformation vector:

$$\varepsilon^{(n)} = (1 - \lambda)^{-1} \left( \varepsilon \lambda^{(n)} + \lambda^{(n)} \varepsilon^{(n)} + R^{(n)} \right) \quad \text{in } V$$  \hspace{1cm} (16)$$

According to the model proposed in [6], the dissipation potential is defined as follows,

$$\varphi = \frac{1}{2} \varepsilon^{T} k e - \kappa \ln \left( \frac{c}{1 - \lambda} \right) \leq 0 \quad \text{in } V$$  \hspace{1cm} (17)$$

where $m, c$, and $\kappa$ are material constants. Its incremental form is,

$$\varphi^{(n)} = \varepsilon^{T} k e^{(n)} - \frac{m \kappa}{1 - \lambda} \ln \left( \frac{c}{1 - \lambda} \right) \lambda^{(n)} + R^{(n)} \quad \text{in } V$$  \hspace{1cm} (18)$$

and the expression for the residual term, assuming $m=2$ for simplicity, is:

$$\dot{R} = 0; \quad \ddot{R} = \dot{\varepsilon}^{T} k \dot{e} - \frac{2 \kappa}{(1 - \lambda)^{2}} \left[ \ln \left( \frac{c}{1 - \lambda} \right) + 1 \right] \dot{\lambda}^{2};$$

$$\dddot{R} = 3 \dot{\varepsilon}^{T} k \dot{e} - \frac{6 \kappa}{(1 - \lambda)^{3}} \left[ \ln \left( \frac{c}{1 - \lambda} \right) + 1 \right] \ddot{\lambda} \dot{\lambda} - \frac{4 \kappa}{(1 - \lambda)^{3}} \left[ \ln \left( \frac{c}{1 - \lambda} \right) + 1.5 \right] \dddot{\lambda}; \quad \ldots$$  \hspace{1cm} (19)$$

Inactive damage

The damage process is inactive when the dissipation potential is locally negative. Its gradients are unconstrained and the gradients of the damage parameter are null, as stated below, where symbol $^\sim$ is used to define a sign-unrestricted variable:
The increment on the ‘damage’ addend of the deformation vector is null,
\[ \varepsilon^{e(n)} = 0 \] (22)
according to definition (16), and the material point responds elastically, according to equation (15), which is now written in the (damaged) flexibility format, with \( f = k^{-1} \):
\[ \varepsilon^{(n)} = f \sigma^{(n)} \] (23)

The step that exposes the activation of damage is determined by the following condition, according to the incremental definitions (1) and (6):
\[ \varphi + \sum_{n=1}^{k} \varphi^{(n)} \frac{\Delta t^{n}}{n!} = 0 \] (24)

**Active damage**

The damage process is inactive when the dissipation potential is locally null. It remains active during increment \( \Delta t \) if its gradients are null and the gradient of the damage parameter is non-negative:
\[ \varphi = 0 \quad \text{and} \quad \varphi^{(n)} = 0 \] (25)
\[ 0 \leq \lambda < 1 \quad \text{and} \quad \dot{\lambda} \geq 0 \] (26)

Definition (23) still holds for the ‘elastic’ addend of the deformation vector, and its ‘damage’ addend is determined from definition (16), written in form,
\[ \varepsilon^{d(n)} = n \lambda^{(n)} + f, \sigma^{(n)} + R^{(n)} \] (27)
where the outward normal vector to the ‘damage surface’ is defined by,
\[ n = (1 - \lambda)^{-1} \varepsilon \] (28)
and:
\[ f = (1 - \lambda)^{-1} \lambda f \] (29)
\[ R^{(n)} = (1 - \lambda)^{-1} R^{(n)} \] (30)

Definitions (14), (23) and (27) yield the following expression for the dissipation potential variation (18),
\[ \varphi^{(n)} = n^{T} \sigma^{(n)} + h \lambda^{(n)} + R^{(n)} \] (31)
where the damage ‘hardening’ parameter is defined by,
\[ h = (1 - \lambda)^{-1} \left[ \varepsilon^{T} k \varepsilon - m \kappa \ln \left( \frac{c}{1 - \lambda} \right) \right] \] (32)
Internal Report, DECivil, 2008

and:

\[ R^{(n)}_\varphi = \varepsilon^T k R^{(n)}_e + R^{(n)} \]  

(33)

The damage process deactivates instantaneously if,

\[ \dot{\lambda} < 0 \]  

