RAYLEIGH WAVES IN SATURATED POROELASTIC MEDIA

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1. APPROXIMATION IN TIME

It is assumed that all variables, and their time derivatives, present in the modelling of the elastodynamic response of saturated porous media, say the displacement $u$, the velocity $v$ and the acceleration $a$, are separated in time and space,

$$u(x,t) = \sum_{n=1}^{N} T_n(t) u_n(x)$$

$$v(x,t) = \sum_{n=1}^{N} T_n(t) v_n(x)$$

$$a(x,t) = \sum_{n=1}^{N} T_n(t) a_n(x)$$

where $T_n(t)$ defines a complete time approximation basis with support $0 \leq t \leq \Delta t$.

The non-periodic, high-order time integration procedure described in [1] yields time integration rules of the form,

$$v_n(x) = i \omega_n u_n(x) - i \omega_{0n} u_0(x)$$

$$a_n(x) = i \omega_n v_n(x) - i \omega_{0n} v_0(x)$$

where $i$ is the imaginary unit, $u_0$ and $v_0$ are the initial conditions of the problems, and $\omega_n$ and $\omega_{0n}$ are (complex) algorithmic frequencies. In the applications reported below, this time integration procedure is implemented in a single time step using a wavelet time approximation basis defined on the interval $[2,3]$.

The commonly used periodic (or periodically extended) spectral decomposition method is recovered using a Fourier basis:

$$T_n(t) = e^{i \omega_n t}$$

Equations (4) and (5) hold with real forcing frequencies,

$$\omega_n = \pm 2n \pi / \Delta t$$

$$\omega_{0n} = 0$$

as periodicity renders the discretization independent of the initial conditions of the problem.

2. BOUNDARY VALUE PROBLEM
The discretization of the time dimension of the equations governing the elastodynamic response of a saturated poroelastic body $V$ with boundary $\Gamma$ generates a set of $N$ fully uncoupled boundary value problems [1], which can be stated in matrix form as follows:

$$D\sigma_n + b_n + (\omega_n^2 p - i\omega_n c)u_n = \theta \quad \text{in } V$$

$$\varepsilon_n = D^\star u_n \quad \text{in } V$$

$$\sigma_n = k \varepsilon_n \quad \text{in } V$$

$$N\sigma_n = \bar{T}_n \quad \text{on } \Gamma_t$$

$$u_n = \bar{u}_n \quad \text{on } \Gamma_n$$

(9) \hspace{2cm} (10) \hspace{2cm} (11) \hspace{2cm} (12) \hspace{2cm} (13)

Subscript $n$, which identifies the forcing frequency for a particular spectral decomposition, is omitted from this point onwards to simplify the notation.

It is noted that the boundary value problem (9)-(13) holds, with appropriate adjustments in the definitions of the generalized body force vector and damping matrix, when the discretization of the time dimension is based on trapezoidal rules, e.g. [4]. In particular, the Newmark-type integration methods,

$$u = u_0 + \Delta t v_0 + \alpha \Delta t^2 a_0 + \beta \Delta t^2 a$$

$$v = v_0 + (1 - \gamma) \Delta t a_0 + \gamma \Delta t a$$

yield a single-frequency spectrum, $N = 1$, and equivalent imaginary algorithmic frequencies $\omega \leftrightarrow -i\gamma(\beta \Delta t)^{-1}$ and $\omega^2 \leftrightarrow -(\beta \Delta t^2)^{-1}$ for the velocity and acceleration integration rules, respectively.

2.1 Domain conditions

In the (spectral decomposition of the) equilibrium, compatibility and elasticity conditions (9), (10) and (11), vectors $\sigma_n$ and $\varepsilon_n$ collect the independent components of the (total) stress and (solid) strain tensors extended to include the fluid pressure and the fluid content, respectively, and vector $u_n$ collects the displacement components of the solid skeleton and the seepage displacement components:

$$\sigma^T = \{ \sigma_{xx} \quad \sigma_{yy} \quad \sigma_{xy} \quad \pi \}$$

$$\varepsilon^T = \{ \varepsilon_{xx} \quad \varepsilon_{yy} \quad 2\varepsilon_{xy} \quad \zeta \}$$

$$u^T = \{ u_x \quad u_y \quad w_x \quad w_y \}$$

(14) \hspace{2cm} (15) \hspace{2cm} (16) \hspace{2cm} (17) \hspace{2cm} (18)
where $\sigma_{ij}$ and $\varepsilon_{ij}$ are the components of the total stress tensor and of the (small) strain tensor, $\pi$ and $\zeta$ are the pore fluid stress and the fluid content and $u_i$ is the (small) displacement of the solid skeleton and $w_j$ is the pore fluid seepage displacement. Vector,

$$
\mathbf{b} = \tilde{\mathbf{b}} + i\omega_0 \rho \mathbf{v}_0 + i\omega_0 (i\omega \rho + c) \mathbf{u}_0
$$

