Hybrid-Mixed Elements for Elastic Non-Linear Porodynamics

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Abstract

The purpose of this paper is to present the formulation of hybrid-mixed displacement and stress elements for the dynamic analysis of biphasic media with elastic non-linear behaviour. The medium is composed of a porous solid skeleton, fully saturated by a compressible fluid obeying to the Darcy law. The solid matrix (i.e. the particle that forms the solid skeleton) is considered incompressible. Biot’s theory of porous media is used to derive the governing equations of the problem. Non-linear effects are present in the response of the medium due to the dependency of the fluid density on the pore pressure and of the pore fraction on the volumetric deformation.

Both hybrid-mixed models are constructed on the direct approximation of the generalized displacement and stress fields in the element, complemented by the independent approximation of the traction field on the Dirichlet boundary of the element (displacement model), or of the displacement field on the Neumann boundary of the element (stress element). The trial functions are used to enforce weakly the domain equations and essential boundary conditions. The well-known Newmark method is used for the integration in time of the space-discretized equations.

Keywords: Saturated porous media, hybrid-mixed elements, non-linear elastic problems.

1. Introduction

The roots of all hybrid variants of the finite element method can be traced back to the pioneering work of (Veubeke, 1965) and (Pian and Tong, 1969). These authors showed that finite elements can be formulated by relaxing the continuity conditions on the essential boundaries of the mesh, as opposed to the conventional (conforming) elements, that choose the shape functions in such way to satisfy a priori the essential boundary conditions. The main advantages common to all these formulations include more flexibility in constructing the approximation basis, more balanced accuracy in terms of the computed displacement and stress fields, and (with certain exceptions) the possibility to fit these formulations into the standard
Direct Stiffness Method framework that is currently used in most commercially available finite element software. As discussed in (Freitas, 1999), the hybrid elements can be classified in three categories, namely hybrid-mixed elements, hybrid elements and hybrid-Trefftz elements. The distinction between these categories is given by the choice of the domain approximation functions. In the hybrid-mixed formulations, the choice of the shape functions is totally free, as they must not satisfy locally any domain equation. Conversely, the distinctive feature of the hybrid-Trefftz elements is that the domain approximation functions must be so chosen as to satisfy locally all of the domain equations. The hybrid elements represent an intermediate situation, where some, but not all, of the domain equations must be locally satisfied.

In the recent years, significant research effort has been dedicated by our group to the modelling of continuous media subjected to static and dynamic excitations, using non-conventional (hybrid) finite element formulations. The developments have avoided, right from the beginning, a competitive attitude towards the conventional elements, preferring instead to regard the hybrid elements as special-purpose finite elements, that may be very well suited in situations that hinder the behaviour of the conventional elements. Indeed, problems involving stress concentrations, as those occurring around wedges, notches, cracks or point loads are consistently handled by simply enriching the approximation basis with functions with the desired behaviour near the stress concentration zone, without the need of computationally expensive mesh refinements. Moreover, incompressible or nearly-incompressible media are adequately modelled using (problem-dependent) approximation functions that contain meaningful physical information on the modelled phenomenon, as typical of the hybrid-Trefftz formulation. In dynamic problems, the same elements have been shown to be virtually frequency-insensitive, thus lifting the need of calibrating their sizes with the wavelength of the propagating wave. Finally, non-conventional elements are exceptionally robust to the mesh distortions that frequently occur when meshing topologically complex domains using automatic meshing techniques.

The distinguishing feature of the non-conventional formulations discussed here is the use of generalized, i.e. non-nodal, variables. It is our view that the over-use of nodal variables may unnecessarily limit the choice of approximation bases and the effectiveness of $p$-adaptive techniques. Conversely, the use of hierarchical, element-dependent, variables is particularly well suited to $p$-adaptive processes, as the finite element governing systems result strongly localized. For the same reason, the sparsity of these systems is particularly high, endorsing the use of special, computationally effective, sparse storage and solving techniques.

For single-phase materials with elastic behaviour, hybrid-Trefftz displacement and stress elements were first reported by (Freitas et al., 1999) and used to solve two-dimensional plane strain problems. The elements exhibit superior convergence rates, under both domain and boundary refinements and marginal sensitivity to mesh distortion. In the same year, (Freitas and Cismasiu, 1999) reported the modelling of singular stress fields caused by cracks or point loads using hybrid-Trefftz displacement elements enriched with additional shape functions known to model accurately these particular effects. The hybrid-Trefftz stress formulation was next
extended to three-dimensional elasticity by (Freitas and Bussamra, 2000). In the following year, the same authors reported in (Bussamra et al., 2001) the application of the same elements to problems defined on materials with elasto-plastic behaviour. The article shows that, when hybrid-Trefftz elements are applied to elasto-plastic problems, some of the terms present in the finite element solving system can no longer be expressed by boundary integrals. However, the trial functions’ richness in physical information still endorses accurate and robust results, even for very low degrees of h-refinement. In a hybrid-mixed context, (Castro and Freitas, 2001) reported the application of computer-friendly (wavelet) bases for solving two-dimensional plane strain applications.

In the same period, the hybrid formulations were extended to dynamic, two-dimensional, single-phase applications. A hybrid stress element for elastodynamic problems was reported by (Freitas and Wang, 2001). It uses an HHT time integration scheme and orthogonal (Legendre) polynomials for the approximation in space. As typical of the hybrid stress formulation, the domain stress and displacement approximations are linked by the condition that they should respect locally the equilibrium equation. A hybrid-Trefftz displacement element was formulated for spectral problems in (Freitas and Cismasiu, 2003), with special emphasis on the modelling of unbounded media. Two alternative approaches were adopted to model unbounded media, namely a finite element with absorbing boundaries and an unbounded element that satisfies explicitly the Sommerfeld condition. For the latter, it was shown that the integration on the side situated at infinity is dispensable. In the same context, (Freitas and Cismasiu, 2001) reported on the formulation and implementation of hybrid-Trefftz displacement and stress elements for elastoplasticity. The paper also covers topics related to the convergence and robustness properties of the elements and to automatic p-adaptivity. Moreover, the hybrid-Trefftz elements are shown to be particularly well suited to shape optimization problems, due to their high topological flexibility and ability to produce accurate solutions using exceptionally coarse meshes.