(34)

and the step that exposes a delayed deactivation is defined by the following conditions:

\[ \sum_{n=1}^{k} \lambda^{(n)} \frac{\Delta t^{n-1}}{(n-1)!} = 0 \quad \text{if } \dot{\lambda} > 0 \]  

(35)

\[ \sum_{n=2}^{k} \lambda^{(n)} \frac{\Delta t^{n-2}}{(n-2)!} = 0 \quad \text{if } \dot{\lambda} = 0 \text{ and } \ddot{\lambda} > 0 \]  

(36)

**Full damage**

Equation (13) shows that the following conditions hold,

\[ \lambda = 1 \quad \text{and} \quad \lambda^{(n)} = 0 \]  

(37)

to ensure a null variation in the stress field:

\[ \sigma^{(n)} = 0 \]  

(38)

Thus, the variation of ‘elastic’ addend of the deformation is also null,

\[ \varepsilon^{(n)}_e = 0 \]  

(39)

while the variation of its ‘damage’ addend remains undetermined.

**NONLOCAL CONSTITUTIVE RELATIONS**

The explicit gradient dependent model of Comi [7] is adopted here to illustrate the implementation of non-local damage constitutive relations. It consists simply in correcting definition (17) for the dissipation potential with a term dependent on the Laplacian of the damage parameter, in the following form, where \( c_g \) is the diffusion coefficient:

\[ \varphi = \frac{1}{2} \varepsilon^T k \varepsilon - \kappa \ln^m \left( \frac{c}{1-\lambda} \right) + c_g \nabla^2 \lambda \leq 0 \quad \text{in } V \]  

(40)

In order to support the implementation of the alternative finite element formulations presented below, it is convenient to preserve duality in the description of the static and kinematic conditions of the constitutive relations. This objective is attained by introducing auxiliary variables, namely the gradient of the damage parameter field and the corresponding flux [3,4].

The resulting expressions for the constitutive relations are summarized in Tables 2 and 3, which hold also for the local model, by setting \( c_g = 0 \) and removing the corresponding boundary conditions.
The role of the damage association condition (43) presented in Table 2, which collects also results (14) and (23), is to introduce the plastic radiation distribution $\sigma^*$ induced by the damage parameter multiplier gradient field $\gamma^*$, defined by equation (45) in Table 3. In the same table, equation (44) is the variation of the modified dissipation potential (40), with condition (25) holding for active damage.

### Table 2: Association conditions.

<table>
<thead>
<tr>
<th>Strain decomposition</th>
<th>‘Elastic’ association</th>
<th>Damage association</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon^{(n)} = \varepsilon_e^{(n)} + \varepsilon_d^{(n)}$ in $V$ (41)</td>
<td>$\varepsilon_e^{(n)} = f \sigma^{(n)}$ in $V$ (42)</td>
<td>$\sigma^* = c_g \gamma^{(n)}$ in $V_d$ (43)</td>
</tr>
</tbody>
</table>

### Table 3: Conditions for active damage.

<table>
<thead>
<tr>
<th>Static phase</th>
<th>Kinematic phase</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varphi^{(n)} = \left[ n^T \quad \nabla^T \right] \begin{bmatrix} \sigma^{(n)} \ \sigma_e^{(n)} \end{bmatrix} + h\lambda^{(n)} + R^{(n)}$ in $V_d$ (44)</td>
<td>$\left{ \begin{bmatrix} \varepsilon_d^{(n)} \ \lambda^{(n)} \end{bmatrix} = \begin{bmatrix} n_x \ \nabla \end{bmatrix} \lambda^{(n)} + \begin{bmatrix} f \sigma^{(n)} + R^{(n)} \ 0 \end{bmatrix} \right}$ in $V_d$ (45)</td>
</tr>
</tbody>
</table>

$n^T \sigma_e^{(n)} = \overline{t}^{(n)}$ on $\Gamma_{dN}$ (46) 

$\lambda^{(n)} = \overline{\lambda}^{(n)}$ on $\Gamma_{dD}$ (47)

The Neumann and Dirichlet boundary conditions (46) and (47) are consistent with the differential equations (44) and (45) characterizing the gradient-dependent damage model being implemented. Hence, in equation (46) vector $n$ denotes the unit outward normal vector to the boundary $\Gamma_d$ of the damaged domain $V_d$. On this boundary the following continuity condition is also enforced, where $t_e = n^T \sigma_e$ is the damage flux:

$$t_e^{(n)} \lambda^{(n)} = 0$$ (48)

The implementation of the algorithm presented in [3,4] can still be used to support the identification of the (moving) damage boundary $\Gamma_d$ and of the boundaries $\Gamma_{dN}$ and $\Gamma_{dD}$ whereon the Neumann and Dirichlet conditions (46) and (47) are enforced.