(19)

combines the effect of the initial conditions of the (non-periodic) problem, $\mathbf{u}_0$ and $\mathbf{v}_0$, with the body force effect, defined by vector,

$$
\tilde{\mathbf{b}}^T = \left\{ \rho f_x, \rho f_y, \rho_a f_x, \rho_a f_y \right\}
$$

(20)

where $f_i$ is the component of the body force per unit mass density, and $\rho$ and $\rho_w$ are the mass density of the mixture and of the fluid, respectively.

The divergence and gradient operators, $\mathbf{D}$ and $\mathbf{D}^T$, are adjoint in geometrically linear problems,

$$
\mathbf{D} = \begin{bmatrix}
\partial_x & 0 & \partial_y & 0 \\
0 & \partial_y & 0 & \partial_x \\
0 & 0 & \partial_x & 0 \\
0 & 0 & 0 & \partial_y 
\end{bmatrix}
$$

(21)

with $\mathbf{D}^T = \mathbf{D}^T$ in Cartesian coordinates for geometrically linear problems, and the (real, symmetric) elastic stiffness, mass and damping matrices are the following,

$$
\mathbf{k} = \begin{bmatrix}
\kappa M & \kappa M - 2\mu & 0 & \alpha M \\
\kappa M - 2\mu & \kappa M & 0 & \alpha M \\
0 & 0 & \mu & 0 \\
\alpha M & \alpha M & 0 & M 
\end{bmatrix}
$$

(22)

$$
\mathbf{\rho} = \begin{bmatrix}
\rho & 0 & \rho_w & 0 \\
0 & \rho & 0 & \rho_w \\
\rho_w & 0 & \rho_w a/n^w & 0 \\
0 & \rho_w & 0 & \rho_w a/n^w 
\end{bmatrix}
$$

(23)

$$
\mathbf{c} = \frac{\zeta}{n^{w^2}} \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 
\end{bmatrix}
$$

(24)

where,

$$
\kappa = \alpha^2 + (\lambda + 2\mu)/M
$$

(25)
\( \lambda \) and \( \mu \) are the Lamé constants of the solid, \( \alpha \) and \( M \) are the first and second Biot coefficients, \( \alpha \) is the scalar tortuosity correction factor, \( n^w \) is the volume fraction of the liquid and \( \xi \) is the dissipation.

### 2.2 Domain conditions

In the Neumann condition (12), vector \( \mathbf{T} \) collects the components of the (total) surface forces, and the pressure prescribed on the boundary, and the boundary equilibrium matrix \( \mathbf{N} \) collects the adequate components, \( n_x \) and \( n_y \), of the unit outward normal vector, \( \mathbf{n} \):

\[
\begin{bmatrix}
 n_x & 0 & n_y & 0 \\
 0 & n_y & n_x & 0 \\
 0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
 \sigma_{xx} \\
 \sigma_{yy} \\
 \sigma_{xy} \\
 \tau \\
\end{bmatrix}
= \begin{bmatrix}
 \bar{T}_x \\
 \bar{T}_y \\
 \bar{\pi} \\
\end{bmatrix}
\]  

The Dirichlet condition (13) is written as follows, where \( w = n_x w_x + n_y w_y \) is the normal component of the seepage displacement:

\[
\begin{bmatrix}
 u_x \\
 u_y \\
 w \\
\end{bmatrix}
= \begin{bmatrix}
 \bar{u}_x \\
 \bar{u}_y \\
 \bar{w} \\
\end{bmatrix}
\]  

The Dirichlet conditions can be alternatively expressed in terms of prescribed velocity or acceleration terms, \( \mathbf{u} = \mathbf{v} \) or \( \mathbf{u} = \mathbf{a} \), in which case the prescribed term in equation (13) is replaced by the following expressions:

\[
\begin{align*}
\mathbf{u} &\leftarrow -i \omega^{-1}(\mathbf{v} + i \omega \mathbf{u}_0) \\
\mathbf{u} &\leftarrow -\omega^{-2}(\mathbf{a} + i \omega \mathbf{v}_0) + \omega^{-1} \omega \mathbf{u}_0 
\end{align*}
\]

The notation used to express the boundary conditions (12) and (13) accounts for mixed conditions, as it implies that the Neumman and Dirichlet boundaries, \( \Gamma_t \) and \( \Gamma_u \), are complementary and disjoint in each dimension of the space: \( \Gamma = \Gamma_t \cup \Gamma_u \) and \( \emptyset = \Gamma_t \cap \Gamma_u \).

