HYBRID-TREFFTZ ELEMENTS FOR ELASTIC UNSATURATED POROUS MEDIA

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The displacement and stress models of the hybrid-Trefftz finite elements are formulated for static and dynamic problems defined on elastic unsaturated porous media. The mathematical formulation is based on the theory of mixtures with interfaces. All models consider the full coupling between the solid, fluid and gas phases, including the effects of relative (seepage) accelerations.

Keywords: Hybrid-Trefftz Finite Elements; Unsaturated Porous Media; Poroelasticity.

1. Introduction

The analysis of elastic wave propagation through soils is an important issue in Geomechanics. Traditional theories treating the soil as a single-phase material are insufficient when the relative motion between the phases influences significantly the overall behavior of the medium. In such cases, alternative biphasic or triphasic theories must be considered for the analysis of saturated and unsaturated soils, respectively.

Unsaturated soils are assemblages of solid particles that form a porous solid skeleton, whose voids are filled with two immiscible fluids, namely a liquid (typically water) and a gas (or non-wetting fluid, typically air). The mechanical response of most unsaturated soils under realistic conditions is largely influenced by the microscopic interaction at the interfaces between the solid, liquid and gas phases.
Therefore, single-phase theories are typically inadequate for modelling such materials and more rigorous formulations must be adopted instead.

For biphasic (i.e. saturated) soils, the Biot’s theory [Biot (1956)] is frequently used to construct the mathematical model for wave propagation problems. Biot’s theory is relatively simple, well understood and thoroughly confirmed by experiments. Due to these merits, various attempts existed to reduce the triphasic problem to a biphasic problem, under certain sets of conditions. For nearly saturated soils, an equivalent biphasic model was proposed, where the compressibility of the liquid phase is increased to account for the presence of air bubbles [Smeulders (1992)]. An heuristic relation between the equivalent compressibility of the liquid and the saturation was established. An equivalent biphasic theory valid for a wider saturation range was derived by [Berryman et al. (1988)] by extending a model initially suggested by [Brutsaert (1964)]. The theory assumes that pore pressure variations are equal in both fluids, so that the capillary pressure may be considered constant. Posteriorly, [Lo et al. (2007)] demonstrated that this assumption is acceptable for low frequency excitations, ‘much smaller’ than a critical frequency that depends on the viscosity of the fluids and the permeability of the solid skeleton. Under these conditions, the two fluids can be represented by a single fluid with volume-averaged material properties, and the Biot’s theory applied to model its behavior.

Biot’s theory, however, does not rigorously take into account the microscopic structure of the porous media, which may play an important role in the macroscopic behavior of unsaturated soils. To overcome this limitation, two classes of approaches were suggested, namely the theory of mixtures and the averaging theory. The theory of mixtures applies the classical equations of continuum mechanics at the macroscopic level and uses the concept of volume fractions to account for the motion of individual phases. Volume fractions are treated as internal variables and additional equations (closure equations) are formulated to account for their variation (e.g. [Loret and Khalili (2000)]). The models based on the theory of mixtures were later improved to include the effects of the drag and capillary forces on the interfaces between the various phases in [Muraleetharan and Wei (1999)] and [Wei and Muraleetharan (2002)]. Conversely, the averaging theory applies the classical equations of continuum mechanics at the microscopic level and uses averaging operators to obtain the macroscopic results. This approach is rooted in the work of Hassanizadeh and Gray (1979a; 1979b; 1980). Averaged quantities include the densities of the three phases, the stress tensor and the seepage velocities of the fluid phases [Oettl et al. (2004)]. Under dynamic loading conditions, all triphasic models predict the existence of three compressional waves, characterized by pore pressure gradients in the fluid phases and a single shear, pressure-free, wave. All waves are dispersive and depend on the permeability and compressibility of the medium. Two of the compressional waves are strongly affected by the degree of saturation [Wei and Muraleetharan (2002)].

Comprehensive reviews of the vast literature available on the mathematical mod-
els for unsaturated soils include (de Boer et al. 1991; 1996) and (Scheng et al. 2008; 2011).

The finite element method is probably the most widely used tool for finding approximate solutions for the problems defined by the mathematical formulations presented above. Due to the physical complexity of the dynamic response of unsaturated soils, the finite element models are typically based on various simplifying assumptions, such as rigid solid skeleton, static gas phase, quasi-static loading conditions and negligible seepage accelerations. The rigid solid skeleton assumption is adequate for modelling pure flow problems. It assumes that the displacements, velocities and accelerations of the solid phase are negligible and is thus not adequate for modelling wave propagation phenomena [Wu and Forsyth (2001)]. More common are the models that neglect the motion of the gas phase [Calari and Abati (2009)]. The air in the pores is assumed to remain at the atmospheric pressure throughout the whole analysis domain and at all times. Consequently, static gas phase models are only capable of accounting for two compressional waves. For consolidation problems, it is quite often reasonable to neglect the velocities and accelerations of all phases (the quasi-static assumptions). Such models (e.g. [Oettl et al. (2004)]; [di Rado et al. (2009)]) are not adequate for transient dynamic problems, but they are very well suited for conventional (conforming displacement) finite elements, as they avoid the necessity of calibrating the leading size of the element with the wavelength of the propagating wave. Finally, a large number of finite element models have been developed on the assumption of negligible seepage accelerations (e.g. [Khoei and Mohammadnejad (2011)]). This assumption endorses the use of the solid displacement and pore pressures as primary (nodal) variables of the conventional finite elements, leading to the so-called \( u - p_w - p_g \) formulation. Besides the inherent limitation of the simplifying assumption, hybrid elements constructed in this way violate the Zienkiewicz-Taylor patch test (or the Babuska-Brezzi condition) when the same order of approximation functions is used for both displacement and pressure fields in the domain. Fully coupled \( (u - u_w - u_g) \) models are rarely implemented using conventional elements (e.g. [Ravichandran (2009)]), mainly due to the fact that the pore pressure is commonly of more interest than the fluid displacements. However, such models have the advantage of not neglecting the seepage accelerations.

Except for the pure flow and quasi-static models, modelling the wave propagation through unsaturated soils using conventional elements requires the calibration of their size with the wavelength of the propagating wave. It is recalled that the 'thumb rule' commonly adopted to reduce the effect of the interpolation error states that at least six (but preferably ten) elements per wavelength should be used [Pluymers et al. (2007)]. This is a very serious problem in modelling the dynamic response of unsaturated soils, as two of the three compressional waves tend to have very low wavelengths, even for rather low excitation frequencies [Wei and Muraleetharan (2002)]. If the permeability of the medium is low, these waves are strongly attenuated, so their influence is restricted to the near-field. However, as
shown in Section 2, when the soil has higher permeability these waves are much less
dissipative, and therefore may significantly influence the dynamic response of the
medium.

The objective of this paper is to present the formulation of hybrid-Trefftz ele-
ments for modelling the elastodynamic response of unsaturated soils. The model is
fully coupled (i.e. does not use any of the simplifying assumptions mentioned above)
and includes all relevant compressional waves.