From 2003, the work of (Degrande et al., 1998) motivated the extension of the non-conventional formulations to dynamics of multiphase media. Two physically distinct (but mechanically similar) research lines were initiated, namely the modelling of articular cartilage and of saturated soils. Despite the fact that both are porous media, the specific issues relevant to the two areas are different: the cartilage tissue faces the issue of full incompressibility, while unboundedness and frequency sensitivity are typical difficulties associated with the modelling of saturated soils. In the cartilage context, (Freitas and Toma, 2009) developed hybrid-Trefftz stress elements which are inherently incompressible due to the choice of approximation bases that satisfy locally the incompressibility condition. The extension of this work to three-dimensional axisymmetric problems was reported in (Freitas and Toma, 2009). The theoretical framework of the hybrid-Trefftz stress and displacement elements for the plane-strain analysis of saturated soils under dynamic excitations was reported in (Freitas and Moldovan, 2011) and (Freitas et al., 2011), respectively. The energy statements associated with the formulations were recovered and sufficient conditions for the uniqueness of the finite element solutions were stated. A novel time integration technique, based on the weak enforcement of the governing equations in the current
time step, was used for the time-discretization of highly transient problems. The technique, first reported in (Freitas, 2008), has the merit of generating time-discretized problems characterized by spectral-like equations, independently on the functions that are used in the time basis, with the only distinction that the (generalized) frequency now results complex. This feature endorses the use of the same finite element formulation in space for harmonic, periodic and transient problems alike. The modelling of semi-infinite media using elements with absorbing boundary was also reported. The convergence and robustness properties of these elements were studied in detail in (Moldovan and Freitas, 2012). It was shown that the use of shape functions that embody the oscillatory characteristics of the medium lifts the well-known necessity of calibrating the leading dimension of the element to the wavelength of the highest frequency wave considered in the analysis, thus endorsing the use of the same finite element mesh for all (pseudo-)spectral problems resulted from the time discretization of a transient problem. The application of these elements to real-world scale problems was reported in (Moldovan and Freitas, 2008), including consolidation and fully transient analyzes on finite and infinite domains. Finally, (Freitas et al., 2010) formulated the full range of non-conventional elements (hybrid-mixed, hybrid and hybrid-Trefftz) for biphasic media and assessed their relative advantages and drawbacks.

After the formulation and implementation of non-conventional elements for poroelastodynamic problems, the investment into the development of hybrid-Trefftz elements for biphasic elastostatics was motivated by the idea of constructing the domain approximation of elements used to model media subjected to transient excitation by merging the Trefftz bases associated with static and dynamic problems. This has a two-fold motivation. On the one hand, despite the robustness of the Trefftz elements to high frequency excitations, very small frequencies pose the difficulty of the dynamic problem converging to a static problem, which the Trefftz basis may not be prepared to handle. To overcome this, enriching the dynamic basis with functions tailored for the static problem seems an appealing prospect. On the other hand, this enrichment is instrumental to compensate the lack of static (null frequency) modes in the pseudo-spectral time discretization (except when harmonic functions are used to construct the time basis, in which case the method coalesces to a Fourier decomposition technique). The formulation of hybrid-Trefftz elements for (fully coupled) poroelastostatic problems was reported in (Moldovan, 2010). A full description of the alternative displacement and stress models, including implementation issues and performance assessment is expected to be published shortly (Moldovan et al., 2012).

The favourable properties of the non-conventional elements motivate the continuation of the investment in their development. Our objective is to develop non-conventional elements for a wide range of continua, with linear and non-linear behaviour, test them thoroughly and integrate them into a mainstream finite element platform where they can be used in stand-alone analyses or together with conventional elements.

To reach a more general practical applicability of non-conventional elements, there are two research lines that are currently being explored. The first line aims at expanding the hybrid-
Treffitz formulation to non-saturated (i.e. three-phase) media. The objective of the second line is the formulation of non-conventional elements for elastic or elasto-plastic non-linear problems.

This document presents the formulation of hybrid-mixed displacement and stress elements for biphasic media with non-linear elastic behaviour. The behaviour of the medium is described by the Biot’s theory of porous media (Biot, 1956), which is casted in a solid displacement – fluid seepage (u-w) form. What typifies the material behaviour considered here is the assumption that the fluid density and the porosity may change due to variations in pore pressure and volume of the solid skeleton, respectively. The solid matrix is assumed incompressible, as typical of geotechnical and indeed biomechanical applications, but the compressibility of the pore fluid is considered in the analysis. Consequently, the fundamental equations governing the elastic response of the medium (equilibrium, compatibility and the constitutive law) must be complemented with the storage equation, stating that an increase in the volume of the solid skeleton triggers an increase in the fluid volume fraction, and with the state equation, asserting that an increase in the fluid suction leads to a reduction of the fluid density.

Both displacement and stress formulations are constructed on the independent approximations of the displacement and stress fields in the domain of the element. As typical of hybrid-mixed elements, none of the domain equations is locally satisfied. Instead, they are enforced weakly, in a Galerkin-residual fashion. What distinguishes the hybrid-mixed displacement and stress models is the (independent) boundary approximation. The boundary tractions and the pore pressure are approximated on the Dirichlet and internal boundaries of the displacement element, and the resulting basis is used to enforce weakly the displacement compatibility on the same boundaries. Conversely, the stress element involves the approximation of the solid displacement and fluid seepage fields on the Neumann and internal boundaries and the weak Galerkin enforcement of the boundary equilibrium on the same boundaries.

When compared with the hybrid formulations, the hybrid-mixed elements described here present two fundamental advantages. First, the independent approximation of the domain displacement and stress fields offer significant flexibility in the choice of the approximation functions, opening the possibility of using orthogonal bases that simplify the integration process. Moreover, for the stress element, this means lifting the need of enriching the regular stress and displacement bases with particular solutions designed to satisfy locally the domain equilibrium equation, thus saving the computational time associated with the (Gauss) point-wise calculation of these particular solutions (which change at every iteration). Second, the hybrid-mixed element also permits avoiding the storage of the final stresses and of their first and second derivatives in all integration points when passing from one time step to the next. Indeed, due to the independent stress approximation, only the generalized stress solution vector needs to be stored at the end of a time step. The price to be paid for these lesser memory requirements is the increase in the dimension of the solving system, which now collects not only domain displacement, but also domain stress modes.

This report opens with a description of the mathematical model that governs the problem, followed by the derivation of the corresponding (u-w) formulation, which serves as basis for
both hybrid-mixed models. Next, the displacement and stress models are derived through a hybrid-mixed discretization in space followed by an implicit time integration of the generalized displacements and stresses, using a Newmark scheme. Finally, a brief description of a possible implementation strategy is presented.

2. Mathematical model

According to the Biot’s theory (Biot, 1956), the biphasic medium is seen as an homogeneous combination of a solid phase and a fluid phase. Each of the phases is regarded as a continuum, with its own motion and state of stress. The responses of the phases influence each other. The coupling is mainly present in the constitutive laws of the medium.

From these fundamental principles, the equations that govern the behaviour of the medium can be derived. These equations are summarized next.