It is possible, also, to account for Robin- and Sommerfeld-type boundaries.

### 3. WAVE EQUATION

The wave equation is obtained combining the domain conditions (9) to (11) to eliminate the stress and strain terms as independent variables:

\[
\mathbf{D} \mathbf{k} \mathbf{D}^* \mathbf{u} + \mathbf{b} + (\omega^2 \mathbf{\rho} - i \omega \mathbf{c}) \mathbf{u} = \mathbf{0}
\]  

The homogeneous form (\( \mathbf{b} = \mathbf{0} \)) of this system of differential equations can be reduced to the the Helmholtz equation,
\[ \nabla^2 \Phi_j + \beta_j^2 \Phi_j = 0 \]  
(31)

where the (complex) wave number \( \beta_j \) depends on the forcing frequency, \( \omega \), assuming that the displacement field, \( u \), is derived from the gradient and rotational terms of scalar potential functions, \( \Phi_j \). As shown below, this generates three sets of solutions, namely two \( P \)-wave families (\( j = 1, 2 \)) and a single \( S \)-wave family (\( j = 3 \)).

### 3.1 \( P \)-wave solutions

The \( P \)-wave solution is obtained assuming that the displacement field in the solid and fluid phases of the mixture, \( u_s \) and \( u_f \), respectively, is the gradient of potential \( \Phi_j \), in form,

\[
\begin{align*}
\{ u_s \} &= \{ \nabla \Phi_j \} \\
\{ u_f \} &= \{ \gamma_j \nabla \Phi_j \}
\end{align*}
\]

with \( j = 1, 2 \)  
(32)

where \( \gamma_j \) is a proportionality coefficient and \( \nabla \) the gradient vector:

\[
\nabla = \begin{bmatrix} \partial_x \\ \partial_y \end{bmatrix}
\]

(33)

Substitution of assumption (32) in equation (30) leads to the following system of differential equations (see Appendix A),

\[
\begin{align*}
(\kappa + \gamma_j \alpha) M \nabla^2 \Phi_j + \omega^2 (\rho + \gamma_j \rho_u) \Phi_j &= 0 \\
(\alpha + \gamma_j) M \nabla^2 \Phi_j + \omega^2 (\rho_w + \gamma_j \rho_{w_2}) \Phi_j &= 0
\end{align*}
\]

(34)

where:

\[
\rho_{w_2} = \frac{\rho_u a}{n^2} - \frac{i \xi}{\omega n^2}
\]

(35)

The Helmholtz equation (31) for the wave number and the proportionality coefficient:

\[
\beta_j^2 = \frac{\rho_u \gamma_j}{\kappa + \alpha \gamma_j} \omega^2 = \frac{\rho_{w_2} \gamma_j}{\alpha + \gamma_j} \omega^2
\]

(36)

(\( \rho_u - \alpha \rho_{w_2} \)) \( \gamma_j^2 \) + (\( \rho - \kappa \rho_u \)) \( \gamma_j \) + (\( \alpha \rho - \kappa \rho_u \)) = 0

(37)

This latter equation has two independent roots, thus leading to two independent \( P \)-wave solutions.

### 3.2 \( S \)-wave solutions

The \( S \)-wave solution is obtained assuming that the displacement field is the anti-gradient of potential \( \Phi_j \):

\[
\begin{align*}
\{ u_s \} &= \{ \tilde{\nabla} \Phi_j \} \\
\{ u_f \} &= \{ \gamma_j \tilde{\nabla} \Phi_j \}
\end{align*}
\]

(38)
\[ \tilde{\mathbf{V}} = \begin{bmatrix} \partial_y \\ -\partial_x \end{bmatrix} \]  

Substitution of assumption (38) in equation (30) leads to the system of differential equations (see Appendix B),

\[
\begin{align*}
\mu \nabla^2 \Phi + \omega^2 (\rho + \gamma_3 \rho_w) \Phi &= 0 \\
\omega^2 (\rho_w + \gamma_3 \rho_{w2}) \Phi &= 0
\end{align*}
\]

and the Helmholtz equation (31) is recovered for the following definitions of the wave number and the proportionality coefficient:

\[
\beta^2_j = (\rho + \rho_w \gamma_3) \frac{\omega^2}{\mu} \quad (41)
\]

\[
\gamma_3 = -\rho_w / \rho_{w2} \quad (42)
\]

4. WAVE MODELLING

According to the results above, the solution of the Helmholtz equation (31) produces three sets of waves propagating in a poroelastic medium, with a particular set of material properties, when subject to a given forcing frequency.