Hybrid-Trefftz elements require the domain approximation functions to satisfy
locally all governing equations. Consequently, the major feature of the hybrid-Trefftz
elements is the richness of information contained in the approximation basis. This
trait allows these elements to yield highly accurate solutions with a relatively small
number of degrees of freedom and considerably enhances their robustness to is-
sues hindering the behavior of conventional elements (e.g. large stress gradients,
topology and mesh distortion, incompressible media, high excitation frequency).
In particular, hybrid-Trefftz elements are very well suited to problems involving
propagation of highly oscillating waves, as their size is not wavelength-dependent.
Moreover, all coefficients of the solving system are defined by boundary integrals,
thus eliminating the geometrical constraints the conventional elements must typ-
ically observe. The price paid, however, is the lack of flexibility in choosing the
approximation functions, which may be numerically heavy to implement and not
allow analytic integration. The formulation of hybrid-Trefftz elements for biphasic
(saturated) soils was reported in [Freitas et al. (2011)] and [Freitas and Moldovan
(2011)]. A comprehensive assessment of the convergence and robustness of these
elements is reported in [Moldovan and Freitas (2012)].

2. Mathematical model

The mathematical model adopted in this paper was suggested by Wei and Mu-
raleetharan (2002). This is a thermodynamically consistent model, based on the
theory of mixtures with interfaces. It explicitly includes the dynamic compatibility
conditions on the interfaces between the three phases and endorses the distinction
between the microscopic fluid flow, caused by the capillary relaxation process (par-
ticularly relevant in swelling soils) and the macroscopic fluid flow, driven by the
pore pressure gradients caused by the application of exterior loads to the medium.
The capillary relaxation process is associated to changes in the volume fractions of
the fluids. When this process is omitted from the analysis (i.e. the capillary equi-
librium is reached immediately), the model becomes linear elastic. Also, the model
recovers the Biot’s theory when the saturation of the medium is increased to 1.0.

In this paper, the linear-elastic model obtained by neglecting the effect of the
capillary relaxation is considered. Its equations are fully derived and commented in
[Wei and Muraleetharan (2002)]. They are restated here for the sake of convenience,
but their interpretation is limited to aspects considered relevant for their finite
element solution.
Consider the triphasic medium $V$ represented in Figure 1, consisting of a matrix of solid particles in contact with each other (the solid skeleton), with the pores filled with a wetting fluid (water) and a non-wetting fluid (air). The boundary $\Gamma$ of the medium is formed by the complementary Neumann ($\Gamma_\sigma$) and Dirichlet ($\Gamma_u$) sides, where the displacements of each phase (collected in vector $u$), the tractions in the solid phase and the pore pressures (collected in vector $t$) are prescribed, $\Gamma = \Gamma_\sigma \cup \Gamma_u$ and, $\phi = \Gamma_\sigma \cap \Gamma_u$. The elastodynamic equations governing the response of the triphasic medium are,

\begin{align*}
D \dot{\varepsilon}(x,t) + b &= \rho_0 \ddot{u}(x,t) + d_0 \dot{u}(x,t), \quad \text{in } V \\
\varepsilon(x,t) &= D^* u(x,t), \quad \text{in } V \\
\sigma(x,t) &= k \varepsilon(x,t) \text{ (or } \varepsilon(x,t) = f \sigma(x,t)), \quad \text{in } V \\
N \sigma(x,t) &= t \Gamma(x,t), \quad \text{on } \Gamma_\sigma \\
u(x,t) &= u \Gamma(x,t), \quad \text{on } \Gamma_u \\
u(x,0) &= u_0(x), \quad \text{in } V \\
\dot{u}(x,0) &= v_0(x), \quad \text{in } V
\end{align*}

In the above expressions, the independent components of the (total) stress tensor in the solid phase $\sigma^S$ and the pore pressures in the wetting and non-wetting phases, $\pi^W$ and $\pi^N$, are collected in vector $\sigma = \{ \sigma^S, \pi^W, \pi^N \}^T$. The strain tensor in the solid phase $\varepsilon^S$ and the fluid contents are collected in vector $\varepsilon = \{ \varepsilon^S, \zeta^W, \zeta^N \}^T$. Vector $u = \{ u^S, u^W, u^N \}^T$ collects the displacement components in each phase. The initial displacement and velocity vectors are given by $u_0 = \{ u_0^S, u_0^W, u_0^N \}^T$ and $v_0 = \{ v_0^S, v_0^W, v_0^N \}^T$, respectively. Vector $b = \{ \rho^S g^S, \rho^W g^W, \rho^N g^N \}^T$ lists the body force components, depending on the mass densities of the phases, $\rho^S$, $\rho^W$ and $\rho^N$. 

![Fig. 1. Domain, Neumann and Dirichlet boundaries.](image-url)
and the components of the gravitational acceleration, $\mathbf{g}$. The body force component is discarded from this point onwards, to keep the presentation concise. Its influence is included in the formulation of hybrid-Trefftz elements, for instance, in [Moldovan et al (2011)].

Gradient and divergence operators $\mathbf{D}$ and $\mathbf{D}^*$ are adjoint in geometrically linear applications. The components of the outward normal to the boundary of the medium are organized in matrix $\mathbf{N}$. The expressions of material damping ($d_0$), mass ($\rho_0$), stiffness ($k$) and flexibility ($f$) matrices can be found in the Appendix A.

3. Integration in time

Governing equations (1) to (7) are first integrated in time, using the mixed procedure reported in [Freitas (2008)].

The method is based on the expansion of all unknown fields (say, $\alpha(x,t)$) in an arbitrary time basis, constructed on $N$ scalar functions $W_n(t)$. According to the Galerkin principle, the same functions are then used as test functions in the weak form enforcement of the governing equations,

$$\alpha(x,t) = \sum_{n=1}^{N} W_n(t) \alpha_n(x), \text{ in } V$$

According to the Galerkin principle, the same functions are used as test functions in the weak form enforcement of the governing equations. This procedure is applied to the velocity and acceleration definitions, on the current time interval $\Delta t$,

$$\int_0^{\Delta t} \tilde{W}_m(v - \dot{u}) \, dt = 0$$
$$\int_0^{\Delta t} \tilde{W}_m(a - \ddot{v}) \, dt = 0$$

where $\tilde{W}_m$ represents the complex conjugate of expression $W_m$.

It is convenient to define the square matrices $\mathbf{H}$ and $\mathbf{G}$, with the generic terms given by,

$$H_{mn} = \frac{1}{\Delta t} \int_0^{\Delta t} \tilde{W}_m W_n \, dt$$
$$G_{mn} = \tilde{W}_m (\Delta t) W_n (\Delta t) - \int_0^{\Delta t} \tilde{W}_m W_n \, dt$$

Equations (9) and (10) can be uncoupled if the time basis is constructed such that matrices $\mathbf{H}$ and $\mathbf{G}$ are related through a diagonal matrix of constants, $\Psi$ [Freitas (2008)],

$$\mathbf{H} \Psi = \mathbf{G}$$
to yield the following velocity and acceleration estimates,

\[ \Delta t v_n = \Psi_n u_n - \psi_n u^0, \text{ for } n = \{1, N\} \]  
\[ \Delta t a_n = \Psi_n v_n - \psi_n v^0, \text{ for } n = \{1, N\} \]

where,

\[ \psi_n = \sum_{m=1}^{N} H_{nm} \hat{W}_m(0) \]  

\( \Psi_n \) is the \( n^{th} \) diagonal term of matrix \( \Psi \) and \( H_{nm} \) denotes the general term of the inverse of matrix \( H \). Note that condition (13) should not be seen as a limitation of the method. Following a procedure detailed in [Freitas (2008)], this condition can be secured for any type of functions collected in the time basis (e.g. polynomial, trigonometric or wavelet functions).