2.1. Primary equations

The mathematical model is constructed on four primary equations, derived from the fundamental laws of continuum mechanics and four secondary equations, which are obtained from the primary equations. The primary equations are: the balance of mass, the balance of momentum, the state equation and the constitutive law of the solid phase.

2.1.1. Balance of mass

The total mass associated to the phase \( \alpha = \{ s, w \} \) that occupies a volume \( V \) is equal to,

\[
M^\alpha = \int_V \rho^\alpha \, dV
\]  

(1)

where \( \rho^\alpha \) is the apparent (average) mass density of phase \( \alpha \). The apparent density relates to the effective density \( \rho_\alpha \) of the material that forms phase \( \alpha \) through the volume fraction \( n^\alpha = V^\alpha / V \) of the respective phase (clearly, \( n^w + n^s = 1 \)) as,

\[
\rho^\alpha = n^\alpha \cdot \rho_\alpha
\]  

(2)

The total mass balance equation states that the total mass of phase \( \alpha \) must be constant,

\[
\frac{D^\alpha M^\alpha}{Dt} = 0
\]  

(3)
where \( \frac{D^\alpha}{Dt} = \frac{\partial}{\partial t} + \left( v^\alpha \right)^T \cdot \nabla \) is the total (material) derivative operator, following the motion of phase \( \alpha \), \( v^\alpha \) represents the velocity vector of the same phase, and \( \nabla \) is the gradient operator. Substitution of definition (1) into the mass balance equation (3) yields the following expression for the mass balance equation,

\[
\frac{D^\alpha \rho^\alpha}{Dt} + \rho^\alpha \left( \nabla \cdot v^\alpha \right) = 0 \tag{4}
\]

It should be noted that the volume integral present in equation (1) vanishes as the mass balance must be satisfied for any given volume (i.e. the integrand must be equal to zero).

Mass balance equation (4) can be re-casted using the effective densities defined by expression (2) to yield,

\[
\nabla \cdot v^\alpha = - \frac{1}{\rho_s} \frac{D^\alpha \rho^\alpha}{Dt} - \frac{1}{n^\alpha} \frac{D^\alpha n^\alpha}{Dt} \tag{5}
\]

As the solid matrix is considered incompressible, the first term in the right-hand side of equation (5) written for the solid phase vanishes.

2.1.2. Balance of momentum

The balance of momentum of phase \( \alpha \) is written as,

\[
\nabla \cdot \sigma^\alpha + \dot{p}^\alpha + \rho^\alpha b = \rho^\alpha a^\alpha \tag{6}
\]

In equation (6), \( \sigma^\alpha \) is the partial stress vector corresponding to phase \( \alpha \), \( a^\alpha \) is the acceleration vector of phase \( \alpha \), \( b \) is the body force per unit mass and,

\[
\dot{p}^s = \dot{p}^w = -\xi \left( v^s \cdot v^w \right) + \pi \left( \nabla n^w \right) \tag{7}
\]

is the momentum supplied to phase \( \alpha \) by the other phase. In equation (7),

\[
\xi = \frac{\rho_w g}{k} \left( n_w \right)^2 \tag{8}
\]
is the dispersion (drag) and \( \pi \) is the pore pressure, subjected to the usual sign convention in continuum mechanics (positive \( \pi \) means swelling occurs). Finally, in equation (8), \( g \) is the acceleration of gravity and \( k \) is the hydraulic conductivity.

For the solid phase, the partial stress vector can be written as,

\[
\sigma^s = \sigma' + n^s (\delta \pi)
\]  
(9)

where \( \sigma' \) is the effective stress and \( \delta \) is the Dirac tensor, \( \delta_{ii} = 1 & \delta_{ij} = 0 \). For the fluid phase, the partial stress is simply,

\[
\sigma^w = n^w (\delta \pi)
\]  
(10)

From expressions (9) and (10), the well-known definition of total stress is recovered as,

\[
\sigma^t = \sigma^s + \sigma^w = \sigma' + \delta \pi
\]  
(11)

2.1.3. State equation

In the fluid phase, the state equation expresses the variation of the fluid density as a function of the variation of the pore pressure,

\[
\frac{1}{K_w} \frac{D^w \pi}{Dt} = - \frac{1}{\rho_w} \frac{D^w \rho_w}{Dt}
\]  
(12)

where \( K_w \) represents the bulk modulus of the fluid phase. Clearly, if the fluid was considered incompressible, the state equation (12) would be rendered dispensable, which is, in fact, the case for the solid phase, where the state equation simply states that \( \rho_s \) must be constant.

2.1.4. Constitutive law in the solid phase

Strains in the solid phase are caused by two factors: the effective stress and the pore pressure. The effective stress represents the load directly applied to the solid skeleton, thus causing strains through particle rearrangement mechanisms, on the one hand, and through deformation at the solid particle (i.e. solid matrix) level, on the other hand. Conversely, the pore pressure does not cause strains through particle rearrangements, its only acting mechanism being the (hydrostatic) state of deformation it causes in the solid matrix. Under the assumption of solid
matrix incompressibility, however, the solid matrix deformations are considered null. Thus, the constitutive law in the solid phase is reduced to,

\[ \sigma^s = k^s \cdot \varepsilon^s \]  

(13)

In the above expression, \( \varepsilon^s \) is the strain vector in the solid phase and \( k \) is the solid phase stiffness matrix,

\[
    \begin{bmatrix}
        \lambda + 2\mu & \lambda & 0 \\
        \lambda & \lambda + 2\mu & 0 \\
        0 & 0 & \mu
    \end{bmatrix}
\]  

(14)

Coefficients \( \lambda \) and \( \mu \) present in the stiffness matrix of the solid phase are the Lamé’s coefficients and are considered invariable. Alternatively, the constitutive law can be casted in terms of total stresses using expressions (11) and (13) as,

\[ \sigma^t = \sigma^s + \delta \pi = k^s \cdot \varepsilon^s + \delta \pi \]  

(15)

2.2. Derived equations

The mathematical model describing the problem is completed by the storage equation, fluid constitutive law, equilibrium equation and Darcy’s law which are derived from the fundamental equations introduced in the previous section.

2.2.1. Storage equation

The storage equation reflects the global balance of mass. It is obtained by summing up the mass balance equations (5) for both phases, to yield,

\[ \nabla \cdot \left[ n^w (\mathbf{v}^w - \mathbf{v}^s) \right] + \nabla \cdot \mathbf{v}^s = -\frac{n^w}{\rho_w} \cdot \frac{Dw}{Dt} \rho_w \]  

(16)

where properties \( n^w + n^s = 1 \) and \( \frac{D^s \rho}{Dt} = 0 \) have been used.