**HYBRID-MIXED STRESS ELEMENT**

The hybrid-mixed stress (HMS) element is derived from the direct, and independent, approximation of the stress and displacement fields in the domain of the element,

$$\sigma^{(n)} = S s^{(n)} \quad in \; V$$ (49)

$$u^{(n)} = U q^{(n)} \quad in \; V$$ (50)

and of the displacement field on its Neumann boundary,
\[ u^{(n)} = Z a^{(n)} \text{ on } \Gamma_N \] (51)

which is defined as the portion of the boundary whereon the displacements are not known. Thus, the Neumann boundary of a stress element combines its inter-element boundary with the Neumann boundary of the mesh the element may contain.

In the extension to continuum damage modelling, the damage parameter and the damage flux fields are also approximated independently, the former in the damage cells superimposed to the element, wherein damage is under processing,

\[ \lambda^{(n)} = D \lambda^{(n)} \text{ in } V_d \] (52)

and the latter on the Dirichlet boundary field,

\[ t^{(n)} = T \tau^{(n)} \text{ on } \Gamma_{dD} \] (53)

The dual transformations of approximations (49) to (51) define (free) generalized deformations,

\[ e^{(n)} = \int S^T e^{(n)} \, dV \] (54)

and (prescribed) generalized body and boundary forces:

\[ \bar{Q}^{(n)} = \int U^T b^{(n)} \, dV \] (55)

\[ \bar{p}^{(n)} = \int Z^T \bar{t}^{(n)} \, d\Gamma_N \] (56)

Similarly, the dual transformations of approximations (52) and (53) define (free) generalized dissipation potentials and (prescribed) generalized boundary damage parameters:

\[ \phi^{(n)} = \int D^T \phi^{(n)} \, dV_d \] (57)

\[ \bar{\lambda}^{(n)} = \int T^T \bar{\lambda}^{(n)} \, d\Gamma_{dD} \] (58)

It is noted that on inter-element boundaries, the ‘prescribed’ term in the definition (56) identifies with boundary forces that equilibrate the stress field approximated in the connecting element. Similarly, the ‘prescribed’ term in definition (58) in equation (58) represents the boundary value of the damage parameter approximation (52) implemented in the connecting cell.

The dual variables defined above ensure the invariance of the inner product in the finite element mapping, e.g.,

\[ s^T e = \int \sigma^T \varepsilon \, dV \] (59)

and are used to enforce on average the local conditions summarized in Tables 1 to 3. The results thus obtained are summarized in Tables 4 and 5.
Internal Report, DECivil, 2008

Table 4: Equilibrium and compatibility conditions (HMS element).

<table>
<thead>
<tr>
<th>Equilibrium</th>
<th>Compatibility</th>
<th>‘Elasticity’</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[-A \quad +B] s^{(n)} = \begin{cases} \bar{Q}^{(n)} \ \bar{p}^{(n)} \end{cases}$</td>
<td>$e^{(n)}_e + e^{(n)}_d = \begin{cases} q^{(n)} \ d^{(n)} \end{cases} + \bar{e}^{(n)}$</td>
<td>$e^{(n)}_e = F s^{(n)}$</td>
</tr>
</tbody>
</table>

(60)  
(61)  
(62)

Table 5: Finite element damage relations.

<table>
<thead>
<tr>
<th>Static phase</th>
<th>Kinematic phase</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi^{(n)} = \begin{bmatrix} N^T &amp; C^T \end{bmatrix} \begin{cases} s^{(n)} \ t^{(n)} \end{cases} + (H - G) \lambda^{(n)} + \phi^{(n)} + R_p^{(n)}$</td>
<td>$\begin{cases} e^{(n)}_d \ \lambda^{(n)} \end{cases} = \begin{bmatrix} N \ C \end{bmatrix} \lambda^{(n)} + \begin{cases} F, s^{(n)} + R_p^{(n)} \ 0 \end{cases}$</td>
</tr>
</tbody>
</table>

(63)  
(64)

(65)  
(66)  
(67)  
(68)

Compatibility equation

Definition (54) is used to enforce on average the local compatibility condition (9):

$e^{(n)} = \int S^T D^T u^{(n)} dV$

Under decomposition (41), the equation above is integrated by parts, and the resulting boundary term is uncoupled into the Neumann and Dirichlet parts:

$e^{(n)}_e + e^{(n)}_d = -\int (D S)^T u^{(n)} dV + \int (N S)^T u^{(n)} d\Gamma_N + \int (N S)^T u^{(n)} d\Gamma_D$  

(69)

The finite element compatibility equation (61) is recovered substituting above the independent domain and boundary displacement approximations (50) and (51), and enforcing the Dirichlet condition (11), to yield the following expressions for the domain and boundary compatibility matrices and for the (prescribed) generalized strains:

$A = \int (D S)^T U dV$  

(70)  

$B = \int (N S)^T Z d\Gamma_N$  

(71)  

$\bar{\epsilon}^{(n)} = \int (N S)^T \bar{u}^{(n)} d\Gamma_D$

Equilibrium equation

Definitions (55) and (56) are used to enforce on average the local domain and boundary equilibrium conditions (8) and (10), respectively:

$\int U^T D \sigma^{(n)} dV + \bar{\sigma}^{(n)} = 0$  

$\int Z^T N \sigma^{(n)} d\Gamma_N = \bar{p}^{(n)}$
The finite element equilibrium equation (60), the dual of the element compatibility condition (61), is obtained substituting above the stress approximation (49) and recalling definitions (70) and (71).

‘Elasticity’ equation
Equation (62) is obtained using the generalized strain expression (54), written for the ‘elastic’ addend, to enforce the local definition (42),

$$e^{(n)}_e = \int S^T f \sigma^{(n)} \, dV$$

for the assumed stress field (49), to yield the following definition for the element (symmetric) flexibility matrix:

$$F = \int S^T f S \, dV$$

(72)

Damage relations
Enforcement of the local definition (44) in expression (57), together with approximations (49) and (52) on the stress and damage parameter fields, yields the following definition for the generalized dissipation potential:

$$\phi^{(n)} = N^T s^{(n)} + H \lambda^{(n)} + R_p^{(n)} + \int D^T \nabla^T \sigma^{(n)} \, dV_d$$

(73)

$$N = \int S^T n \cdot D \, dV_d$$

(74)

$$H = \int D^T h \cdot D \, dV_d$$

(75)

$$R_p^{(n)} = \int D^T R_p^{(n)} \, dV_d$$

(76)

Result (63) is obtained integrating by parts the domain integral term present in equation (73),

$$\int D^T \nabla^T \sigma^{(n)} \, dV_d = \int (\nabla D)^T \sigma^{(n)} \, dV_d + \int D^T n \sigma^{(n)} \, d\Gamma_{dn} + \int D^T n \sigma^{(n)} \, d\Gamma_{dd}$$

and enforcing next the association condition (43), under definitions (45) and (52) for the gradient of the damage parameter approximation, together with the flux boundary condition (46) and the flux approximation (53), while recalling result (74), to yield:

$$G = \int (\nabla D)^T e_g \cdot (\nabla D) \, dV_d$$

(77)

$$C = \int T^T D \, d\Gamma_{dd}$$

(78)

$$\phi^{(n)} = \int D^T \bar{\sigma}^{(n)} \, d\Gamma_{dn}$$

(79)

It is noted that the finite element hardening (75) and damage gradient (77) matrices are symmetric.
The finite element description (64) of the kinematic phase of the damage process, the dual transformation of the description of its static phase (63) is obtained enforcing the local conditions (45) in definitions (54) and (58) for the generalized ‘damage’ addend of the generalized deformations and for the generalized prescribed boundary damage parameter, while enforcing the stress approximation (49) and recalling results (74) and (78), with:

\[
F_s = \int S^T f_s \, dV \tag{80}
\]

\[
R_e^{(n)} = \int S^T R_e^{(n)} \, dV_d
\]

At a given instant, \( t \), in the incremental damage process, there is a direct knowledge of the modes which are non-active, \( \phi_N < 0 \), and will be constrained to remain so, \( \phi_N + \Delta \phi_N \leq 0 \), and the modes that are currently active, \( \phi_A = 0 \), and that are expected to remain active during the increment, \( \phi_A + \Delta \phi_A = 0 \), as stated by conditions (65) to (68) in Table 5, where \( n \geq 1 \) and \( m \geq 2 \).