Following the same strategy, the weak enforcement of equations (1) to (5), using the time basis \( W_n(t) \) for testing, and the enforcement of initial conditions (6) and (7) in the resulting expressions, yield a series of \( N \) uncoupled problems in space variables only,

\[ D\sigma_n + \omega_n^2 \rho_n u_n = F_n^0, \text{ in } V \]  
\[ \varepsilon_n = D^* u_n, \text{ in } V \]  
\[ \sigma_n = k \varepsilon_n, \text{ in } V \]  
\[ \varepsilon_n = f \sigma_n, \text{ in } V \]  
\[ N \sigma_n = t_{\Gamma_n}, \text{ on } \Gamma \sigma \]  
\[ u_n = u_{\Gamma_n}, \text{ on } \Gamma_u \]

where the pseudo-spectral frequency, defined as,

\[ \omega_n = -\frac{i \Psi_n}{\Delta t} \]  

is generally complex, \( i \) is the imaginary unit, and

\[ \rho_n = \rho_0 = \frac{1}{\omega_n} \]  
\[ F_n^0 = -\frac{\psi_n}{\Delta t} \left( \hat{\omega}_n \rho_n u^0 + \rho_0 v^0 \right) \]  
\[ u_{\Gamma_n} = \frac{1}{\Delta t} \sum_{m=1}^{N} H_{nm} \int_{0}^{\Delta t} \hat{W}_m u_{\Gamma} dt \]  
\[ t_{\Gamma_n} = \frac{1}{\Delta t} \sum_{m=1}^{N} H_{nm} \int_{0}^{\Delta t} \hat{W}_m t_{\Gamma} dt \]

Equations (17) to (20) can be reduced to the equivalent Navier equation,

\[ DkD^* u_n + \omega_n^2 \rho_n u_n = F_n^0, \text{ in } V \]
The method has the merit of generating time-discretized problems defined by spectral-like equations, independently of the functions that are used in the time basis, thus endorsing the use of the same finite element formulation in space for harmonic, periodic and transient problems. This feature is essential to Trefftz modelling and hindered by available implicit and explicit methods. The resulting time-discretized problems are characterized by complex generalized frequencies which depend on the size of the time step and of the choice of the time basis. The method is unconditionally stable and, depending on the adopted time basis, may endorse the use of very large time steps [Moldovan and Freitas (2008)].

4. Closed-form solutions in space

The general solution of the original problem defined by equations (1) to (7) is obtained as a linear combination of the functions satisfying the homogeneous problem (in space), defined by,

\[ DkD^*u(x) = 0, \text{ in } V \]  

and each of the pseudo-spectral problems defined by the Navier equation (28).

The solution space of these equations is derived next, in cylindrical coordinates \((r - \theta)\), for the static (homogeneous) and spectral (non-homogeneous) problems, respectively. The choice of cylindrical coordinates is motivated by the need to avoid spurious, Gibbs-like oscillations of the solution fields of the Trefftz elements in the vicinity of the boundary of the element, when trigonometric functions are used to construct the space basis.

4.1. Solutions of the homogeneous (static) problem

Using definitions (A.3) to (A.10), the Navier equation associated to the homogeneous problem (29) is written as,

\[ \begin{align*}
(M_{SS} + n^S\mu^S)\nabla\nabla^*u^S + n^S\mu^S\nabla^2u^S + M_{SW}\nabla\nabla^*u^W + M_{SN}\nabla\nabla^*u^N &= 0 \\
M_{SW}\nabla\nabla^*u^S + M_{WV}\nabla\nabla^*u^W + M_{WN}\nabla\nabla^*u^N &= 0 \\
M_{SN}\nabla\nabla^*u^S + M_{WN}\nabla\nabla^*u^W + M_{NN}\nabla\nabla^*u^N &= 0
\end{align*} \]  

where \(\nabla = \left\{ \frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \theta} \right\}^T\) designates the divergence operator, \(\nabla^* = \left\{ \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial}{\partial \theta} \right\}^T\) is the curl operator, and \(\nabla^2 = \nabla^T\nabla^* = (\nabla^*)^T\nabla = \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2}\) is the Laplace operator.

Navier equation (30) is satisfied by biharmonic and harmonic displacement potentials, generating coupled and uncoupled displacement modes, respectively, to be included in the approximation basis \(U_s = \{U^S, U^W, U^N\}^T\), along with two pore pressure modes, one for each of the fluid phases. The trial functions collected in the approximation basis \(U_r = \{U^S, U^W, U^N\}^T\) do not generate strains and are due to rigid body and pure flow modes. The expressions of the displacement and stress fields corresponding to each of these modes are given next.
4.1.1. Biharmonic solutions

When biharmonic displacement potentials are used to construct the solution set \( U^b_s \), they generate one single constant stress mode, plus a series of complex conjugate stress modes of higher degree.

**Constant stress mode** The constant stress mode is given by the displacement field,

\[
U^{b}_{s0} = \begin{bmatrix}
1 \\
0 \\
-\alpha W \\
0 \\
-\alpha N \\
0
\end{bmatrix} r^{(31)}
\]

where,

\[
-\alpha W = \frac{M_{SN} M_{WN} - M_{SW} M_{NN}}{M_{WW} M_{NN} - M^2_{WN}} \tag{32}
\]

\[
-\alpha W = \frac{M_{SW} M_{WN} - M_{SN} M_{WW}}{M_{WW} M_{NN} - M^2_{WN}} \tag{33}
\]

and the hydrostatic stress field,

\[
S^{b}_{s0} = 2 (\bar{\lambda} + n S \mu^S) \begin{bmatrix}
1 \\
1 \\
0 \\
0 \\
0
\end{bmatrix} \tag{34}
\]

\[
\bar{\lambda} = M_{SS} - M_{SW} \alpha^W - M_{SN} \alpha^N \tag{35}
\]

**Higher degree stress modes** The higher degree stress modes yield the following expressions for the displacement field,

\[
U^{b}_{s_n} = \frac{1}{2(n+1)n S \mu^S} \begin{bmatrix}
\left[\bar{\lambda} + 3n S \mu^S - (n+1)(\bar{\lambda} + n S \mu^S)\right] \\
\mp i \left[\bar{\lambda} + 3n S \mu^S + (n+1)(\bar{\lambda} + n S \mu^S)\right] \\
-2\alpha W n S \mu^S \\
\pm 2\alpha W n S \mu^S \\
-2\alpha N n S \mu^S \\
\pm 2\alpha N n S \mu^S
\end{bmatrix} r^{n+1} \exp \pm i n \theta \tag{36}
\]
while the corresponding stress field is,

\[
\mathbf{S}^b_{s_n} = \begin{bmatrix}
-n(2)(\lambda + n^S\mu^S) \\
(n+2)(\lambda + n^S\mu^S) \\
\pm in(\lambda + n^S\mu^S) \\
0 \\
0
\end{bmatrix} r^n \exp \pm in\theta
\]  

(37)

4.1.2. **Harmonic solutions**

When harmonic displacement potentials are used to construct the solution set \( \mathbf{U}^h_s \), they generate pure deviatoric modes with static fluid behavior. For degrees \( n \geq 0 \), pairs of complex conjugate solutions of the Navier equation (30) are obtained and collected in set \( \mathbf{U}^h_s \),

\[
\mathbf{U}^h_s = \frac{1}{2(n+1)n^S\mu^S} \begin{bmatrix}
1 \\
\pm i \\
0 \\
0
\end{bmatrix} r^{n+1} \exp \pm i(n+2)\theta
\]  

