Hybrid-Trefftz displacement and stress elements for biphasic elastostatics

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\section*{Abstract}

The displacement and stress models of the hybrid-Trefftz finite element formulation are obtained from the corresponding hybrid models by restricting the domain trial functions to a solution subset of the governing Navier differential equation. This reduces the problem to the boundaries of the elements, a trademark feature of both Trefftz and boundary element methods, with the added advantage that the solving system results sparse and Hermitian, as typical of hybrid finite elements. Moreover, the approximation bases are regular, as fundamental solutions of the governing boundary value problem need not to be included. The hybrid-Trefftz displacement and stress elements are applied here to elastostatic problems defined on biphasic media (e.g. water-saturated soils). A significant battery of tests is designed to investigate the effects of both \textit{p}- and \textit{h}-refinements on the convergence of the finite element predictions to known analytic solutions, and to assess the stability of the elements when confronted with issues that typically hinder the application of conventional finite elements, such as incompressibility of the solid phase and gross mesh distortion. The results of a more complex simulation are also presented and compared with those obtained using commercial finite element software.

\textit{Keywords:} Trefftz methods, porous media, elasticity

\section{1. Introduction}

The development of the hybrid-Trefftz displacement and stress elements for static poroelasticity complements the previously reported hybrid-Trefftz stress element for spectral poroelastodynamics [1]. The mathematical model assumes an elastic solid phase fully permeated by a compressible liquid phase obeying Darcy’s law. The Biot’s theory of porous media [2] is used to formulate the governing differential equations the problem.

The hybrid-Trefftz displacement and stress models are obtained directly from the corresponding (pure) hybrid models [3] by restricting the domain approximation functions to a free-field solution of the differential equation governing the problem. This approach is in line with various other formulations reported in [4, 5, 6, 7] and presents the advantage that the mathematical formalism and the layout of the solving system are identical to those of hybrid elements. This trait allows for the direct implementation of the hybrid-Trefftz elements in software designed to accommodate hybrid elements without significant adaptation.

The article opens with the general description of the biphasic elastostatic problem, followed by the description of the governing equations of the hybrid displacement and stress models. The hybrid displacement (stress) model is constructed on the independent approximations of the domain displacements (stresses) and of the Dirichlet (Neumann) boundary tractions (displacements). The typifying feature of the hybrid element is that the domain displacement (stress) approximation basis is constrained to satisfy locally the domain compatibility (equilibrium) equation. Conversely, no restrictions are enforced on the boundary traction (displacement) basis, except for completeness and linear independence. The finite element equations are obtained through the weak enforcement of the domain equilibrium (compatibility) and elasticity equations, using the displacement (stress) trial functions for weighting, and on the weak enforcement of the Dirichlet (Neumann) and interelement boundary conditions, using the boundary traction (displacement) approximation for weighting.

The hybrid-Trefftz models are obtained as particularizations of the corresponding hybrid models for domain approximation functions locally satisfying all domain equations. These functions are typically derived by constructing either Navier or Beltrami equations and by subsequently solving them for the displacement and stress potential functions, respectively. The first approach is used here, generating Navier-compliant harmonic and biharmonic displacement potentials out of which the domain strain and stress bases are obtained enforcing successively the domain compatibility and elasticity conditions. The boundary approximation remains, however, unrestricted.

The central objective of this work is to assess the performance of the proposed models in terms of accuracy, convergence and robustness. The convergence patterns under \textit{p}- and \textit{h}-refinements are presented first. They show that the convergence under (hierarchic) \textit{p}-refinement is faster than under \textit{h}-refinement, thus allowing accurate results to be attained using relatively coarse meshes. This is a consequence of the particular choice of domain approximation functions, which are physically meaningful and problem-specific. For the same reason, both models present good stability when the mesh is grossly distorted and when the medium is nearly incompressible, conditions that typically hinder the behaviour of conventional finite elements. Finally, a more complex problem is solved to prove the good agreement of the results with the predictions of the finite element software \textit{ABAQUS}TM [8].