Depending on the problem being analysed, it is convenient to describe the propagation of waves either in Cartesian or in polar co-ordinates. In the first case, the Helmholtz equation takes the form,

\[
\partial_{xx} \Phi + \partial_{yy} \Phi + \beta^2_j \Phi = 0 \quad (43)
\]

and the results thus obtained model the propagation of plane waves. The propagation of cylindrical waves is modelled using the solution of the Helmholtz equation in polar co-ordinates:

\[
\partial_{rr} \Phi_j + r^{-1} \partial_r \Phi_j + r^{-2} \partial_{\theta\theta} \Phi_j + \beta^2_j \Phi_j = 0 \quad (44)
\]

4.1 Plane waves

The general solution of equation (43) is the following, where \(i\) is the imaginary unit:

\[
\Phi_j = e^{i(-k_j x + \eta_j y)} \quad (45)
\]

\[
k_j^2 + \eta_j^2 = \beta^2_j \quad (46)
\]

The displacement field is obtained applying definition (32) for \(P\)-waves (\( j = 1, 2 \)),

\[ u_+ = \begin{bmatrix} u_x \\ u_y \end{bmatrix}, \quad \Phi_j = -ik_j \quad \text{and} \quad u_j = \gamma_j u_+ \quad (47) \]

and definition (38) for \(S\)-waves:
\[ u_s = \left\{ \begin{array}{l} u_x \\ u_y \\ u_\theta \end{array} \right\}_s = \left\{ \begin{array}{l} i \eta_j \\ k_j \end{array} \right\} \Phi_j \quad \text{and} \quad u_j = \gamma_j u_s \quad (48) \]

The stress and pressure fields are defined using the constitutive relations (11), after determining the deformations compatible with the displacement fields (47) and (48) using equation (10). The following results are obtained for P-waves (\( j = 1, 2 \)) and for S-waves:

\[
\sigma = \left\{ \begin{array}{l} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{yy} \\ \pi \end{array} \right\} = \left\{ \begin{array}{l} [\lambda + (\alpha + \gamma_j) \alpha M \beta_j^2 - 2 \mu k_j^2] \\ [-\lambda + (\alpha + \gamma_j) \alpha M \beta_j^2 - 2 \mu \eta_j^2] \\ 2 \mu k_j \eta_j \\ - (\alpha + \gamma_j) M \beta_j^2 \end{array} \right\} \Phi_j \quad (49) \]

\[
\sigma = \left\{ \begin{array}{l} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{yy} \\ \pi \end{array} \right\} = \left\{ \begin{array}{l} +2 \mu k_3 \eta_3 \\ -2 \mu k_3 \eta_3 \\ \mu (k_3^2 - \eta_3^2) \\ 0 \end{array} \right\} \Phi_3 \quad (50) \]

These results can be obtained from Appendices A and B, under definitions (45) and (46).

4.2 Cylindrical waves
The general solution of equation (44) is written in form,

\[ \Phi_j = W_n(\beta_j r) e^{i n \theta} \quad \text{with} \quad n = 0, \pm 1, \pm 2, \ldots \quad (51) \]

and inserted in the Helmholtz equation (31) to show that \( W_n \) is the Bessel function of order \( n \), that is, the solution of the Bessel’s differential equation:

\[ (\beta_j r)^2 W_n^{(1)} + (\beta_j r) W_n^{(1)} + [n^2 - \beta_n^2] W_n = 0 \quad (52) \]