(38)

The corresponding stress modes are given by,

\[
\mathbf{S}^h_{s_n} = \begin{bmatrix}
1 \\
-1 \\
\pm i \\
0 \\
0
\end{bmatrix} r^n \exp \pm i(n+2)\theta
\]  

(39)
4.1.3. Constant fluid pressure modes

If a constant pressure is applied to one of the fluids, it generates a hydrostatic stress field $S^f_s$ and a radial fluid displacement field, $U^f_s$, given by,

$$U^W_s = \frac{1}{2(M_{NN}M_{WW} - M_{WN}^2)} \begin{bmatrix} 0 \\ 0 \\ M_{NN} \\ 0 \\ -M_{WN} \\ 0 \end{bmatrix} r$$  \hspace{1cm} (40)

$$S^W_s = \begin{bmatrix} \alpha^W \\ \alpha^W \\ 0 \\ 1 \\ 0 \end{bmatrix}$$  \hspace{1cm} (41)

$$U^N_s = \frac{1}{2(M_{NN}M_{WW} - M_{WN}^2)} \begin{bmatrix} 0 \\ 0 \\ -M_{WN} \\ 0 \\ M_{WW} \\ 0 \end{bmatrix} r$$  \hspace{1cm} (42)

$$S^N_s = \begin{bmatrix} \alpha^N \\ \alpha^N \\ 0 \\ 0 \\ 1 \end{bmatrix}$$  \hspace{1cm} (43)

4.1.4. Null stress modes

Besides the stress generating modes described above, the following rigid-body displacement modes defined on the solid phase also satisfy the Navier equation,

$$U^S_r = \begin{bmatrix} 0 & \cos \theta & \sin \theta \\ \sin \theta & \cos \theta & r \sin 2\theta \\ -\sin \theta & \cos \theta & r \cos 2\theta -r \sin 2\theta \end{bmatrix}$$  \hspace{1cm} (44)

In each of the fluid phases, five linear free flow modes are identified,

$$U^f_r = \begin{bmatrix} \cos \theta & \sin \theta & 0 & r \sin 2\theta & r \cos 2\theta \\ -\sin \theta & \cos \theta & r & r \cos 2\theta & -r \sin 2\theta \end{bmatrix}$$  \hspace{1cm} (45)

along with $n + 2$ free flow modes of degree $n \geq 2$

$$U^f_r = \begin{bmatrix} \sin^n \theta & \cos \theta & \sin \theta \cos^n \theta & (k + \cos^2 \theta - n \sin^2 \theta) \sin^k \theta \cos^{n-k-1} \theta \\ -\sin^{n+1} \theta & \cos^{n+1} \theta & - \sin^n \theta & - (1 + n) \sin^{k+1} \theta \cos^{n-k} \theta \end{bmatrix} r^n$$  \hspace{1cm} (46)

with $0 \leq k \leq n - 1$ and $f = W, N$. For the sake of conciseness, only the non-zero terms of the stress-free modes are given here. They correspond to the solid phase.
displacement, (44), and to the motion of each of the fluid phases, (45) and (46), while the displacements in the other phases remain null.

4.2. Solutions of the non-homogeneous (spectral) problem

To obtain its general solution, Navier equation (28) is written as,

\[
\left\{
\begin{align*}
(M_{SS} + n^S \mu^S) \nabla^s \mathbf{u}^S + n^S \mu^S \nabla^s \mathbf{u}^S + M_{SW} \nabla^s \mathbf{u}^W + M_{SN} \nabla^s \mathbf{u}^N + \\
\omega^2 \left( \rho_{SS} \mathbf{u}^S + \rho_{SW} \mathbf{u}^W + \rho_{SN} \mathbf{u}^N \right) &= 0 \\
M_{SW} \nabla^s \mathbf{u}^S + M_{WW} \nabla^s \mathbf{u}^W + M_{WN} \nabla^s \mathbf{u}^N + \\
\omega^2 \left( \rho_{SW} \mathbf{u}^S + \rho_{WW} \mathbf{u}^W \right) &= 0 \\
M_{SN} \nabla^s \mathbf{u}^S + M_{WN} \nabla^s \mathbf{u}^W + M_{NN} \nabla^s \mathbf{u}^N + \\
\omega^2 \left( \rho_{SN} \mathbf{u}^S + \rho_{NN} \mathbf{u}^N \right) &= 0
\end{align*}
\right.
\]  

(47)

where,

\[
\begin{align*}
\rho_{SS} &= n^S \rho^S - \frac{i}{\omega} (\mu^W + \mu^N) \\
\rho_{WW} &= n^W \rho^W - \frac{i}{\omega} \mu^W \\
\rho_{NN} &= n^N \rho^N - \frac{i}{\omega} \mu^N \\
\rho_{SW} &= \frac{i}{\omega} \mu^W \\
\rho_{SN} &= \frac{i}{\omega} \mu^N
\end{align*}
\]  

(48) - (52)

Definitions given in the Appendix A were used to obtain the above expressions. Note that the index \( n \) associated to each spectral problem was dropped, for simplicity.

4.2.1. Compressional waves

Problem (47) admits as solution three compressional waves and one shear wave. The displacements corresponding to the compressional waves are determined by solving the following eigenproblem, for the wave numbers \( \beta_P, j = 1, 2, 3 \) and the corresponding phase multipliers \( \gamma_P^j, \alpha = \{ S, W, N \} \),

\[
\begin{pmatrix}
\frac{-\beta_P^2}{\omega^2} & M_{SS} & M_{SW} & M_{SN} \\
M_{SW} & M_{WW} & M_{WN} & M_{NN} \\
M_{SN} & M_{WN} & M_{NN} & M_{NN}
\end{pmatrix}
\begin{pmatrix}
\rho_{SS} \\
\rho_{WW} \\
\rho_{NN} \\
\rho_{SW}
\end{pmatrix}
+ \begin{pmatrix}
\rho_{SS} & \rho_{SW} & \rho_{SN} \\
\rho_{SW} & \rho_{WW} & 0 \\
\rho_{SN} & 0 & \rho_{NN}
\end{pmatrix}
\begin{pmatrix}
\gamma_S^j \\
\gamma_W^j \\
\gamma_N^j
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]  

(53)

The displacement field corresponding to each compressional wave is given by,

\[
U_P = \frac{1}{2} \begin{pmatrix}
\beta_P^1 (J_{n-1} (\beta_P r) - J_{n+1} (\beta_P r)) \\
i\beta_P^1 (J_{n-1} (\beta_P r) + J_{n+1} (\beta_P r)) \\
\gamma_P^j \beta_P^1 (J_{n-1} (\beta_P r) - J_{n+1} (\beta_P r)) \\
i\gamma_P^j \beta_P^1 (J_{n-1} (\beta_P r) + J_{n+1} (\beta_P r)) \\
\gamma_P^j \beta_P^1 (J_{n-1} (\beta_P r) - J_{n+1} (\beta_P r)) \\
i\gamma_P^j \beta_P^1 (J_{n-1} (\beta_P r) + J_{n+1} (\beta_P r))
\end{pmatrix} \exp \imath \theta
\]  

(54)
(a) P1 wave, 100Hz.  
(b) P1 wave, 500Hz.  
(c) P2 wave, 100Hz.  
(d) P2 wave, 500Hz.  
(e) P3 wave, 100Hz.  
(f) P3 wave, 500Hz.