\section{2. Problem definition}

Let \( V \) represent the biphasic body under analysis and let \( \Gamma \) be its surface (Figure 1). Boundary \( \Gamma \) is formed by the complementary Neumann \( \Gamma_{\nu} \) and Dirichlet \( \Gamma_{\nu} \) sides, whereon the components of the traction \( t_{\nu} \) or displacement \( u_{\nu} \) vectors are

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prescribed, $\Gamma = \Gamma_e \cup \Gamma_u$ and, $\phi = \Gamma_e \cap \Gamma_u$. The mathematical model considered for the analysis is the solid displacement - fluid seepage (u-w) variant of the Biot Theory of Porous Media [2], assuming an elastic solid phase fully permeated by a liquid phase with a flow governed by Darcy’s law. Both phases are considered compressible.

$$\nabla \cdot \mathbf{b} = 0, \text{ in } V$$

$$\varepsilon = \mathbf{D} \cdot \mathbf{u}, \text{ in } V$$

$$\mathbf{u} = \mathbf{u}_F, \text{ on } \Gamma_e$$

$$\mathbf{u} = \mathbf{u}_R, \text{ on } \Gamma_u$$

The equations governing the response of the body under static conditions are,

$$\mathbf{D} \sigma + \mathbf{b} = 0, \text{ in } V$$ (1)

$$\varepsilon = \mathbf{D} \cdot \mathbf{u}, \text{ in } V$$ (2)

$$\mathbf{u} = \mathbf{u}_F, \text{ on } \Gamma_e$$ (3)

$$\mathbf{u} = \mathbf{u}_R, \text{ on } \Gamma_u$$ (4)

where the independent components of the (total) stress and strain tensors in the solid phase and the pore fluid pressure ($\pi$) and fluid content ($\zeta$) are organized in vectors $\mathbf{u}(x,y) = \{\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \sigma_{x\gamma}, \sigma_{y\gamma}, \pi\}$ and $\varepsilon(x,y) = \{\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{x\gamma}, \varepsilon_{y\gamma},\zeta\}$, respectively. Vector $\mathbf{u}(x,y) = \{u_x, u_y, w_y\}$ collects the solid skeleton displacement and the fluid seepage, and $\mathbf{b}(x,y) = \{b_x, b_y, \rho_u b_x, \rho_w b_y\}$ lists the body force components, depending on the mass densities of the mixture and of the liquid phase, $\rho$ and $\rho_w$, respectively. However, the body force component is discarded from this point onwards, to keep the presentation concise. Its influence is included in the formulation, for instance, in [10].

3. Hybrid and hybrid-Trefftz models

The hybrid-Trefftz displacement (stress) formulation is derived from the corresponding hybrid model by requiring that the domain displacement (stress) field satisfies locally all domain equations. As for the hybrid formulation, the equilibrium (compatibility) equations are enforced on average using the functions in the displacement (stress) approximation basis as weighting functions. Independently of the basis adopted in the domain, the boundary tractions (displacements) are also approximated and used to enforce in a weak form the displacement (traction) continuity, while the boundary traction (displacement) continuity is enforced explicitly.

3.1. Finite element discretization

Assume that the analyzed domain $V$ is discretized into finite elements, as presented in Figure 2. Let $V_e$ and $V_u$ represent the domain and the boundary of a typical finite element, respectively.

For a displacement (stress) element, the concept of Dirichlet (Neumann) boundary, previously defined as the exterior boundary where the displacements (tractions) are known, is now extended to include the portion of the boundary which is shared with another element, while the complementary Neumann (Dirichlet) boundary continues restrained to the part of the boundary where the applied tractions (displacements) are known, $\Gamma_e^n = \Gamma_u \cup \Gamma_t$, $\{\Gamma_u = \Gamma_e \cup \Gamma_t\}$, where $\Gamma_t$ denotes the interelement boundary.