**Solving system**

The finite element solving system is obtained combining the equation summarized in Tables 4 and 5 to eliminate the generalized deformations as explicit variables:

\[
\begin{bmatrix}
F + F_s & A & -B & N & \cdot \\
A^T & \cdot & \cdot & \cdot & \cdot \\
-B^T & \cdot & \cdot & \cdot & \cdot \\
N^T & \cdot & H - G & C^T & \cdot \\
\cdot & \cdot & \cdot & C & \cdot \\
\end{bmatrix}
\begin{bmatrix}
s^{(n)} \\
q^{(n)} \\
d^{(n)} \\
\lambda^{(n)} \\
\tau^{(n)} \\
\end{bmatrix}
= 
\begin{bmatrix}
\bar{e}^{(n)} \\
-\bar{Q}^{(n)} \\
-\bar{p}^{(n)} \\
\phi^{(n)} \\
\bar{\lambda}^{(n)} \\
\end{bmatrix}
- 
\begin{bmatrix}
R_e^{(n)} \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\end{bmatrix}
+ 
\begin{bmatrix}
\bar{\phi}^{(n)} \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\end{bmatrix}
\tag{81}
\]

This system is linear, recursive, symmetric and highly sparse. Moreover, it is well-suited to adaptive refinement and parallel processing.

Suitability to adaptive refinement results from the option of uncoupling approximations (49) to (56) from the approximation used to describe the geometry (and topology) of the element and is strengthened by the use of generalized (non-nodal) variables. In consequence, the refinement in the approximation consists simply in adding the relevant approximation modes in approximations (49)-(56) and the corresponding rows and columns of system (81).

Suitability to parallel processing is consequent upon the fact that the variables present in this system (81) which are associated with domain approximations are strictly element- or cell-dependent, namely the generalized stresses and displacements, \( s \) and \( q \), and the generalized damage parameter, \( \lambda \), while the variables associated with boundary approximations are shared by at most two connecting elements or cells, namely the generalized boundary displacements \( d \) and the generalized damage fluxes, \( \tau \).
Linearity of system (81) is consequent upon the definitions given above for the structural matrices, which are either constant or depend on the state variables at the instance the load increment is implemented. Moreover, the systems are recursive, because the residual terms present satisfy condition (5).

System (81) is solved at every increment for the active damage modes, as identified in Table 5. The control on the time increment, $\Delta t$, is implemented as stated above for the local conditions, using now the definitions for the generalized dissipation potentials, $\phi$, and damage parameters, $\lambda$, associated with each damage mode, using the procedure defined in Ref. [4]. This procedure does not cover, however, the situation of full damage, which is analysed below in the text.

**HYBRID STRESS ELEMENTS**

The hybrid stress (HS) element is obtained by direct specialization, simply by constraining the stress approximation (49) to satisfy locally the domain equilibrium condition (8), to yield,

$$\mathbf{D} \mathbf{S} = \mathbf{O} \quad \text{in } \mathcal{V} \quad \text{(82)}$$

it being assumed, for simplicity, that the variation of the body forces is null, $\mathbf{b}^{(n)} = 0$.

Equation (69) shows clearly that the condition above renders irrelevant approximation (50) on the domain displacement field, thus reducing the finite element compatibility condition (61) to form (84) in Table 6. In addition, equation (60) simplifies to the boundary equilibrium equation (83), as the domain equilibrium condition is now locally satisfied.

<table>
<thead>
<tr>
<th>Equilibrium</th>
<th>Compatibility</th>
<th>‘Elasticity’</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbf{B} \mathbf{s}^{(n)} = \mathbf{p}^{(n)}$</td>
<td>$\mathbf{e}^{(n)} + \mathbf{d}^{(n)} = \mathbf{B} \mathbf{d}^{(n)} + \mathbf{\bar{e}}^{(n)}$</td>
<td>$\mathbf{e}^{(n)} = \mathbf{F} \mathbf{s}^{(n)}$</td>
</tr>
</tbody>
</table>

Table 6: Equilibrium and compatibility conditions (HS and HTS elements).

The finite element descriptions obtained above for the constitutive relations, equations (62) to (68), remain valid for the hybrid stress element, as they are unaffected by the local equilibrium constraint (82). The finite element solving system is obtained in the same manner, combining now the equations summarized in Tables 5 and 6, to yield:

$$
\begin{bmatrix}
\mathbf{F} + \mathbf{F}_c & -\mathbf{B} & \mathbf{N} & \cdot \\
-\mathbf{B}^T & \cdot & \cdot & \cdot \\
\mathbf{N}^T & \cdot & \mathbf{H} - \mathbf{G} & \mathbf{C}^T \\
\cdot & \cdot & \mathbf{C} & \cdot
\end{bmatrix}
\begin{bmatrix}
\mathbf{s}^{(n)} \\
\mathbf{d}^{(n)} \\
\mathbf{\lambda}^{(n)} \\
\mathbf{\tau}^{(n)}
\end{bmatrix}
=
\begin{bmatrix}
\mathbf{\bar{e}}^{(n)} \\
-\mathbf{\bar{p}}^{(n)} \\
\mathbf{\varphi}^{(n)} \\
\mathbf{\bar{\lambda}}^{(n)}
\end{bmatrix}
+
\begin{bmatrix}
\mathbf{R}^{(n)}_c \\
\cdot \\
\cdot \\
\cdot
\end{bmatrix}

\quad (86)
$$
This system preserves the properties summarized above for the HMS element solving system (81). It involves substantially less variables, the domain displacement degrees-of-freedom, $q$, and is less prone to involve spurious modes, in consequence of limiting the number of independent finite element approximations.

**HYBRID-TREFFTZ STRESS ELEMENTS**

The hybrid-Trefftz stress (HTS) element is obtained by constraining further the approximations to satisfy locally all domain conditions of the element. This so-called Trefftz constraint can be written in the following form using the results summarized in Tables 1 and 2 for active damage conditions (assuming $b^{(n)} = 0$):

$$
\begin{align*}
\left[\mathbf{D} \left[ \lambda(I - \lambda)k \right] (\mathbf{D}^* \mathbf{u}^{(n)} - n_\lambda \mathbf{\lambda}^{(n)} - R_e^{(n)}) \right] & = 0 \\
n^T \left[ \lambda(I - \lambda)k \right] (\mathbf{D}^* \mathbf{u}^{(n)} - n_\lambda \mathbf{\lambda}^{(n)} - R_e^{(n)}) + (h + c_\lambda \nabla^2) \mathbf{\lambda}^{(n)} + R_\lambda = 0
\end{align*}
\text{ in } V \quad (87)
$$

Under undamaged conditions, this system simplifies to the Navier equation:

$$
\mathbf{D} \mathbf{k} \mathbf{D}^* \mathbf{u}^{(n)} = 0 \quad \text{in } V \quad (88)
$$

The differential system of equations (87) is highly non-linear, and will not have, in general, analytical solutions. On the contrary, system (88) is linear and the sets of analytical solutions for alternative structural elements are well-established in the literature on Elasticity.

**Elastic elements**

Most of the elements in the finite element mesh respond elastically during the loading process. It is straightforward to constrain the domain approximation basis to satisfy the Trefftz condition (88), and rather convenient in terms of accuracy and rates of convergence because this basis embodies the physics of the response being modelled.

In the hybrid finite element context discussed here, this corresponds to constrain further the stress approximation (49) to satisfy not only the domain equilibrium condition (8), as stated by equation (82), but to be associated with (elastic) strain and displacement fields that also satisfy locally the domain compatibility and elasticity conditions (9) and (42).

This is equivalent to assume that there exists a (dependent) displacement approximation (50) that locally satisfies the Navier condition (88), to yield the following relations:

$$
\mathbf{D} \mathbf{k} \mathbf{D}^* \mathbf{U} = \mathbf{O} \quad (89)
$$

$$
\mathbf{S} = \mathbf{k} \mathbf{D}^* \mathbf{U} \quad (90)
$$

The elastic part of the solving system (86) holds for the hybrid-Trefftz stress element:

$$
\begin{bmatrix}
\mathbf{F} & -\mathbf{B} \\
-\mathbf{B}^T & \cdot
\end{bmatrix}
\begin{bmatrix}
\mathbf{s}^{(n)} \\
\mathbf{d}^{(n)}
\end{bmatrix}
= 
\begin{bmatrix}
\mathbf{c}^{(n)} \\
-\mathbf{p}^{(n)}
\end{bmatrix} \quad (91)
$$
The essential difference is that conditions (89) and (90) can now be used to obtain the following boundary integral expression for the element flexibility matrix (72):

\[ F = \int (N S)^T U \, d\Gamma \]  

(92)

This result is obtained inserting condition (90) in definition (72) and integrating by parts to enforce constraint (89):

\[ F = \int (k D^* U)^T D^* U \, dV = -\int (D k D^* U)^T U \, dV + \int (N k D^* U)^T U \, d\Gamma \]

Inelastic elements

The damage process may involve part or the full domain of particular elements in the finite element mesh. It is not feasible, in either case, to extend the Trefftz constraint to these situations due to the strongly non-linear nature of the supporting system (87).