Fig. 2. Wavelength variation with saturation and permeability for compressional waves.

where $J_n(z)$ represents the Bessel function of the second kind and order $n$, and $n$ is an integer between $-N$ and $N$. The equilibrated domain stress field corresponding
Fig. 3. Attenuation variation with saturation and permeability for compressional waves.

to the displacement field (54) is,

\[
S_p = \frac{1}{2} \left[ \begin{array}{c}
S_{p_1}^r \\
S_{p_1}^w \\
\end{array} \right] 
\begin{array}{c}
\iota n^S \mu^S \left( J_{n-2} \left( \beta_{p_j} r \right) \right) - J_{n+2} \left( \beta_{p_j} r \right) \\
-2 \left( M_{SW} + \gamma_{p_j}^W M_{WW} + \gamma_{p_j}^N M_{NN} \right) J_n \left( \beta_{p_j} r \right) \\
-2 \left( M_{SN} + \gamma_{p_j}^W M_{WN} + \gamma_{p_j}^N M_{NN} \right) J_n \left( \beta_{p_j} r \right) \\
\end{array} \right] \exp \left( \iota n \theta \right) \tag{55}
\]
where:

\[ S_{r}^{p} = n_{S}^{S} \mu_{S}^{S} \left[ 2J_{n} (\beta_{P_{j}}r) + J_{n-2} (\beta_{P_{j}}r) + J_{n+2} (\beta_{P_{j}}r) \right] - 2 \left( M_{SS} + 2n_{S}^{S} \mu_{S}^{S} \right) J_{n} (\beta_{P_{j}}r) \]  

(56)

\[ S_{g}^{p} = n_{S}^{S} \mu_{S}^{S} \left[ 2J_{n} (\beta_{P_{j}}r) - J_{n-2} (\beta_{P_{j}}r) - J_{n+2} (\beta_{P_{j}}r) \right] - 2 \left( M_{SS} + 2n_{S}^{S} \mu_{S}^{S} \right) J_{n} (\beta_{P_{j}}r) \]  

(57)

It is important to understand in detail the characteristics of the compressional waves propagating through unsaturated soils, as reflected by the solutions derived above. To discuss this issue, the variations of the wavelength \( (\lambda = \frac{\omega}{2\pi}) \) and attenuation (defined as \( \Im (\beta) \)) of the three compressional waves are presented in Figures 2 and 3, respectively, as functions of the hydraulic conductivity and saturation of the medium. The problem is defined on a Massilon sandstone, whose geomechanical properties are defined in APPENDIXXXX2. The plots are presented for two frequencies of the propagating wave, namely 100Hz and 500Hz. The saturation range used in the plots (10% \( \leq S \leq 95\% \)) corresponds to the limits of the adopted mathematical model, which is only valid between the residual and the air-entry degrees of saturation. It should be noted that for \( S \leq 10\% \) the water is mainly adsorptive, while for \( S \geq 95\% \), the air is present in bubble form and the medium can be analyzed using biphasic models (e.g. [Smeulders (1992)]). The hydraulic conductivity range is taken between \( 10^{-2}\m/s \), corresponding to pervious soils (e.g. well sorted sands, sand and gravel) and \( 10^{-6}\m/s \), which corresponds to semi-pervious soils (e.g. very fine sands, loess, silt).

As expected, the wavelength values of all compressional waves are significantly larger for the low frequency case (left hand side of Figure 2) than for the higher frequency case (right hand side of the same Figure). While the variation trend of the wavelength is remarkably similar for any given type of wave, very significant differences exist between the three compressional waves.

The wavelength and the phase velocity \( (v = \lambda \frac{\omega}{2\pi}) \) of the fast propagating \( P_{1} \) wave exhibit very low sensitivity to variations of the permeability and saturation of the medium. This is consistent with the physical nature of this compressional wave, which propagates essentially through direct contact between the solid particles and is thus insensitive to those properties of the medium that condition the movement of the fluid constituents. Moreover, as \( \lambda \cdot \omega \approx \text{constant} \), the phase velocity of \( P_{1} \) waves is also insensitive to the frequency of the travelling wave (i.e. the medium is not dispersive for \( P_{1} \) waves). The wave presents virtually no attenuation (note that the vertical axis in Figure 3 is logarithmic), especially for the lower frequency case.

Conversely, the second compressional wave is strongly influenced by the saturation and permeability of the medium, which suggests that the propagation of this wave is determined by the fluid phases. Both the wavelength and the phase velocity of the \( P_{2} \) wave reach a neat maximum for the smallest saturation and permeability tested. Compared to the \( P_{1} \) wave, the second compressional wave propagates...
much slower and its wavelength is significantly smaller. The phase velocity of the P2 wave is proportional with its frequency, especially at lower saturation values (i.e. the medium is dispersive in respect to the P2 wave). The P2 wave is much more attenuated than the P1 wave, especially for low permeability media. This happens because P2 waves are transmitted through fluid motion which is obviously hindered by the low permeability of the medium. However, unlike P1 waves, the attenuation of the P2 wave does not depend significantly on the frequency.

Finally, the third compressional wave is only significant for mid-range saturation values. When saturation is below 10% or above 90%, the phase velocity of the P3 wave is extremely small and the attenuation large. This means that the P3 wave is conditioned by the existence of capillary pressures, which essentially constitute its mean of propagation. When the fluid phases tend to disconnect (the liquid phase disconnects when the medium is close to dry, while the opposite is true for the gas phase), the capillary pressures vanish and the P3 wave disappears. The third compressional wave propagates very slowly and is more attenuated than any other wave propagating through unsaturated soils, which means that its contribution to the dynamic response of the medium is only relevant at close field. The phase velocity of the P3 wave increases with the frequency (i.e. the medium is P3-dispersive), but the attenuation is essentially frequency-independent.

The features of the three compressional waves have important consequences on their computational modelling using conventional finite elements, mainly because of the well-known need of calibrating the leading dimension of the finite element such as to have at least six (but preferably ten) finite elements per wavelength [Pluymers et al. (2007)]. Due to the extremely small wavelength of the P3 wave, the correct inclusion of this wave in the finite element model is practically impossible when conventional elements are used and large domains need to be modelled. This is especially problematic for high permeability media, where the attenuation is smaller, but has a lesser impact on the results at far-field and/or for low permeability soils. The same problem may hinder the modelling of the P2 wave, especially at higher frequencies. Indeed, high frequencies greatly reduce the wavelength of the second compressional wave, but (unlike the case of P1 waves) do not increase significantly the attenuation of the same wave. This means that the wave may still propagate deep into the medium, but it becomes much harder to model due to its reduced wavelength.