3.2. Hybrid displacement element

3.2.1. Approximation bases

The displacement model is derived by directly approximating the displacement field in the domain of the element as:

$$\mathbf{u} = U_1 \mathbf{X}_1 + U_2 \mathbf{X}_2, \text{ in } V_e$$ (6)

In equation (6), vector $\mathbf{X}_1$ collects the generalized displacements associated with the strain producing displacement modes $U_1$, while vector $\mathbf{X}_2$ lists the weights corresponding to the rigid body motion and to the pure flow modes $U_2$.

Consistent with the domain approximation of the displacements, compatible strains are determined from condition (2):

$$\varepsilon = \mathbf{E}_1 \mathbf{X}_1, \text{ in } V_e$$ (7)

$$\mathbf{E}_1 = \mathbf{D}^\star \mathbf{U}_1$$ (8)

The traction (Cauchy stress) field in the solid phase and the pore pressure are independently approximated on the Dirichlet boundary $\Gamma_e^n$, as:

$$t = Z \mathbf{P}, \text{ on } \Gamma_e^n$$ (9)

The functions used to define the (strictly hierarchical) boundary basis $Z$ must only observe the restrictions of completeness and linear independence.

3.2.2. Finite element equations

The hybrid displacement element domain equations (10) are derived by the Galerkin weighted residual enforcement of the domain equilibrium (1) and elasticity (3) equations and by directly enforcing the Neumann boundary equilibrium condition (4) in the resulting boundary integrals [11].

$$\left\{ \begin{array}{l}
K \mathbf{X}_1 - B_1 \mathbf{P} = \mathbf{x}_1 \\
-B_2 \mathbf{P} = \mathbf{x}_2
\end{array} \right.$$ (10)

Figure 1: Finite elements, Neumann and Dirichlet boundaries.

Figure 2: Finite elements, Neumann and Dirichlet boundaries.
where:
\[
K = \int E^t_k E_1 \, dV^e \\
B_i = \int U^t_i Z \, d\Gamma^e_u \\
x_i = \int U^t_i \tau_r \, d\Gamma^e_u
\]
with \(i = \{1, 2\}\). The Galerkin weighted residual procedure is used again to enforce weakly the Dirichlet boundary compatibility equation (5), using the traction approximation functions defined by equation (9) as test functions, to give,
\[
- B_1^t X_1 - B_2^t X_2 = -p
\]
\[
p = \int Z^t u_r \, d\Gamma^e_u
\]
where \(A^*\) represents the conjugate transpose of matrix \(A\).

### 3.2.3. Governing system

The hybrid displacement element governing system is obtained by combining the domain (10) and boundary (14) conditions:
\[
\begin{bmatrix}
K & 0 & -B_1 \\
0 & 0 & -B_2 \\
-B_1^t & -B_2^t & 0
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2 \\
P
\end{bmatrix}
= \begin{bmatrix}
x_1 \\
x_2 \\
0
\end{bmatrix}
\tag{16}
\]

The governing system is Hermitian, highly sparse and strongly localized, as vectors \(X_1\) and \(X_2\) are strictly element dependent, while the static degrees of freedom \(P\) are shared by at most two neighboring elements. It yields unique estimates for the displacement field in the solid phase and for the fluid seepage, including their rigid body and free flow components, and for the Cauchy stresses and pore pressure on the internal and external boundaries of the finite element mesh. The stress field in the domain can be calculated from the domain displacement field enforcing successively the compatibility (2) and elasticity (1) equations.

### 3.3. Hybrid stress element

#### 3.3.1. Approximation bases

The stress field is directly approximated in the domain of the element as:
\[
\sigma = S_1 Y_e, \quad \text{in } V^e
\tag{17}
\]

In the above definition, vector \(Y\) collects the generalized stresses associated with the functions listed in the basis \(S_1\). As typical of hybrid stress elements, the functions used for the domain stress approximation are bounded to satisfy locally the homogeneous form of the equilibrium condition (1),
\[
\mathcal{D}S_1 = 0
\tag{18}
\]

The displacement field is independently approximated on the Neumann boundary \(\Gamma^u\), as:
\[
u = ZQ, \quad \text{on } \Gamma^u
\tag{19}
\]

Except for completeness and linear independence, no other restrictions are imposed at this stage on the functions used to define the (strictly hierarchical) boundary basis \(Z\).