One possibility is to use the Trefftz stress approximation in the implementation of the hybrid stress solving system (86). The element flexibility matrix is still defined by equation (92). The alternative is to implement either of the domain approximations, namely the stress approximation (49) or the damage parameter approximation (52) using approximate solutions of the homogeneous form of the Trefftz constraint system (87):

\[
\begin{bmatrix}
D & (I - D_j)k & (D^* U - n, D_j)
\end{bmatrix} = 0 \\
n^T [(I - D_j)k] (D^* u^{(n)} - n, D_j) + (h + c_g \nabla^2)D_j = 0
\]

(93)

IMPLEMENTATION OF THE DAMAGE MODEL

When the local damage model is used in the analysis, boundary conditions (46) and (47) are removed from the formulation and the diffusion coefficient, \( c_g \), is set to zero. It can be readily seen that, under these assumptions, the flux approximation (53) is rendered irrelevant and the finite element solving systems (81) and (86) simplify to form,

\[
\begin{bmatrix}
F + F_e & A & -B & N
\end{bmatrix}
\begin{bmatrix}
s^{(n)}
q^{(n)}
d^{(n)}
A^{(n)}
\end{bmatrix} = \begin{bmatrix}
\bar{\varepsilon}^{(n)}
-\bar{Q}^{(n)}
-\bar{p}^{(n)}
\phi^{(n)}
\end{bmatrix} - \begin{bmatrix}
R_e^{(n)}
0
0
\phi^{(n)} + R_\phi^{(n)}
\end{bmatrix}
\]

(94)

for the hybrid-mixed stress model, and, for the hybrid and hybrid-Trefftz stress models, to:

\[
\begin{bmatrix}
F + F_e & -B & N
-B^T & \cdot & \cdot & H
N^T & \cdot & H
\end{bmatrix}
\begin{bmatrix}
s^{(n)}
d^{(n)}
A^{(n)}
\end{bmatrix} = \begin{bmatrix}
\bar{\varepsilon}^{(n)}
-\bar{Q}^{(n)}
-\bar{p}^{(n)}
\phi^{(n)}
\end{bmatrix} - \begin{bmatrix}
R_e^{(n)}
0
0
\phi^{(n)} + R_\phi^{(n)}
\end{bmatrix}
\]

(95)

It is noted, also, that the simplest approximation (52) for the damage parameter field, namely the constant mode, can only be used in the implementation of the local model.
However, this approximation is inconsistent with the non-local damage model, as a consequence of the continuity condition (47). Although this model can be applied assuming a null diffusion coefficient, \( c_s = 0 \), that is, under local modelling conditions, it must be implemented using higher order bases, so designed as to include the constant field mode to represent full damage.

Full damage is the only situation that is not covered by a direct adaptation of the control rules of the plasticity relations proposed in Ref. [4]. Its control is assessed below for three alternative situations, namely quasi-fully damaged material points, and fully-damaged material points and material zones, the latter being modelled at cell level. It is recalled that definitions (44) to (47) no longer apply under full damage conditions, and should be replaced by conditions (37) to (39).

**Quasi-fully damaged material point**

Results (74) to (76) and (80), together with definitions (19), (28) to (30) and (32), expose situations of potential numerical instability when the damage parameter approaches the full damage condition, \( \lambda \to 1 \).

The simplest way to weaken these situations of quasi-singularity is to define the damage parameter modes in approximation (52) in form,

\[
D = (1 - \lambda) \bar{D}
\]

replacing thus results (74) to (80) by those presented in the Appendix. Consequent upon this definition for the approximation basis, the kernel functions of the hardening matrix are integrable under full damage conditions, namely, \( \lim_{\lambda \to 1} (1 - \lambda)^2 h = 0 \). This same property extends to the residual term (79) under the additional condition (37). The modified form of matrix (80), see Appendix, is integrable, but not necessarily real, under full damage conditions, \( \lambda = 1 \), unless the stress field at that point is null. This situation is analysed next.