It is convenient to define at this point the seepage rate as,

\[ \dot{w} = n^w (\mathbf{v}^w - \mathbf{v}^s) \]  

(17)
and the fluid content rate as,

\[ \dot{\zeta} = \nabla \cdot \dot{w} \]  

(18)

With definitions (17) and (18), the storage equation (16) is rewritten as,

\[ \dot{\zeta} = \frac{1}{n^s} \cdot \frac{D^s n^s}{Dt} - \frac{n^w}{\rho_w} \cdot \frac{D^w \rho_w}{Dt} \]  

(19)

where the solid phase mass balance equation (5) was used to express the solid phase velocity gradient as a function of the pore fraction variation.

2.2.2. Constitutive law of the fluid phase

The constitutive law of the fluid phase is obtained inserting the state equation (12) into the storage equation (16), to yield,

\[ \frac{D^w \pi}{Dt} = - \dot{\varepsilon}^s \]  

(20)

where the volumetric strain rate is defined as,

\[ \dot{\varepsilon}^s = \delta \cdot \dot{\varepsilon}^s = \nabla \cdot \dot{v}^s \]  

(21)

In expression (20), the pressure rate multiplier is the inverse of the second Biot’s coefficient,

\[ Q = \frac{K_w}{n^w} \]  

(22)

Using definition (22), the fluid phase constitutive law can be written in the alternative form,

\[ \frac{D^w \pi}{Dt} = Q \left( \delta \cdot \dot{\varepsilon}^s \right) + Q \dot{\zeta} \]  

(23)

Finally, expression (23) can be inserted into equation (19) to yield a somewhat simpler expression for the storage equation,
Alternatively, expression (24) can be casted in terms of the volume fraction of the fluid phase, using the addition to unity property of the volume fractions,

\[
\frac{D n^s}{Dt} = (1 - n^w) \dot{e}^s
\]  

(25)

It is simple to interpret the physical meaning of equation (24). Under the assumption of solid matrix incompressibility, a negative volumetric deformation of the solid phase can only be caused by the rearrangement of the solid particles, which leads to an increase of the volume fraction associated with the solid phase, since the same volume of solid particles now occupy less volume in the mixture.

2.2.3. Equilibrium equation in the solid phase

The equilibrium equation is written here in terms of total stresses. It is obtained summing up the momentum balance equations (6) for the two phases to yield,

\[
\nabla \cdot \sigma^f + \rho \dot{b} = n^s \rho_s \dot{a}^s + n^w \rho_w \dot{a}^w
\]  

(26)

where \( \rho \) is the mixture density, defined as,

\[
\rho = n^s \rho_s + n^w \rho_w
\]  

(27)

Using the seepage rate variable defined by expression (17), the fluid phase acceleration vector \( \dot{a}^w \) can be written in terms of the seepage acceleration \( \frac{D a^w}{Dt} \) as,

\[
\dot{a}^w = \dot{a}^s + \frac{1}{n^w} a^{ws} - \frac{1 - 2 n^w}{(n^w)^2} \dot{e}^s \cdot \dot{\mathbf{w}}
\]  

(28)

Result (28) is substituted in the equilibrium equation (26), to yield,

\[
\nabla \cdot \sigma^f + \rho \dot{b} = \rho \dot{a}^s + \rho_w a^{ws} - \frac{\rho_w}{n^w} \left( 1 - 2 n^w \right) \dot{e}^s \cdot \dot{\mathbf{w}}
\]  

(29)
Note that the geometrically non-linear term present in the equilibrium equation (29) does not necessarily vanish under the assumption of infinitesimal deformations. However, if the strain rates are also considered infinitesimal, the last term of equation (29) can be neglected and the equilibrium equation simplifies to,

\[ \nabla \cdot \sigma^t + \rho \mathbf{b} = \rho \mathbf{a}^s + \rho_w \mathbf{a}^{ws} \]  

\[ (30) \]

2.2.4. Darcy’s law

The (generalized) Darcy’s law is obtained from the balance of momentum equation written for the fluid phase. Using expressions (7) and (10), equation (6) becomes,

\[ \nabla \pi + \rho_w \mathbf{b} = \rho_w \mathbf{a}^w + \frac{\xi}{n_w} \left( \mathbf{v}^w - \mathbf{v}^s \right) \]  

\[ (31) \]

Using equation (28), the seepage acceleration is inserted into the Darcy’s law (31),

\[ \nabla \pi + \rho_w \mathbf{b} = \rho_w \mathbf{a}^s + \frac{\rho_w}{n_w} \mathbf{a}^{ws} + \frac{\xi}{n_w^2} \dot{\mathbf{w}} - \frac{\rho_w}{n_w^2} \left( 1 - 2 \frac{n_w}{n} \right) \dot{\mathbf{e}}^s \cdot \dot{\mathbf{w}} \]  

\[ (32) \]

As in the previous section, the assumption of infinitesimal strain rates vanishes the geometrically non-linear term present in the Darcy’s law, to yield,

\[ \nabla \pi + \rho_w \mathbf{b} = \rho_w \mathbf{a}^s + \frac{\rho_w}{n_w} \mathbf{a}^{ws} + \frac{\xi}{n_w^2} \dot{\mathbf{w}} \]  

\[ (33) \]

3. Governing equations of the hybrid-mixed formulations

The governing equations are obtained from the (u-w) formulation of the Biot’s theory of porous media, derived in Section 2, by assuming infinitesimal deformations. However, the strain rates are not assumed to be infinitesimal. It should be noted that under the infinitesimal deformation assumption, the distinction between the reference and the actual configuration vanishes. This simplifies all the material derivatives present in the mathematical model to simple time derivatives.

For convenience, the equations are organized in three main classes, labelled equilibrium, compatibility and constitutive equations. These are the fundamental equations of the formulation,
and are fully discretized. Storage and state equations are secondary equations. They are only discretized in time and used to update the pore fraction, fluid and mixture densities and the second Biot’s coefficient after every iteration.

Boundary equations are specified on the Dirichlet and Neumann boundaries of the medium, as well as on the interior boundaries of the finite elements.