#### 3.3.2. Finite element equations

The hybrid stress element domain statement (20) is obtained through the Galerkin weighted residual enforcement of the domain compatibility (2) and elasticity (3) equations, in which the Dirichlet boundary compatibility condition (5) is locally imposed [11].
\[
\mathcal{F}Y - Aq = y
\tag{20}
\]

where:
\[
\mathcal{F} = \int S^t_1 f S_1 \, dV^e
\tag{21}
\]
\[
A = \int (NS_1)^t Y \, d\Gamma^e_u
\tag{22}
\]
\[
y = \int (NS_1)^t u_r \, d\Gamma^e_u
\tag{23}
\]

On the Neumann boundaries, the equilibrium condition (4) is enforced weakly, using the displacement approximation functions defined in (19) as test functions, to give,
\[
- A^* Y = -q
\tag{24}
\]
\[
q = \int Z^t \tau_r \, d\Gamma^e_u
\tag{25}
\]

#### 3.3.3. Governing system

The governing system of the stress element is obtained by combining the domain and boundary statements, (20) and (24), respectively:
\[
\begin{bmatrix}
\mathcal{F} & -A \\
-A^* & 0
\end{bmatrix}
\begin{bmatrix}
Y \\
Q
\end{bmatrix}
= \begin{bmatrix}
y \\
-q
\end{bmatrix}
\tag{26}
\]

The governing system has the same properties as system (16) of the displacement element. It yields unique estimates for the stress and pore pressure fields in the domain of the element, for the normal and tangential components of the boundary displacements and for the boundary-normal fluid seepage. The domain displacement field calculated from the domain stress estimate will generally violate, however, both the boundary and the interelement compatibility conditions, as it misses the strain-free components associated with the rigid body modes in the solid phase and with the fluid free flow. These components are typically calculated in the post-processing phase, as reported in [10].

### 3.4. Hybrid-Trefftz elements

If the trial functions used for the domain and stress approximations are selected from the free-field solution set of the governing Navier (or Beltrami) differential equation, the hybrid models naturally converge to their Trefftz counterparts. This condition is the typical trait of Trefftz formulations, and is enforced by requiring that the domain displacement, strain and stress approximations (6), (7) and (17) satisfy the fundamental domain equations (1), (2) and (3).

Bound to satisfy the same equations, the displacement, strain and stress approximation bases result unique for both models. They collect functions that satisfy the homogeneous Navier equation,
\[
\mathcal{D}k \mathcal{D}^* u = 0, \quad \text{in } V
\tag{27}
\]

obtained by merging the fundamental equations, and must satisfy simultaneously the following conditions:
\[
\mathcal{D}S_1 = 0
\tag{28}
\]
\[
S_1 = kE_1
\tag{29}
\]
\[
E_1 = \mathcal{D}^* U_1
\tag{30}
\]
\[
0 = \mathcal{D}^* U_2
\tag{31}
\]
Condition (27) is satisfied by displacement functions defined as the irrotational and solenoidal terms of scalar potentials $\Phi$, which satisfy the equation $\nabla^2 \Phi = 0$. They generate coupled and uncoupled displacement modes, respectively, to be included in the approximation basis $U_1$. Strain-free trial functions, corresponding to rigid body and pure flow modes, are also collected and casted in the approximation basis $U_2$. Finally, singular potentials may be included in the approximation basis in order to model singular fields associated, for instance, with notches and cracks. As this paper deals with regular fields only, they are not included in the bases used here, although one of the singular modes is exploited in Section 4.1 to generate analytic solutions for the convergence tests.

Trefftz compliant strain and stress bases are derived from $U_1$ through conditions (30) and (29). The explicit expressions of the Trefftz-compliant displacement and stress modes are listed in Appendix C.