The hybrid-Trefftz elements presented in Section 5 are aimed to give a consistent response to these modelling issues. Due to their physically-meaningful approximations, hybrid-Trefftz elements are insensitive to wavelength variations and are not bounded to observe the condition described in [Pluymers et al. (2007)]. Moreover, as their domain approximation is strictly hierarchical, hybrid-Trefftz elements allow the analyst to only include in the basis waves that are needed, when are needed, discarding, for instance, far-field P3 waves in impervious media.
4.2.2. Shear waves

A single shear wave solves the Navier equation (47). It is characterized by wave number and phase multipliers given by,

\[ \beta_s^2 = \omega^2 \rho_s \mu_s + \gamma_s^W \rho_{sw} + \gamma_s^N \rho_{sn} \]

\[ \gamma_s^W = -\frac{\rho_{sw}}{\rho_{ww}} \]

\[ \gamma_s^N = -\frac{\rho_{sn}}{\rho_{nn}} \]

The displacement and stress fields corresponding to the shear wave are given by,

\[ U_s = \frac{1}{2} \begin{bmatrix} i\beta_s^{-1}(J_{n+1}(\beta_{sr}) + J_{n-1}(\beta_{sr})) \\ \beta_s^{-1}(J_{n+1}(\beta_{sr}) - J_{n-1}(\beta_{sr})) \\ i\gamma_s^W \beta_s^{-1}(J_{n+1}(\beta_{sr}) + J_{n-1}(\beta_{sr})) \\ \gamma_s^W \beta_s^{-1}(J_{n+1}(\beta_{sr}) - J_{n-1}(\beta_{sr})) \\ i\gamma_s^N \beta_s^{-1}(J_{n+1}(\beta_{sr}) + J_{n-1}(\beta_{sr})) \\ \gamma_s^N \beta_s^{-1}(J_{n+1}(\beta_{sr}) - J_{n-1}(\beta_{sr})) \end{bmatrix} \exp(i\theta) \]

\[ S_s = \frac{1}{2} \begin{bmatrix} \mu_s (J_{n-2}(\beta_{sr}) - J_{n+2}(\beta_{sr})) \\ -i\mu_s (J_{n-2}(\beta_{sr}) - J_{n+2}(\beta_{sr})) \\ -n^5 \mu_s (J_{n-2}(\beta_{sr}) + J_{n+2}(\beta_{sr})) \end{bmatrix} \exp(i\theta) \]

Fig. 4. Wavelength variation with saturation and permeability for shear waves.

As shear waves can only propagate through the solid phase, their vibration and attenuation characteristics, presented in Figures 4 and 5, are qualitatively similar to those of the P1 waves. Dispersion of these waves is practically absent, and the attenuation (represented logarithmically in Figure 5) is proportional to the frequency. S waves propagate at a lower phase velocity than P1 waves and are similarly attenuated. Consequently, conventional finite elements are typically able to recover correctly the propagation of the shear waves through unsaturated soils.
5. Hybrid-Trefftz elements

Hybrid-Trefftz elements are formulated next for spectral problems of type (28), resulting from the discretization in time of the original problem. They are derived from the corresponding (pure) hybrid elements by restricting their domain approximation basis to functions satisfying locally domain equations (1) to (3). Consequently, their bases collect functions belonging to the solution sets derived in Sections 4.1 and 4.2.

Hybrid-Trefftz elements are encoded into two alternative models, namely the displacement model and the stress model. The fundamental distinction between the models is that the displacement model produces domain solutions that are locally compatible, while the solutions obtained with the stress model are locally equilibrated.

In the following derivations, index \( n \) denoting each of the spectral problems defined by equations (17) to (22) is dropped, for simplicity.

5.1. Finite element discretization

Consider that the domain \( V \) represented in Figure 1 is discretized into finite elements and let the domain of a generic element be denoted by \( V^e \) and its boundary by \( \Gamma^e \). Boundary \( \Gamma^e \) is formed, in general, by the complementary Neumann (\( \Gamma^e_\sigma \)) and Dirichlet (\( \Gamma^e_u \)) parts, where equations (21) and (22) are prescribed, respectively, and the interior (\( \Gamma^e_i \)) part, where the boundary equilibrium and compatibility conditions (65) and (66) must be satisfied,

\[
\begin{align*}
t^i + t^k &= 0 \text{ on } \Gamma^e_i \\
u^i &= u^k \text{ on } \Gamma^e_i
\end{align*}
\]

where \( i \) and \( k \) denote the two finite elements that share the interior boundary \( \Gamma^e_i \) (Figure 6).
As the domain approximations are strictly hierarchical for both displacement and stress models, hybrid-Trefftz elements may have arbitrary numbers of edges and nodes. Also, due to their excellent robustness to gross mesh distortion, they are not bounded by the regularity restrictions specific to the conventional elements.

5.2. Hybrid-Trefftz displacement element

5.2.1. Approximation bases

The displacement model of the hybrid-Trefftz element is built on the direct approximation of the displacement field in the domain of the element and of the traction field on its essential (extended Dirichlet) boundary.

The Trefftz-compliant domain basis used in this work is constructed using subsets of the dynamic (spectral), static, and rigid body solutions of domain equations (1) to (3), as derived in Section 4,

\[ u = U_dX_d + U_sX_s + U_rX_r \text{ in } V^e \]  

(67)

where \( U_d = (U_{P_1}, U_{P_2}, U_{P_3}, U_S) \).

It should be noted that only the static terms \( U_s \) and \( U_r \) need to be included in the Trefftz basis of a static problem, that is, of type (29). Similarly, only the spectral terms collected in matrix \( U_d \) are needed to construct the Trefftz basis of spectral problems of type (28) when the initial condition term is absent. However, transient problems require the presence of both static and harmonic parts of the Trefftz basis in order for the approximation to be complete.

In equation (67), vectors \( X_d, X_s \) and \( X_r \) collect the generalized displacements associated with the modes present in the displacement basis. They are not associated with the nodes (or other geometric features) of the element and have no particular physical meaning. This option endorses the use of elements with different \( p \)-refinements in the same mesh and enhances the flexibility in the construction of the domain basis, which may thus be biased towards those modes that are physically meaningful for each particular problem (for instance, far-field elements with
low permeability may not include modes corresponding to the highly evanescent P3 wave which is not expected to have any influence in the region).

A dependent approximation of the domain stress field $\sigma$ is obtained from definition (67) using compatibility and elasticity equations (18) and (19) as,

$$\sigma = S_dX_d + S_sX_s \text{ in } V^e$$

(68)

Clearly, bases $S_d = (S_{P_1} S_{P_2} S_{P_3} S_S)$ and $S_s$ correspond to the dynamic (spectral) and static solutions derived in Section 4. Thus, the functions present in bases (67) and (68) satisfy the (Trefftz) equations listed in Table 1,

<table>
<thead>
<tr>
<th>Equilibrium equations</th>
<th>Compatibility and elasticity equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{D}S_d + \omega^2 \rho U_d = 0$</td>
<td>$S_d = k\mathcal{D}^*U_d$</td>
</tr>
<tr>
<td>$\mathcal{D}S_s = 0$</td>
<td>$S_s = k\mathcal{D}^*U_s$</td>
</tr>
<tr>
<td>$0 = \mathcal{D}^*U_r$</td>
<td></td>
</tr>
</tbody>
</table>

Table 1. Trefftz constraints - displacement element.

The traction (Cauchy stress) in the solid phase and the fluid pore pressures are independently approximated on the Dirichlet and interior boundaries of the element (extended Dirichlet boundary), $\Gamma_u^e = \Gamma_u \cup \Gamma_i^e$ as,

$$t = ZP \text{ on } \Gamma_u^e$$

(69)

Except for completeness and linear independence, no restrictions are enforced on the functions collected in the (hierarchical) boundary basis $Z$.