### 3.1. Equilibrium equation

The (generalized) equilibrium equation is obtained by collecting the solid phase equilibrium equation (29) and the Darcy’s law (32),

\[
\begin{align*}
\nabla \sigma^I + \rho b &= \rho a^s + \rho_w a^{ws} - \frac{\rho_w}{n_w} \left(1 - 2n_w\right) \dot{e}^s \cdot \dot{w} \\
\nabla \pi + \rho_w b &= \rho_w a^s + \frac{\rho_w}{n_w} a^{ws} + \frac{\xi}{\left(n_w\right)^2} \dot{w} - \frac{\rho_w}{\left(n_w\right)^2} \left(1 - 2n_w\right) \dot{e}^s \cdot \dot{w}
\end{align*}
\]

For concision sake, equations (34) can be written in the synthetic form,

\[
D \sigma + b_m = \rho a + c v - e_{nl} v
\]

where \(\sigma\) is the generalized stress vector, collecting the components of the stress field in the solid phase and the pore pressure, \(D\) is the differential equilibrium operator and \(b_m = \left(\rho b \quad \rho_w b\right)^T\) is the mixture body force vector. The explicit expressions of the density \(\rho\), damping \(c\) and strain rate \(e_{nl}\) matrices are given in the Appendix.

### 3.2. Compatibility equation

The compatibility equation of the biphasic medium is obtained by merging the compatibility equations for each of its phases,

\[
\begin{align*}
\dot{e}^s &= \nabla v^s \\
\dot{\zeta} &= \nabla \cdot \dot{w}
\end{align*}
\]

Again, it is convenient to group both compatibility equations in a single expression,

\[
\dot{e} = D^s v
\]
where the generalized strain rate vector collects the strain rate components in the solid phase and the fluid content rate, $D^*$ represents the compatibility differential operator and $v = \begin{pmatrix} v^s & \dot{w} \end{pmatrix}^T$ is the generalized velocity vector.

### 3.3. Constitutive law

Equations (15) and (23) are used to derive the constitutive law of the biphasic material. As stiffness matrix $k^s$ associated with the solid phase is assumed to be invariable, expression (15) can be derived in time to allow for the insertion of the pore pressure rate given by equation (23). The constitutive laws of the solid and fluid phases result as follows:

\[
\begin{align*}
\dot{\sigma}^s &= \left( k^s + \delta\delta Q \right) \dot{e}^s + \left( \delta Q \right) \dot{\zeta} \\
\dot{\pi} &= Q \left( \delta \cdot \dot{e}^s \right) + Q \dot{\zeta}
\end{align*}
\]

(38)

With the notations used above for the generalized stress and strain vectors, expression (38) simplifies to,

\[
\dot{\sigma} = k \cdot \dot{\varepsilon}
\]

(39)

The explicit expression of the stiffness matrix $k$ of the biphasic material is given in the Appendix.

### 3.4. State and storage equations

To complete the mathematical model, the state and storage equations are included in forms (40) and (41).

\[
\begin{align*}
\dot{n}^w &= \left( 1 - n^w \right) \dot{e}^s \\
\dot{\rho}^w &= -\frac{\rho^w}{K^w} \dot{\pi}
\end{align*}
\]

(40)  (41)

### 3.5. Boundary conditions

The Dirichlet $\left( \Gamma_u^e \right)$ and Neumann $\left( \Gamma_{\sigma}^e \right)$ boundaries of a finite element are defined as its exterior boundaries where the displacements and tractions (Cauchy stresses) are prescribed, respectively (Figure 1). Thus, equations (42) and (43) are associated to these boundaries,
\[
\mathbf{u} = \mathbf{u}_r \quad \text{on } \Gamma_u^e
\]
\[
\mathbf{N}\sigma = \mathbf{t}_r \quad \text{on } \Gamma_{\sigma}^e
\]

where \( \mathbf{u} \) is the generalized displacement field, collecting the solid phase displacements and the fluid seepage, and \( \mathbf{N} \) is the outward normal matrix associated to the respective boundary.

![Finite element discretization](image)

On the interior boundary \( \Gamma_i^e \) separating elements \( i \) and \( j \), equations (44) and (45) are required to enforce inter-element compatibility and equilibrium conditions,

\[
\mathbf{u}^i - \mathbf{u}^j = 0 \quad \text{on } \Gamma_i^e
\]
\[
\mathbf{N}^i \mathbf{\sigma}^i + \mathbf{N}^j \mathbf{\sigma}^j = 0 \quad \text{on } \Gamma_i^e
\]

4. Hybrid-mixed displacement formulation

The hybrid-mixed displacement model is constructed on the independent approximations of the displacement and stress fields in the domain of the element and of the traction field on its Dirichlet boundary. All bases are constructed hierarchically using generalized variables. No domain equation must be observed \textit{a priori} by the functions collected in the domain bases. The discretization in time of the space-discretized equations is performed using the well-known Newmark method.

4.1. Finite element approximations

In the domain of the finite element, the generalized displacement and stress fields are approximated as,

\[
\mathbf{u}(\mathbf{x}, t) = \mathbf{U}(\mathbf{x}) \cdot \mathbf{X}(t) \quad \text{in } V^e
\]
\( \sigma(x,t) = S(x) \cdot Y(t) \quad \text{in} \quad V^e \) \hspace{1cm} (47)

Besides the above domain approximations, the traction field is approximated on the extended Dirichlet boundary of the element, defined as \( \Gamma_u^e = \Gamma_u^e \cup \Gamma_i^e \), to yield,

\[ t(s,t) = Z(s) \cdot p(t) \quad \text{on} \quad \Gamma_u^e \] \hspace{1cm} (48)

where \( s \) represents the side coordinate of the respective boundary.

No restrictions are enforced on the choice of the trial functions collected in the displacement, stress and boundary traction bases \( U(x) \), \( S(x) \) and \( Z(s) \). They are constructed hierarchically and are not related in any way to the nodes of the element. Consequently, the generalized displacements, stresses and tractions \( X(t) \), \( Y(t) \) and \( p(t) \) do not have any particular physical meaning. However, the abandon of the nodal variable concept allows much more flexibility in defining the levels of \( p \)-refinement, which may be different for each element and boundary, according to the need of the analyst.

### 4.2. Discretization in space

The problem described in Section 3 is semi-discretized in space by enforcing in a weak form the domain equations (35), (37) and (39), using the functions collected in the displacement and stress bases (46) and (47). Also, the essential boundary conditions (42) and (44) are enforced weakly on the Dirichlet and interior boundaries of the element using the functions collected in the boundary traction basis (48) for weighting.

#### 4.2.1. Domain statements

**Equilibrium**

The generalized equilibrium equation is enforced in the following form,

\[ \int U^* \left( D \sigma + b_m \right) dV^e = \int U^* \left( \rho a + c v - e_{ni} v \right) dV^e \] \hspace{1cm} (49)

where the transpose conjugate of matrix \( U \) is denoted by \( U^* \).