When inserted in definitions (11) and (21) Trefftz conditions (28) to (30) reduce the finite element stiffness and flexibility matrices to the same boundary integral expression,

$$K = \mathcal{F} = \int U_1^* N S_1 \, d\Gamma$$

(32)

The governing systems for the hybrid-Trefftz displacement and stress models preserve the forms (16) and (26), and the respective algebraic properties.

4. Numerical tests

The set of numerical simulations presented below is designed to assess the convergence of both models under $p$- and $h$-refinement, and the sensitivity of the models to issues known to hinder the numerical performance of the conventional finite elements, namely gross mesh distortions, and nearly-incompressible situations may be modelled: recovery of the exact solution, if the exact field is present in the finite element basis; or convergence to the exact solution, if the exact field is not included in the domain basis. The second option is used here, as the boundary tractions $t_1$ and $t_2$ (boundary displacements $u_1$ and $u_2$), presented in Figure 3, are defined using the singular biharmonic potential $\Phi = r^2 \log r$, which is never included in the basis. Pore pressure results null for the applied load pattern.

The geometry of the domain is defined by $r_{\text{min}} = 10\text{m}$, $r_{\text{max}} = 20\text{m}$ and $\phi = 0.5\text{rad}$. The potential function is defined in respect to point $O$ (Figure 3), exterior to the medium. For illustrative purposes, the stress fields corresponding to the performed tests are shown in Figure 4.

The single-, two- and four-element meshes presented in Figure 5 are used to assess the influence of $h$-refinement procedures on the convergence patterns, while the complementary $p$-refinement process is implemented by gradually increasing the dimension of the domain approximation basis.

4.1. Convergence to analytic solutions

The convergence tests presented here essentially consist of applying on the structural body presented in Figure 3 of a load pattern for which the exact solution is known a priori. For a displacement (stress) element, this load pattern consists of boundary tractions (displacements) applied on the whole perimeter of the medium. These forces (displacements) are simply the boundary values of some Navier-compliant domain field, which consequently represents the analytic solution of the problem. Two situations may be modelled: recovery of the exact solution, if the exact field is present in the finite element basis; or convergence to the exact solution, if the exact field is not included in the domain basis. The second option is used here, as the boundary tractions $t_{r_1}$ and $t_{r_2}$ (boundary displacements $u_{t_1}$ and $u_{t_2}$), presented in Figure 3, are defined using the singular biharmonic potential $\Phi = r^2 \log r$, which is never included in the basis. Pore pressure

$$\epsilon = 1 - \frac{E_{\text{FE}}}{E}$$

(33)

where $E$ and $E_{\text{FE}}$ are the exact mechanical energy and its finite element approximation, respectively, which are computed according to expressions (52) and (53) of Appendix D.

The convergence patterns of the mechanical energy error measure (33) are presented for both displacement and stress models in Figure 6. The values on the abscissa represent (logarithmically) the total number of degrees of freedom, i.e. the dimension
of the finite element solving system. Each plot contains three solid line graphs, for the $p$-convergence patterns obtained with the single-, two- and four-element meshes and three dotted lines, representing the $h$-convergence patterns obtained using the same degree of refinement in the domain of the elements.

The results demonstrate that the convergence under $p$-refinement is rather high, even for small numbers of degrees of freedom (the total dimension of the finite element solving system never exceeds 134). The $p$-refinement convergence is enhanced by the special nature of the approximation functions, which embody significant information regarding the modelled phenomenon. On the other hand, the impact of $h$-refinement is weaker. Substantial testing experience, however, has proven that there exists a limited amount of accuracy a certain mesh can meet for a particular problem. Once this point is reached, further increasing the degree of $p$-refinement would result in loss of precision because of the consequent ill-conditioning of the governing system. Under such circumstances, a superior level of convergence may significantly improve the quality of the results and ultimately spare computational effort, as the convergence may then be reached for much lower orders of approximation.