5.2.2. Domain statements

The domain statements of the hybrid-Trefftz displacement element are obtained by enforcing weakly the domain equilibrium equation (17), using the displacement basis (67) for weighting,

$$\int U_j^* (\mathcal{D}\sigma + \omega^2 \rho u) \, dV^e = \int U_j^* F^0 \, dV^e$$

(70)

where $U_j^*$ denotes the transpose conjugate of matrix $U_j$ and $j = \{d, s, r\}$.

The integration by parts of the first term of equation (70) forces the emergence of the boundary terms, in which boundary equilibrium equation (21) and Dirichlet boundary approximation (69) are enforced:

$$\int U_j^* \mathcal{D}\sigma \, dV^e = \int U_j^* tF \, d\Gamma_\sigma + \int U_j^* Z d\Gamma_u P - \int (\mathcal{D}^*U_j)^* \sigma \, dV^e$$

(71)
Enforcing approximations (67) and (68) in equations (70) and (71), respectively, and using properties given in Table 1, yields three equations of type,

\[D_{jd} X_d + D_{js} X_s + D_{jr} X_r - B_j P = x_{\Gamma_j} - x_0^j \quad (72)\]

one for each \(j\). In the above equation, the following definitions hold,

\[D_{jd} = \int U_j^* N S_d \, d\Gamma \quad (73)\]

\[D_{js} = \int U_j^* N S_s \, d\Gamma - \omega^2 \int U_j^* \rho U_s \, dV^e \quad (74)\]

\[D_{jr} = -\omega^2 \int U_j^* \rho U_r \, dV^e \quad (75)\]

\[B_j = \int U_j^* Z \, d\Gamma_u \quad (76)\]

\[x_{\Gamma_j} = \int U_j^* t_{\Gamma} \, d\Gamma_\sigma \quad (77)\]

\[x_0^j = \int U_j^* F^0 \, dV^e \quad (78)\]

5.2.3. **Boundary statement**

The boundary statement of the hybrid-Trefftz displacement element is obtained enforcing on average essential conditions (22) and (66) on the extended Dirichlet boundary of the element, using the functions collected in basis \(Z\) for weighting.

On the exterior Dirichlet boundary of the element, this procedure yields,

\[\int Z^* (u - u_\Gamma) \, d\Gamma_u^e = 0 \quad (79)\]

Inserting approximation (67) into the above expression and taking into account definition (76), the following boundary equation is recovered,

\[-\sum_j (B_j^* X_j) = -p_\Gamma \quad (80)\]

\[p_\Gamma = \int Z^* u_{\Gamma} \, d\Gamma_u^e \quad (81)\]

On the interior boundary shared by elements \(i\) and \(k\), compatibility condition (66) is enforced weakly as,

\[\int Z^* (u^i - u^k) \, d\Gamma_i^e = 0 \quad (82)\]

Insertion of definition (76) into the above equation leads to the inter-element compatibility equation,

\[-\sum_j \left[ (B_j^i)^* X_j^i \right] + \sum_j \left[ (B_j^k)^* X_j^k \right] = 0 \quad (83)\]
5.2.4. Solving system

The solving system of the hybrid-Trefftz displacement element is obtained by collecting the domain and boundary statements (72) and (80) (for exterior Dirichlet boundaries) or (83) (for interior boundaries),

\[
\begin{pmatrix}
D_{dd} & D_{ds} & D_{dr} & -B_d \\
D_{sd} & D_{ss} & D_{sr} & -B_s \\
D_{rd} & D_{rs} & D_{rr} & -B_r \\
-B_d^* & -B_s^* & -B_r^* & 0
\end{pmatrix}
\begin{pmatrix}
X_d \\
X_s \\
X_r \\
P
\end{pmatrix}
= \begin{pmatrix}
x_{\Gamma_d} - x_0^d \\
x_{\Gamma_s} - x_0^s \\
x_{\Gamma_r} - x_0^r \\
-p_r
\end{pmatrix}
\tag{84}
\]

System (84) is highly sparse (typically more than 90% of its coefficients are null) and strongly localized, if not condensed on the boundary variables. This means that the system can be stored and solved using highly efficient procedures especially designed for sparse systems. Moreover, system (84) is ideally suited to parallel processing, as the generalized displacements collected in vectors \(X_j\) are strictly element-dependent and the generalized tractions \(P\) can be shared by at most two neighboring elements. Finally, as all bases are strictly hierarchical and no summation of coefficients is required in the assemblage of the global solving system, localized \(p\)-refinement procedures only require the computation of the terms that correspond to the new approximation functions. This trait endorses the application of automatic \(p\)-adaptive procedures for the selection of the orders of the domain and boundary bases.

5.3. Hybrid-Trefftz stress element

5.3.1. Approximation bases

The hybrid-Trefftz stress model is constructed on the direct approximation of the total stress and fluid pressure fields in the domain of the element and of the displacement field on its Neumann and interior boundaries.

Similar to the displacement element, the Trefftz-compliant domain basis is constructed using the spectral and static solutions of the original equations, derived in Section 4,

\[
\sigma = S_d Y_d + S_s Y_s \quad \text{in } V^e
\tag{85}
\]

where \(S_d = (S_{P_1} \, S_{P_2} \, S_{P_3} \, S_{S})\).

Thus, the stress basis (85) adopted for the stress model is the same as for the displacement element presented above. If the original time-domain problem defined by equations (1) to (2) is harmonic or if the initial conditions of the current step of a transient problem are null, only the spectral part of the basis (85) is needed. Conversely, if the original problem is static, basis \(S_s\) is sufficient to consistently recover its solution, except for the rigid-body displacements of the elements, which have no impact on the stress field in the domain of the element. However, the rigid-body modes can be obtained by enforcing the displacement continuity on the
extended Dirichlet boundary, in the postprocessing phase, as reported for biphasic media in [Moldovan et al (2012)].

As typical of hybrid stress elements, a dependent domain displacement is constructed such to satisfy locally equilibrium equation (17),

$$\mathbf{u} = \mathbf{U}_d \mathbf{y}_d + \mathbf{u}_0 \text{ in } V^e$$  \hspace{1cm} (86)

$$\mathbf{U}_d = (\mathbf{U}_P \mathbf{U}_P \mathbf{U}_S)$$  \hspace{1cm} (87)

$$\mathbf{u}_0 = \omega^{-2} \rho^{-1} \mathbf{F}_0$$  \hspace{1cm} (88)

The functions collected in spectral basis $\mathbf{U}_d$ are derived such as to equilibrate locally stress functions $S_d$, as presented in Section 4.2. The particular solution $\mathbf{u}_0$ is needed to equilibrate the initial condition term $\mathbf{F}_0$, as $D S_s = 0$. Table 2 lists the equations satisfied by the functions present in bases (85) and (86).

<table>
<thead>
<tr>
<th>Equilibrium equations</th>
<th>Compatibility and elasticity equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D S_d + \omega^2 \rho U_d = 0$</td>
<td>$S_d = k D^* U_d$</td>
</tr>
<tr>
<td>$D S_s = 0$</td>
<td></td>
</tr>
<tr>
<td>$\omega^2 \rho u_0 = F_0$</td>
<td></td>
</tr>
</tbody>
</table>

Table 2. Trefftz constraints - stress element.

It should be noted that neither compatibility nor elasticity equations are satisfied a priori by the stress and displacement approximations. These conditions are enforced in a weak form in the next section.