Integrating by parts the first term in equation (49) to force the emergence of the boundary fields and inserting the domain approximations (46) and (47), the following expression is obtained,

\[ \int U^* \left( N \sigma \right) d\Gamma^e - \int \left( D^* U \right)^* S dV^e \cdot Y + \int U^* b_m dV^e = \int U^* \rho U dV^e \cdot \bar{X} + \int U^* e_{ni} U dV^e \cdot \bar{X} - \int U^* e_{ni} U dV^e \cdot \bar{X} \] \hspace{1cm} (50)
The boundary integral in equation (50), taken on the entire boundary of the element, is split into its (extended) Dirichlet and Neumann parts, where the traction approximation (48) and natural boundary condition (43) are inserted. After making notations,

\[
B_v = \int \left( D^* U \right)^* S dV^e
\]
\[
M = \int U^* \rho U dV^e
\]
\[
C = \int U^* \iota U dV^e
\]
\[
E_{nl} = \int U^* e_{nl} U dV^e
\]
\[
B_\Gamma = \int U^* Z d\Gamma_u^e
\]
\[
\bar{t}_\nu = \int U^* t_\nu d\Gamma_\sigma^e
\]
\[
\bar{b} = \int U^* b_m dV^e
\]

the space-discretized equilibrium equation becomes,

\[
M \ddot{X} + \left( C - E_{nl} \right) \dot{X} + B_v Y - B_\Gamma p = \bar{t}_\nu + \bar{b}
\]

**Compatibility**

The generalized compatibility equation is enforced in the following form,

\[
\int S^* (\dot{\epsilon} - D^* \nu) dV^e = 0
\]

After inserting the domain displacement approximation (46) into equation (59), the space-discretized equilibrium equation is,

\[
\int S^* \epsilon dV^e = B_v^* \cdot \dot{X}
\]

**Constitutive law**

The constitutive law (39) is enforced using the functions collected in the domain stress basis for weighting,

\[
\int S^* (f \cdot \sigma - \dot{\epsilon}) dV^e = 0
\]
where \( f = k^{-1} \) is the material flexibility matrix of the mixture, given in explicit form in the Appendix.

Domain stress approximation (47) is inserted in equation (61) to yield,

\[
\mathbf{F} \cdot \dot{\mathbf{Y}} = \int \mathbf{S}^* \dot{\mathbf{e}} dV^e
\]

\[
\mathbf{F} = \int \mathbf{S}^* \mathbf{f} \mathbf{S} dV^e
\]  

(62)  

(63)

Finally, the second domain statement is obtained merging the compatibility and constitutive equations (60) and (62),

\[
\mathbf{B}^*_v \cdot \dot{\mathbf{X}} - \mathbf{F} \cdot \dot{\mathbf{Y}} = 0
\]  

(64)

4.2.2. Boundary statement

The essential boundary condition (42) is enforced on the Dirichlet boundary of the element using the functions collected in the traction basis (48) for weighting,

\[
\int \mathbf{Z}^* (\mathbf{u} - \mathbf{u}_\Gamma) d\Gamma_u^e = 0
\]  

(65)

Inserting in expression (65) the domain approximation (46), the boundary statement becomes,

\[
\mathbf{B}^*_v \cdot \mathbf{X} = \bar{\mathbf{u}}_\Gamma
\]  

(66)

\[
\bar{\mathbf{u}}_\Gamma = \int \mathbf{Z}^* \mathbf{u}_\Gamma d\Gamma_u^e
\]  

(67)

Applying the same procedure, compatibility condition (44) reads as follows on the inter-element boundary common to elements \( i \) and \( j \),

\[
\left( \mathbf{B}_\Gamma^i \right)^* \cdot \mathbf{X}^i - \left( \mathbf{B}_\Gamma^j \right)^* \cdot \mathbf{X}^j = 0
\]  

(68)

Domain and boundary statements (58), (64), (66) and (68) conclude the space semi-discretization process. Next, the time-dependent generalized displacements and stresses must be discretized in time.

4.3. Discretization in time
4.3.1. Time integration procedure

The well-known Newmark method is used to integrate in time the space-discretized equations in the current time step. In this procedure, the values of the first and second time derivatives of a generic quantity $\Theta(t)$ at the end of the time interval ($\dot{\Theta}(\Delta t)$ and $\ddot{\Theta}(\Delta t)$) are expressed as functions of the initial and final values of the same quantity ($\Theta_0$ and $\Theta(\Delta t)$), and of the initial values of its first and second derivatives ($\dot{\Theta}_0$ and $\ddot{\Theta}_0$). In the resulting expressions, $\gamma$ and $\beta$ are the Newmark coefficients that are used to calibrate the numerical dissipation of the method and $\Delta t$ is the size of the time step.

$$\dot{\Theta}(\Delta t) = \frac{\gamma}{\beta \Delta t} [\Theta(\Delta t) - \Theta_0^v]$$  \hspace{1cm} (69)

$$\ddot{\Theta}(\Delta t) = \frac{1}{\beta \Delta t^2} [\Theta(\Delta t) - \Theta_0^s]$$  \hspace{1cm} (70)

where

$$\begin{align*}
\Theta_0^v &= \Theta_0 + \frac{\gamma - \beta}{\gamma} \Delta t \cdot \dot{\Theta}_0 + \frac{\gamma - 2\beta}{2\gamma} \Delta t^2 \cdot \ddot{\Theta}_0 \\
\Theta_0^s &= \Theta_0 + \Delta t \cdot \dot{\Theta}_0 - \frac{1 - 2\beta}{2} \Delta t^2 \cdot \ddot{\Theta}_0
\end{align*}$$  \hspace{1cm} (71)

The Newmark method is implicit and unconditionally stable for $2\beta \geq \gamma \geq \frac{1}{2}$.

4.3.2. Domain equations

Approximations (69) and (70) are enforced on the time derivatives of the generalized displacements and stresses present in the domain equations (58) and (64). In this way, the differential equations are reduced to the following fully discretized algebraic forms:

- First domain statement (equilibrium)

$$D_v X + B_v Y - B_T p = \bar{r}_u + \bar{b} + d_0$$  \hspace{1cm} (72)

$$D_v = \frac{1}{\beta \Delta t^2} [M + \gamma \Delta t (C - E_{nl})]$$  \hspace{1cm} (73)
\[ d_0 = \frac{1}{\beta \Delta t^2} \left[ M \dot{X}_0^v + \gamma \Delta t \left( C - E_{nt} \right) X_0^v \right] \]  

(74)

It should be noted that all quantities present in equations (72) to (74) correspond to the end of the time interval, except for those marked with subscript “0”.