4.2. Sensitivity to mesh distortion

To assess the mesh distortion sensitivity of the hybrid-Trefftz displacement and stress elements, the analytic solution problem described in the previous Section is applied to the structure represented in Figure 7. The maximum degree used to construct the domain approximation functions is $N = 15$. Legendre polynomials with a maximum degree of $M = 3$ are used to define the interior boundaries approximations.

The medium is divided in four finite elements, as presented in Figure 7, with the radial side being $L = \frac{x}{2} = 5.0$ m. The elements are successively distorted by decreasing the value of the distortion parameter $\eta$, from $\eta = 1.0$, corresponding to the undistorted mesh, to $\eta = 10^{-2}$, as shown in Figure 7. For each value of $\eta$, the mechanical energy error measure given by definition,

$$\epsilon = 1 - \frac{E_{\text{dist}}}{E_{\text{undist}}}$$

is calculated, where $E_{\text{dist}}$ and $E_{\text{undist}}$ are the mechanical energies computed on the distorted and undistorted meshes, respectively.

Figure 8 presents plots of the variation of the error measure given by expression (34) as a function of the distortion parameter $\eta$.

For both displacement and stress models, the targeted energy is accurately recovered for all tested values of the distortion parameter $\eta$. Although the error of the stress model exhibits a seemingly increasing pattern when the distortion parameter approaches zero, the value of this error remain relatively insignificant throughout. Conversely, the displacement model error has a more stable pattern, with errors of the order of $\epsilon \approx 5 \cdot 10^{-6}$.

4.3. Sensitivity to incompressibility of constituents

Nearly incompressible constituents are quite common in saturated porous media problems, due to the high bulk moduli that often characterize one or both constituents [12]. While the fluid incompressibility do not pose any numerical difficulties under drained conditions, as volume change can still occur through fluid seepage, a nearly incompressible solid skeleton may significantly hinder the behaviour of most conventional finite elements, due to the well-known numerical instability associated with the Poisson coefficient approaching 0.5.

The sensitivity of the hybrid-Trefftz displacement and stress elements to this issue is assessed here, using the same testing conditions as defined in the previous section. The discretization of the medium and the shape functions adopted for the domain and boundary approximations are again the same as those defined in Section 4.2.

The geomechanical characteristics of the medium are defined in Appendix B, except for the Poisson coefficient of the solid matrix, whose value is calibrated through parameter $\eta$ according to definition,

$$\nu = 0.5 - \eta$$

The initial value of $\eta$ is set to $10^{-1}$ and subsequently decreased to $10^{-10}$. For each value of $\eta$, the mechanical energy error is calculated according to definition (33), which is reproduced here for convenience,

$$\epsilon = 1 - \frac{E_{\text{FE}}}{E}$$
The variation of the error measure $e$ as a function of the (logarithm of the) distortion parameter $\eta$ is presented, for both displacement and stress models, in Figure 9. The results are virtually insensitive to the successive increases of the Poisson coefficient $\nu \leq 0.5 - 10^{-10}$. Again, the main reason for this robustness is the physical significance of the Trefftz-compliant approximation functions, which locally respect the domain elasticity equation, despite the high values of the bulk moduli. Significant degradation of the predicted results may, however, be caused by loss of numerical precision when parameter $\eta$ is inferior to the numerical precision (the present tests are run using double precision).

4.4. Drained surface traction test

The surface traction problem presented in Figure 10 is solved to illustrate the ability of the hybrid-Trefftz elements to recover the interelement continuity and the enforced boundary conditions under a more complex loading situation. A similar analysis is performed with the commercial finite element analysis package \textit{ABAQUS} \textsuperscript{TM} \cite{8}, to evaluate the coherence of the results.

The analyzed soil is confined in a square tank with a side length of 4.0m and subjected to a surface load, acting solely on the solid skeleton. Free surface flow is permitted for the fluid phase. The normal displacements in the solid phase and the fluid seepage are restricted along the walls of the container. However, no constraints are imposed on the tangential components of the media motion on the tank walls, which are considered frictionless.