The following expressions are obtained for the displacement and stress fields for P-waves (\( j = 1, 2 \)),

\[ u_s = \left\{ \begin{array}{l} u_r \\ u_\theta \\ u_\phi \end{array} \right\}_s = \left\{ \begin{array}{l} \beta_j^{-1} (W_{n-1} - W_{n+1}) \\ i \beta_j^{-1} (W_{n-1} + W_{n+1}) \end{array} \right\} e^{i n \theta} \quad \text{and} \quad u_j = \gamma_j u_s \quad (53) \]

\[
\sigma = \left\{ \begin{array}{l} \sigma_{rr} \\ \sigma_{r\theta} \\ \sigma_{\theta\theta} \\ \pi \end{array} \right\} = \left\{ \begin{array}{l} \mu \left( 2 W_n + W_{n-2} + W_{n+2} \right) - 2 \left( \kappa + \gamma_j \alpha \right) M W_n \\ \mu \left( 2 W_n - W_{n-2} - W_{n+2} \right) - 2 \left( \kappa + \gamma_j \alpha \right) M W_n \\ i \mu W_{n-2} - W_{n+2} \\ -2 (\alpha + \gamma_j) M W_n \end{array} \right\} e^{i n \theta} \quad (54) \]

and for S-waves:
\[
\mathbf{u}_r = \left\{ \begin{array}{l} u_x \\ u_y \end{array} \right\} = \left\{ \begin{array}{l} i \beta_j^{-j} (W_{n+j} + W_{n-j}) \\ \beta_j^{-j} (W_{n+j} - W_{n-j}) \end{array} \right\} e^{j \theta} \quad \text{and} \quad \mathbf{u}_f = \gamma_j \mathbf{u}_r \tag{55} \]

\[\mathbf{\sigma} = \begin{bmatrix} \sigma_{rr} & \sigma_{rt} \\ \sigma_{tr} & \sigma_{tt} \end{bmatrix} = \begin{bmatrix} \mu (W_{n-2} - W_{n+2}) \\ -i \mu (W_{n-2} - W_{n+2}) \\ -\mu (W_{n-2} + W_{n+2}) \end{bmatrix} e^{j \theta} \tag{56} \]

5. RAYLEIGH WAVES

Rayleigh waves are generally defined as plane waves. Therefore, they can be modelled by a linear combination of results (47) to (50) for displacements,

\[
\begin{align*}
\begin{bmatrix} u_x \\ u_y \end{bmatrix} &= \sum_{j=1}^{\infty} \begin{bmatrix} -ik_j \\ +i \eta_j \end{bmatrix} \Phi_j X_j + \begin{bmatrix} ik_j \\ i \eta_j \end{bmatrix} \phi_j X_j \\
\begin{bmatrix} w_x \\ w_y \end{bmatrix} &= \sum_{j=1}^{\infty} \begin{bmatrix} -ik_j \\ +i \eta_j \end{bmatrix} \gamma_j \Phi_j X_j + \begin{bmatrix} ik_j \\ i \eta_j \end{bmatrix} \gamma_j \phi_j X_j
\end{align*} \tag{57} \tag{58}
\]

and stresses,

\[
\begin{align*}
\begin{bmatrix} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{yy} \\ \pi \end{bmatrix} &= \sum_{j=1}^{\infty} \begin{bmatrix} -[\lambda + (\alpha + \gamma_j) \alpha M] \beta_j^2 - 2 \mu k_j^2 \\ 2 \mu k_j \eta_j \\ -[\lambda + (\alpha + \gamma_j) \alpha M] \beta_j^2 - 2 \mu \eta_j^2 \\ -[\alpha + \gamma_j] M \beta_j^2 \end{bmatrix} \Phi_j X_j + \begin{bmatrix} +2 \mu k_j \eta_j \\ -2 \mu k_j \eta_j \\ \mu (k_j^2 - \eta_j^2) \tag{59} \\
0 \end{bmatrix} \phi_j X_j \end{align*}
\]

where the combination weights \(X_j\) represent the (unknown) generalized amplitudes of the \(P\)- and \(S\)-wave terms.

Rayleigh waves propagate in bounded and unbounded media and are characterized by four conditions: they are associated with null surface forces and their amplitudes decay in depth.

\[ t_x = t_y = \pi = 0 \quad \text{for} \quad y = 0 \]

\[ \lim_{x \to \pm \infty} u_y = 0 \]

Figure 1: Rayleigh waves

Under the notation defined in Figure 1, the free-surface condition is stated using definitions (45) and (59) at \(x = 0\):
According to definitions (45), the decay or amplification of the wave amplitude is controlled by the imaginary part of the wave numbers, while its real part defines the mode of propagation. The decay of the amplitude in depth is stated by condition, 
\[ \Im (\eta_j) < 0 \] (61)
and the propagation of upward and downward waves is controlled as follows:
\[ \Re (\eta_j) \begin{cases} < 0 & : \text{upward movement} \\ > 0 & : \text{downward movement} \end{cases} \] (62)