- Second domain statement (compatibility and constitutive equations)

\[ B^v X - F Y = B^v X_0^v - F Y_0^v \]  

(75)

4.3.3. Solving system

The solving system of the hybrid-mixed displacement element is constructed merging domain equations (72) and (75) and boundary equation (66).

\[
\begin{bmatrix}
D_V & B_V & -B_T \\
B^*_V & -F & \bullet \\
-B^* T & \bullet & \bullet
\end{bmatrix}
\begin{bmatrix}
X \\
Y \\
p
\end{bmatrix}
=
\begin{bmatrix}
\bar{t}_u + \bar{b} + d_0 \\
B^* V X_0^v - F Y_0^v \\
-\bar{u}_u
\end{bmatrix}
\]  

(76)

As the generalized displacements and stresses are strictly element-dependent, system (76) is adequate for localized and/or adaptive $p$-refinement procedures. Moreover, it is Hermitian if the geometrically non-linear strain rate effects are not included in the analysis.

Besides the strain-rate matrix, (material) non-linear terms are present in the mass ($M$) and flexibility ($F$) matrices. In the right-hand side, the body force vector $\bar{b}$ is an additional source of non-linearity.

5. Hybrid-mixed stress formulation

The hybrid-mixed stress model is built on the same domain approximation as the displacement model. The trial functions are still not subjected to any restrictions besides completeness and linear independence and are used to enforce in the weak form all the domain equations. The main difference arises in the boundary approximation. The boundary displacement field is approximated on the Neumann and interior boundaries of the element and used to enforce on average the traction continuity on the same boundaries. The time discretization of the finite element equations is still performed using the Newmark method.

5.1. Finite element approximations
Expressions (46) and (47) are used to approximate the domain displacement and stress fields in the hybrid-mixed stress element, respectively. The approximations are still built hierarchically and involve generalized (non-nodal) variables.

On the extended Neumann boundary of the element, defined as \( \Gamma^e = \Gamma^e_\sigma \cup \Gamma^e_i \), the displacement field is independently approximated as,

\[
u(s,t) = Z(s) \cdot q(t) \quad \text{on } \Gamma^e_\sigma
\]

where vector \( q(t) \) collects the generalized boundary displacements.

### 5.2. Discretization in space

The domain equations are discretized in space by enforcing weakly the compatibility and constitutive equations (37) and (39), using the stress approximation functions for weighting, and the equilibrium equation (35), using the displacement basis for weighting. The essential conditions (43) and (45) are enforced on the extended Neumann boundary of the element using the functions collected in the boundary displacement basis as test functions.

#### 5.2.1. Domain statements

**First domain statement (compatibility equation and constitutive law)**

The compatibility and constitutive equations are enforced using expressions (59) and (61), respectively. Eliminating the strain field from the two equations yields,

\[
\int S^* (f \cdot \tilde{\sigma} - D^* v) dV^e = 0
\]

The second term of the above expression is integrated by parts and natural boundary condition (42) is enforced on the resulting Dirichlet sides. Boundary approximation (77) is also inserted into the extended Neumann boundary term. Finally, displacement and stress approximations (46) and (47) are inserted in equation (78), to yield,

\[
\begin{align*}
F\ddot{Y} + A_\Gamma \dot{X} - A_V \dot{q} &= \tilde{u}_{\Gamma^e_\sigma} \\
A_V &= \int (DS)^* U dV^e \\
A_\Gamma &= \int (NS)^* Z d\Gamma^e_\sigma \\
\pi_{\Gamma^e_\sigma} &= \int (NS)^* \mathbf{u}_\Gamma d\Gamma^e_u
\end{align*}
\]
Second domain statement (equilibrium equation)

Equilibrium equation (35) is enforced in form (49). Substitution of stress approximation (47) in the left-hand side of equation (49) yields,

\[ \mathbf{A}_\Gamma^* \cdot \mathbf{Y} - \mathbf{M} \ddot{\mathbf{X}} - (\mathbf{C} - \mathbf{E}_{nl}) \dot{\mathbf{X}} = -\mathbf{b} \]  

(83)

where notations (52) to (54), (57) and (80) are used.

5.2.2. Boundary statement

The functions collected in the boundary displacement basis (77) are used to enforce on average the essential boundary condition (43) on the Neumann boundary of the element,

\[ \int_{\Gamma^e} \mathbf{Z}^* \left( \mathbf{N} \sigma - \mathbf{t}_\Gamma \right) d\Gamma_{\sigma} = 0 \]  

(84)

Substituting in the above expression the stress approximation (47) yields,

\[ \mathbf{A}_\Gamma^* \cdot \mathbf{Y} = \mathbf{t}_\sigma \]  

(85)

\[ \mathbf{t}_\sigma = \int_{\Gamma} \mathbf{Z}^* \mathbf{t}_\Gamma d\Gamma_{\sigma} \]  

(86)

On the internal boundary shared by elements \( i \) and \( j \), the weak enforcement of equilibrium condition (45) yields,

\[ \left( \mathbf{A}_\Gamma^i \right)^* \cdot \mathbf{Y}^i + \left( \mathbf{A}_\Gamma^j \right)^* \cdot \mathbf{Y}^j = 0 \]  

(87)

5.3. Discretization in time

Equations (79), (83) and (85) are discretized in time using the same method as presented in Section 4.3. Using expressions (69) and (70), the first and second time derivatives of the generalized displacements and stresses present in the domain statements are written as functions of the un-derived values of the same fields at the end of the time interval. To distinguish between field values at the beginning and the end of the time interval, the former quantities are marked with a “0” subscript.

5.3.1. Domain equations
The time integration procedure described above reduces domain equations (79) and (83) to the following algebraic forms:

- **First domain statement (compatibility and constitutive equations)**

\[
F_Y + A_V X - A_\Gamma \dot{q}' = \left( \bar{u}_{\Gamma_o}^v - \bar{u}_{\Gamma_o}^{v_0} \right) + F Y_0^v + A_V X_0^v
\]  
\[
\bar{u}_{\Gamma_o}^v = \bar{u}_{\Gamma_o}^v(0) + \frac{\gamma - \beta}{\gamma} \Delta t \cdot \dot{\bar{u}}_{\Gamma_o}^v(0) + \frac{\gamma - 2\beta}{2\gamma} \Delta t^2 \cdot \ddot{\bar{u}}_{\Gamma_o}^v(0)
\]  
\[
\dot{q}' = \frac{\beta \Delta t}{\gamma} \dot{q}
\]

It should be noted that in the (most common) case of a constant valued enforced displacement on the Dirichlet side, the right-hand side term \( \left( \bar{u}_{\Gamma_o}^v - \bar{u}_{\Gamma_o}^{v_0} \right) \) present in equation (88) is null.

- **Second domain statement (equilibrium equation)**

\[
A_{V,\Gamma}^* Y - D_V X = -\bar{b} - d_0
\]  
Notations (73) and (74) are used in the above equation.