**Fully-damaged material point**

When full damage develops at a particular material point, system (81) must be modified thus to include explicitly the condition of null stress at that point (or points), say \( x = \tilde{x} \),

\[
\begin{bmatrix}
F + F_c & A & -B & N & \cdot & S^T \\
A^T & \cdot & \cdot & \cdot & \cdot & \cdot \\
-B^T & \cdot & \cdot & \cdot & \cdot & \cdot \\
N^T & \cdot & H - G & C^T & \cdot & \cdot \\
\cdot & \cdot & C & \cdot & \cdot & \cdot \\
\bar{S} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{bmatrix}
\begin{bmatrix}
\tilde{s}^{(n)} \\
q^{(n)} \\
d^{(n)} \\
\lambda^{(n)} \\
\epsilon^{(n)} \\
\tilde{\epsilon}_d^{(n)}
\end{bmatrix}
= 
\begin{bmatrix}
\tilde{\tau}^{(n)} \\
-\bar{Q}^{(n)} \\
-\bar{p}^{(n)} \\
\phi^{(n)} \\
\bar{\alpha}^{(n)} \\
0
\end{bmatrix}
\begin{bmatrix}
R^{(n)}_c \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot
\end{bmatrix}
+ \begin{bmatrix}
\tilde{\phi}^{(n)} + R^{(n)}_\phi \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot
\end{bmatrix}
\tag{96}
\]
where matrix $\bar{S}$ and vector $\bar{e}_d^{(n)}$ define the state of stress (49) at the ‘damage’ addend of the deformation field at that point, extracted by a Dirac function in definition (54),

$$\bar{e}_d^{(n)} = \delta(x - \bar{x}) e_d^{(n)} \text{ in } V_d$$

and thus in finite element compatibility condition (69), which now reads:

$$e^{(n)} + e^{(n)} + \bar{e}_d^{(n)} = -\int (D S)^T u^{(n)} dV + \int (N S)^T u^{(n)} d\Gamma_N + \int (N S)^T u^{(n)} d\Gamma_D$$

Similar extensions apply directly to the hybrid and hybrid-Trefftz solving system (86), and to the versions (94) and (95) associated with the local model of damage.

**Fully-damaged material cell**

The procedure described above can be readily extended to model the effect of cells subject to full damage, *i.e.* $\lambda = 1$ in $V_d$. It consists simply in approximating directly the ‘damage’ addend of the deformation field in that cell,

$$e_d^{(n)} = E \bar{e}_d^{(n)} \text{ in } V_d$$

and using the dual transformation to recover result (96) with,

$$\bar{S} = \int E^T S dV_d$$

as the last equation in the system represents now the average enforcement of the null stress condition. Naturally, the strain approximation matrix, $E$, may still combine Dirac functions at particular locations, namely the cell Gauss points, as it is the practice in the implementation of collocation methods.

**CLOSURE**

It cannot be stated, at this stage, whether the reformulation summarized in this report will overcome the limitations reported in Ref. [2] on the implementation of stress elements and of the hybrid-Trefftz formulation in the modelling of continuum damage. Nonetheless, and prior to the numerical implementation and assessment that this report was written to support, it can be stated that specific features have been introduced to address the aforementioned limitations.

Besides preserving symmetry, the formulations presented above allow for the direct control of the continuity of the strain field, the violation of which hampered the quality of the results obtained with some of the formulations tested in Ref. [2]. This is consequent upon the fact that the ‘elastic’ addend of the deformation field is continuous in the domain of the element and the continuity across the element cells of the its ‘damage’ addend is directly enforced.
However, the major reason that justifies the investment in the numerical implementation and testing of the formulations presented here is the option of adopting an approach that, in essence, mirrors the followed in the implementation of gradient-dependent plasticity, which has proved to respond adequately both in terms of quality and stability of the results [3,4]. Special attention is given now to the only situation that distinguishes, in formal terms, the gradient-dependent plasticity model and the continuum damage model addressed here, that is, the alternative situations of full damage, in particular in what concerns the disarming of potential situations of numerical instability.

REFERENCES


APPENDIX

The simplified expressions for results (74) to (80) are the following:

\[ N = \int S^T \epsilon \vec{D} \, dV_d ; \quad H = \int (1-\lambda)^2 \vec{D}^T h \vec{D} \, dV_d \]

\[ R_p^{(n)} = \int (1-\lambda) \vec{D}^T R_p^{(n)} \, dV_d ; \quad G = \int [\nabla (1-\lambda) \vec{D}] c_g [\nabla (1-\lambda) \vec{D}] \, dV_d \]

\[ C = \int (1-\lambda) T^T \vec{D} \, d \Gamma_{dd} ; \quad \vec{\Phi}^{(n)} = \int (1-\lambda) \vec{D}^T \Gamma_{\vec{u}}^{(n)} \, d \Gamma_{dN} ; \quad F_s = \int \lambda (1-\lambda)^{-1} S^T f S \, dV_d \]