Independent on the domain approximations listed above, the displacement fields in the three phases are independently approximated on the Neumann and interior boundaries of the element as,

$$\mathbf{u} = Z \mathbf{Q} \text{ on } \Gamma^e \sigma$$  \hspace{1cm} (89)

Except for completeness and linear independence, no restrictions are enforced on the functions collected in the (hierarchical) boundary basis $Z$.

5.3.2. Domain statement

The domain statement of the hybrid-Trefftz stress element corresponds to the weak enforcement of the compatibility and elasticity equations (18) and (20) in the domain of the element, using stress basis (85) for weighting,

$$\int S^e_j (f \mathbf{a} - D^* \mathbf{u}) \, dV^e = 0$$  \hspace{1cm} (90)
where \( j = \{d, s\} \).

The integration by parts of the second term of equation (70) forces the emergence of the boundary displacement terms, in which boundary continuity equation (22) and Dirichlet boundary approximation (89) are enforced:

\[
\int S_j^* \mathbf{D}^* \mathbf{u} \, dV = \int (N \mathbf{S}_j)^* \mathbf{u} \mathbf{r} \, d\Gamma_u + \int (N \mathbf{S}_j)^* \mathbf{Z} \, d\Gamma_\sigma \mathbf{Q} - \int (\mathbf{D} \mathbf{S}_j)^* \mathbf{u} \, dV (91)
\]

Inserting approximation (85) and (86) in expressions (90) and (91) and using the Trefftz constraints listed in Table 2 yields,

\[
F_{j\alpha} Y_{\alpha} + F_{js} (y_{s} - y_i^0) = 0 \quad (92)
\]

\[
F_{dd} = \int (N S_d)^* \mathbf{U} \, d\Gamma = F_{ds}^* \quad (93)
\]

\[
F_{sd} = \int (N S_s)^* \mathbf{U} \, d\Gamma = F_{ss}^* \quad (94)
\]

\[
A_j = \int (N \mathbf{S}_j)^* \mathbf{Z} \, d\Gamma_\sigma \quad (96)
\]

\[
y_{\alpha} = \int (N \mathbf{S}_j)^* \mathbf{u} \mathbf{r} \, d\Gamma_u \quad (97)
\]

\[
y_i^0 = \int (\mathbf{D} S_i)^* \mathbf{u}_0 \, dV \quad (98)
\]

\[
y_s^0 = 0 \quad (99)
\]

5.3.3. Boundary statements

Essential conditions (21) and (65) are enforced on the Neumann and interior boundaries of the element, respectively, using the functions collected in basis \( \mathbf{Z} \) as test functions.

On the Neumann boundary, this yields,

\[
\int Z^* (N \mathbf{\sigma} - \mathbf{t}_\Gamma) \, d\Gamma_\sigma = 0 \quad (100)
\]

Using approximation (85) and definition (96), equation (100) is written in the following form,

\[
- \sum_j (A_j^* Y_j) = -q_\Gamma \quad (101)
\]

\[
q_\Gamma = \int Z^* \mathbf{t}_\Gamma \, d\Gamma_\sigma \quad (102)
\]

On the interior boundary shared by elements \( i \) and \( k \), equilibrium condition (65) is enforced weakly as,

\[
\int Z^* (\mathbf{t}_i + \mathbf{t}_k) \, d\Gamma_\sigma = 0 \quad (103)
\]
Definition (96) is inserted into expression (103), to yield the inter-element equilibrium equation,

$$-\sum_j \left[ (A^*_j)^* Y^*_j \right] - \sum_j \left[ (A^*_j)^* Y^*_j \right] = 0$$  \hspace{1cm} (104)

5.3.4. Solving system

Solving system (105) is obtained by merging domain equation (92) and boundary statements (101), for exterior Dirichlet boundaries, and (83) for interior boundaries.

$$\begin{bmatrix}
F_{dd} & F_{ds} & -A_d \\
F_{sd} & F_{ss} & -A_s \\
-A_d^* & -A_s^* & 0
\end{bmatrix}
\begin{bmatrix}
Y_d \\
Y_s \\
Q
\end{bmatrix}
= \begin{bmatrix}
y_{d} - y_{0d} \\
-y_{s} \\
-q
\end{bmatrix}$$  \hspace{1cm} (105)

System (105) shares all properties of the solving system (84) of the displacement element. Moreover, as material flexibility matrix \( f \) is Hermitian (a property not shared by the generalized mass matrix \( \rho \) if velocity-induced terms are present), system (105) is Hermitian.

References


### Appendix A. Explicit expressions of matrices and coefficients

The definitions of the matrices and coefficients given below were established in accordance with [Wei and Muraleetharan (2002)]. The notations used in the original article were maintained, whenever possible.

The expression of the damping $d_0$ and mass $\rho_0$ matrices present in equilibrium
equation (1) are,

\[
d_0 = \begin{bmatrix}
\mu^W + \mu^N & 0 & -\mu^W & 0 & -\mu^N & 0 \\
0 & \mu^W + \mu^N & 0 & -\mu^W & 0 & -\mu^N \\
-\mu^W & 0 & \mu^W & 0 & 0 & 0 \\
0 & -\mu^W & 0 & \mu^W & 0 & 0 \\
-\mu^N & 0 & 0 & 0 & \mu^N & 0 \\
0 & -\mu^N & 0 & 0 & 0 & \mu^N 
\end{bmatrix}
\] (A.1)

\[
\rho_0 = \begin{bmatrix}
n^S \rho^S & 0 & 0 & 0 & 0 & 0 \\
0 & n^S \rho^S & 0 & 0 & 0 & 0 \\
0 & 0 & n^W \rho^W & 0 & 0 & 0 \\
0 & 0 & 0 & n^W \rho^W & 0 & 0 \\
0 & 0 & 0 & n^N \rho^N & 0 & 0 \\
0 & 0 & 0 & 0 & n^N \rho^N & 0 
\end{bmatrix}
\] (A.2)

The expression of the stiffness matrix \( k \) present in equation (3) is,

\[
k = \begin{bmatrix}
M_{SS} + 2n^S \mu^S & M_{SS} & 0 & M_{SW} & M_{SN} \\
M_{SS} & M_{SS} + 2n^S \mu^S & 0 & M_{SW} & M_{SN} \\
0 & 0 & n^S \mu^S & 0 & 0 \\
M_{SW} & M_{SW} & 0 & M_{WW} & M_{WN} \\
M_{SN} & M_{SN} & 0 & M_{WN} & M_{NN} 
\end{bmatrix}
\] (A.3)

In expressions (A.1) to (A.3), \( n^\alpha \ (\alpha = S,W,N) \) represent the volume fractions of each phase and,

\[
\mu^f = \frac{(n^f)^2 \nu^f}{k_k^f}, \text{ where } f = W, N
\] (A.4)

In expression (A.4), \( \nu^f \) is the dynamic viscosity of the fluid, \( k \) is the intrinsic permeability and \( k_k^f \) is the relative permeability of the \( f \)–fluid.

The stiffness moduli present in definition (A.3) are given by the following ex-
In the above expressions, $\mu^S$ and $\lambda^S$ are the Lamé's coefficients. Elastic constants $\lambda^S_{pe}$, $\Theta^W$ and $\Theta^N$ can be determined experimentally, following the procedures described in [Wei and Muraleetharan (2002)]. Finally, $K_\alpha$ ($\alpha = S, W, N$) are the bulk moduli of the phases.