5.3.2. **Solving system**

Equations (88), (91) and (85) are merged together into the solving system of the hybrid-mixed stress element.

\[
\begin{bmatrix}
F & A_V & -A_\Gamma \\
A_{V,\Gamma}^* & -D_V & \bullet \\
-\Lambda_{\Gamma,\Gamma}^* & \bullet & \bullet
\end{bmatrix}
\begin{bmatrix}
Y \\
X \\
\dot{p}'
\end{bmatrix}
= \begin{bmatrix}
\left( \bar{u}_{\Gamma_o}^v - \bar{u}_{\Gamma_o}^{v_0} \right) + F Y_0^v + A_V X_0^v \\
-\bar{b} - d_0 \\
-\bar{t}_{\Gamma_o}
\end{bmatrix}
\]  

System (92) has the same algebraic (and algorithmic) properties as system (76) of the hybrid-mixed displacement element.

6. **Notes on the implementation**
Despite the full discretization of the equilibrium, compatibility and constitutive laws, the state and storage equations (40) and (41) are still not included in the model. Next, these equations are discretized in time and used for the algorithmic updating of the non-linear material characteristics in the domain integration points. Finally, a possible implementation strategy, based on the fixed-point (Picard) method, is presented.

6.1. Non-linear quantities and their updates

State and storage equations (40) and (41) link the variations of the fluid density $\rho_w$ and pore fraction $n_w$ to the pore pressure and volumetric strain rates, respectively. On their turn, these variations influence the mixture density $\rho$, according to equation (27), and the second Biot’s coefficient, defined by equation (22). All these quantities need to be updated at every iteration in the solution process.

- **State equation**
  Equation (40) is integrated using the Newmark method, to yield,

$$
\left(1 + \frac{\beta \Delta t}{\gamma} \cdot \dot{\varepsilon}^s \right) \cdot n_w^t = n_w^0 + \frac{\beta \Delta t}{\gamma} \cdot \dot{\varepsilon}^s \\
$$

$$
n_w^0 = n_w(0) + \frac{\gamma - \beta}{\gamma} \Delta t \cdot \dot{n}_w(0) + \frac{\gamma - 2\beta}{2\gamma} \Delta t^2 \cdot \ddot{n}_w(0)
$$

- **Storage equation**
  Equation (69) is used to integrate in time the fluid density rate present in the storage equation (41). The following expression is obtained,

$$
\left(1 + \frac{\beta \Delta t}{\gamma} \cdot \frac{\dot{\pi}}{K_w} \right) \cdot \rho_w = \rho_w^0 \\
$$

$$
\rho_w^0 = \rho_w(0) + \frac{\gamma - \beta}{\gamma} \Delta t \cdot \dot{\rho}_w(0) + \frac{\gamma - 2\beta}{2\gamma} \Delta t^2 \cdot \ddot{\rho}_w(0)
$$

The time-discretized forms (93) and (95) of equations (40) and (41) are used to perform the updating of the non-linear material characteristics.

6.2. Solution algorithm
Both systems (76) and (92) can be separated into their linear and non-linear parts, to yield the following generic system,

$$(A_l + A_{nl}) \cdot x = h_l + h_{nl}$$  \hspace{1cm} (97)$$

In expression (97) the linear terms $A_l$ and $h_l$ include the linear sub-matrices and sub-vectors and the initial values of the non-linear ones. It is recalled that non-linear terms include the mass matrix $M$, the strain-rate matrix $E_{nl}$, the flexibility matrix $F$ (only the fluid-related term) and the body force vector $\mathbf{b}$.

System (97) can be solved by a variety of methods. Due to its simplicity, a strategy based on the fixed-point method is described next.

**Input (from the previous time step)**
- initial values of the pore fraction $n^w$ and of the fluid density $\rho_w$;
- updating quantities $n_{0w}$ and $\rho_{w0}$, computed according to expressions (94) and (96);
- initial values vectors $X_0^n$, $X_0^a$ and $Y_0^v$, given by equation (71).

**Iterative process in the current time step**
- solve $A_l \cdot x^0 = h_l$ for $x^0$;
- compute $\dot{x}^0$ according to expression (69);
- compute $\dot{\sigma}$ and $\dot{\varepsilon}$ using equations (47) and (39);
- update the fluid pore fraction and density in the integration points, using expressions (93) and (95);
- update the second Biot’s coefficient $Q$ and the mixture density $\rho$, according to expressions (22) and (27);
- compute the non-linear terms $A_{nl}^0$ and $h_{nl}^0$;
- solve $\left(A_n + A_{nl}^0\right) \cdot x^1 = h_n + h_{nl}^0$ for $x^1$;
- compare solution vectors $x^0$ and $x^1$. If the difference exceeds the tolerance, enter the next iteration.

**Output (for the next time step)**
The output for the next time step involves the calculation of the same quantities that served as input for the current time step.

**Bibliography**


Appendix

The explicit expressions of the density ($\rho$), damping ($\mathbf{c}$), strain rate ($\mathbf{e}_{nl}$), material stiffness ($\mathbf{k}$) and flexibility ($\mathbf{f}$) matrices are given below.

\[ \mathbf{\rho} = \begin{bmatrix} \rho & \rho_w & \circ \\ \circ & \rho & \rho_w \\ \rho_w & \frac{\rho_w}{n_w} & \circ \\ \circ & \rho_w & \frac{\rho_w}{n_w} \end{bmatrix} \]  \quad (98)

\[ \mathbf{\mathbf{e}} = \begin{bmatrix} \circ & \circ & \circ & \circ \\ \circ & \circ & \circ & \circ \\ \circ & \frac{\xi}{(n_w)^2} & \circ & \circ \\ \circ & \circ & \frac{\xi}{(n_w)^2} & \circ \end{bmatrix} \]  \quad (99)
\[ \epsilon_{nt} = \frac{1 - 2n_w}{n_w} \epsilon^0 . \]  

\[ \mathbf{k} = \begin{bmatrix} \lambda + 2\mu + Q & \lambda + Q & \circ & Q \\ \lambda + Q & \lambda + 2\mu + Q & \circ & Q \\ \circ & \circ & \mu & \circ \\ Q & Q & \circ & Q \end{bmatrix} \]  

\[ \mathbf{f} = \frac{1}{4\mu(\lambda + \mu)} \begin{bmatrix} \lambda + 2\mu & -\lambda & \circ & -2\mu \\ -\lambda & \lambda + 2\mu & \circ & -2\mu \\ \circ & \circ & 4(\lambda + \mu) & \circ \\ -2\mu & -2\mu & \circ & 4\mu + \frac{4\mu(\lambda + \mu)}{Q} \end{bmatrix} \]