The testing material is the same as for the tests reported above, namely a water-saturated Molsand soil.

The medium is meshed in 16 square elements with the leading dimension of $L = 1.0$ m. The domain approximation bases are constructed, for all elements, using polynomial functions of degree 15. For the displacement (stress) model, this generates a total number of 1125 (1024) kinematic (static) degrees of freedom. The boundary approximation is implemented using Legendre polynomials, with a maximum degree of 3, thus generating a total of 384 (312) static (kinematic) degrees of freedom. The total number of degrees of freedom of the model is 1509 (1336).

For each element the local coordinate system is centered on its barycenter, and the local Cartesian axes are taken parallel to its principal directions.

The alternative simulation implemented in \textit{ABAQUS} \textsuperscript{TM} involves the structure’s discretization in 1600 finite elements. The finite-strain CPE4P \cite{8} element is used in the analysis. This is a 4-node bilinear displacement and pore pressure element, with 3 degrees of freedom per node (two nodal displacements and the nodal pore pressure). Thus, the total number of structural degrees of freedom is 5043.

Figures 11 to 13 present a comparison between the solid phase stress fields predicted by the hybrid-Trefftz displacement and stress elements (HTDE and HTSE, respectively) and those computed by \textit{ABAQUS} \textsuperscript{TM}. All stresses obtained with the hybrid-Trefftz model are in good agreement with those predicted by \textit{ABAQUS} \textsuperscript{TM}, despite the significant difference of mesh refinement and the consequent discrepancy in terms of total number of degrees of freedom. They are continuous over the interelement boundaries and model adequately the enforced boundary conditions. It should be noted that the traction discontinuity occurring at $x = -L$ on the free surface of the medium doesn’t
seem to affect the stress fields in the adjacent region, according to the hybrid-Trefftz stress model. However, Figure 13(c) shows that this factor does cause discontinuities in the tangential stresses predicted by $ABAQUS^{TM}$.

### 5. Conclusions

The hybrid-Trefftz displacement and stress elements for elastostatic analysis of biphasic media are presented. They are derived using a hybrid model framework, with the distinction that the domain approximation functions are constrained to satisfy locally all domain equations. As a consequence, all terms present in the finite element solving system are defined by boundary integral expressions. Moreover, as the domain approximation functions embody physically relevant information, accurate results can be achieved using a relatively low number of degrees of freedom and/or coarse finite element meshes and the elements present enhanced robustness to mesh distortion and near-incompressible constituents. The price paid, however, is the lack of flexibility in choosing the approximation functions, which in general become numerically heavier to implement and do not allow analytic integration.

### References


### A. Explicit expressions for material stiffness matrix, outward normal matrix and differential operators

The expression of the material stiffness matrix present in elasticity equation (3) is,

$$
\mathbf{k} = \begin{bmatrix}
k_{11} & k_{12} & 0 & k_{14} \\
k_{12} & k_{22} & 0 & k_{14} \\
0 & 0 & k_{33} & 0 \\
k_{14} & k_{14} & 0 & k_{44}
\end{bmatrix}
$$

The stiffness coefficients are defined by,

$$
k_{11} = k_{22} = k_{12} + 2k_{33} \\
k_{12} = \lambda + \alpha^2 M \\
k_{33} = \mu \\
k_{44} = M
$$

where $\lambda$ and $\mu$ are the Lamé constants and $\alpha$ and $M$ are the first and second Biot coefficients, respectively.

In Cartesian coordinates, the expressions of the equilibrium and compatibility differential operators present in equations (1) and (2) are,

$$
\mathbf{D} = \begin{bmatrix}
\frac{\partial}{\partial x} & 0 & 0 \\
0 & \frac{\partial}{\partial y} & 0 \\
0 & 0 & \frac{\partial}{\partial z}
\end{bmatrix}
$$

and

$$
\mathbf{D}^* = \mathbf{D}^T
$$

The components of the outward normals are organized in matrix $\mathbf{N}$,

$$
\mathbf{N} = \begin{bmatrix}
n_x & 0 & n_y \\
0 & n_y & n_z \\
0 & 0 & 1
\end{bmatrix}
$$

### B. Geomechanical description of water-saturated Molsand soil

The numerical applications presented in this work use a water-saturated Molsand soil with an incompressible solid matrix ($\alpha = 1.0$). The main geomechanical weights of the Molsand soil are summarized next.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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<tbody>
<tr>
<td>$\rho_w$</td>
<td>1000 kg/m$^3$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>2650 kg/m$^3$</td>
</tr>
<tr>
<td>$M$</td>
<td>5.67 $\cdot$ 10$^6$ N/m$^2$</td>
</tr>
<tr>
<td>$E$</td>
<td>2.98 $\cdot$ 10$^3$ N/m$^2$</td>
</tr>
<tr>
<td>$K$</td>
<td>5.97 $\cdot$ 10$^2$ N/m$^2$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>1.0</td>
</tr>
<tr>
<td>$n^v$</td>
<td>0.388</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.333</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>1.0</td>
</tr>
</tbody>
</table>
Higher degree stress modes  The higher degree stress modes yield the following expressions for the displacement field,

\[
U_{bn} = \frac{1}{2(n+1)\mu} \left[ \begin{array}{c} \frac{1}{\pm 2n+1} \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right] r^{n+1} \exp[\pm in\theta]
\]

and stress field,

\[
S_{bn} = \frac{1}{\pm 2(n+1)\mu} \left[ \begin{array}{c} -\frac{1}{n+1} \\ 0 \\ 0 \\ 0 \\ 0 \end{array} \right] r^n \exp[\pm in\theta]
\]

for any degree \( n \geq 1 \).

C.3. Constant pore pressure solution

A hydrostatic load applied on the fluid phase generate a constant pore pressure and a constant total stress field in the biphasic medium. Navier compliant non-null pore pressure modes can be obtained using the following pure seepage displacement field definition,

\[
U_p = \frac{1}{2\pi} \left[ \begin{array}{c} 0 \\ 0 \\ \cos \theta \\ 0 \end{array} \right] r
\]

to yield,

\[
S_p = \left[ \begin{array}{c} \alpha \\ \alpha \\ 0 \\ 0 \end{array} \right]
\]

C.4. Null stress solutions

Besides the stress generating modes described above, the following rigid-body displacement modes defined on the solid phase also satisfy Navier equation (27),

\[
U_{rs} = \left[ \begin{array}{c} 0 \\ 0 \\ \sin \theta \\ 0 \end{array} \right] r
\]

In the fluid phase, five linear free flow modes are identified,

\[
U_{fw} = \left[ \begin{array}{c} 0 \\ 0 \\ \cos \theta \\ 0 \end{array} \right] r
\]

along with \( n+2 \) free flow modes of degree \( n \geq 2 \) [9],

\[
U_{fw} = \left[ \begin{array}{c} \sin^n \theta \cos \theta \\ \sin^n \theta \cos^n \theta \\ -\sin^{n+1} \theta \cos^{n+1} \theta \\ 0 \\ 0 \end{array} \right] r^n
\]

with \( 0 \leq k \leq n - 1 \).

D. Mechanical Energy

The mechanical energy dissipated by the structural system, is defined as:

\[
E = \frac{1}{2} \int \sigma^* : \epsilon \, dV
\]  

(52)

Substituting approximations (6), (7) and (17) into the definition (52), the finite element discrete approximation \( E_{FE} \) of the mechanical energy assumes the following expression,

\[
E_{FE} = \frac{1}{2} \sum_{i=1}^{N_e} X_i^* L_i K_i X_i, \text{for a displacement element}
\]

\[
E_{FE} = \frac{1}{2} \sum_{i=1}^{N_e} Y_i^* F_i Y_i, \text{for a stress element}
\]

where \( N_e \) is the number of elements in the mesh.