Similar relations apply to the \( y \)-component of the movement:
\[ \Re (k_j) \begin{cases} < 0 & : \text{leftward propagation} \\ > 0 & : \text{rightward propagation} \\ = 0 & : \text{standing} \end{cases} \]
\[ \Im (k_j) \begin{cases} < 0 & : \text{leftward decay} \\ > 0 & : \text{rightward decay} \\ = 0 & : \text{undamped} \end{cases} \] (63)

6. RAYLEIGH WAVES IN SINGLE-PHASE MEDIA

The propagation of Rayleigh waves in homogeneous and isotropic single-phase media is addressed first to establish the basis of the procedure adopted in the extension to the propagation in saturated porous media presented in Section 8.

6.1 Basic equations

Equations (9) to (13) still apply to single-phase media, under the simplification that the stress, strain, displacement and body force vectors collect now only the solid phase terms:
\[ \sigma^T = \begin{bmatrix} \sigma_xx & \sigma_yy & \sigma_xy \end{bmatrix} \] (64)
\[ \varepsilon^T = \begin{bmatrix} \varepsilon_xx & \varepsilon_yy & 2\varepsilon_xy \end{bmatrix} \] (65)
\[ u^T = \begin{bmatrix} u_x & u_y \end{bmatrix} \] (66)
\[ \bar{b}^T = \begin{bmatrix} \rho f_x & \rho f_y \end{bmatrix} \] (67)

The differential equilibrium operator is defined by,
\[ D = \begin{bmatrix} \partial_x & 0 & \partial_y \\ 0 & \partial_y & \partial_x \end{bmatrix} \] (68)
with the relation \( D^* = D^T \) still holding for the differential compatibility operator. The simplified forms of the elastic stiffness, mass and damping matrices are the following:
The Neumann and Dirichlet conditions (12) and (13), respectively, simplify to:

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{bmatrix} =
\begin{bmatrix}
\tau_x \\
\tau_y
\end{bmatrix} = \mathbf{c}
\]

\[
\mathbf{c} = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}
\]

\[
\rho = \begin{bmatrix} \rho & 0 \\ 0 & \rho \end{bmatrix}
\]

The wave equation (30) still holds for single-phase media, and the use of Helmholtz displacement potentials still yields the equivalent scalar equation (31). As shown below, the difference now is that it only generates two sets of solutions, namely a P-wave and a S-wave.

### 6.2 P- and S-wave solutions

The P-wave solution is obtained using approximation (32) for the displacement in the solid phase:

\[
\mathbf{u} = \nabla \Phi_P
\]

Implementation of the procedures summarized in Section 3.1 leads to the Helmholtz equation (31) with the following definition for the wave number:

\[
\beta_p^2 = \frac{\omega^2 \rho - i \omega c}{\lambda + 2\mu}
\]

The S-wave solution is obtained using approximation (38) for the displacement in the solid phase,

\[
\mathbf{u} = \vec{\nabla} \Phi_S
\]

to yield the following expression for the wave number in the Helmholtz equation:

\[
\beta_s^2 = \frac{\omega^2 \rho - i \omega c}{\mu}
\]

### 6.3 Rayleigh waves
Definitions (57) and (59) for the displacement and stress fields caused by Rayleigh waves propagating in a single-phase medium simplify to the following:

\[
\begin{align*}
\left\{ \begin{array}{c}
  u_x \\
  u_y 
\end{array} \right\} &= \left\{ \begin{array}{c}
  -i k_p \\
  +i \eta_p 
\end{array} \right\} \Phi_p X_p + \left\{ \begin{array}{c}
  i \eta_s \\
  +ik_s 
\end{array} \right\} \Phi_s X_s \\
\left\{ \sigma_{xx} \right\} &= \left\{ \begin{array}{c}
  -\lambda \beta_p^2 - 2 \mu k_p^2 \\
  -\lambda \beta_p^2 - 2 \mu \eta_p^2 \\
  2 \mu k_p \eta_p 
\end{array} \right\} \Phi_p X_p + \left\{ \begin{array}{c}
  2 \mu k_s \eta_s \\
  -2 \mu k_s \eta_s \\
  \mu (k_s^2 - \eta_s^2) 
\end{array} \right\} \Phi_s X_s
\end{align*}
\]

(78)

\[
\left\{ \sigma_{yy} \right\} = \left\{ \begin{array}{c}
  -\lambda \beta_p^2 - 2 \mu \eta_p^2 \\
  2 \mu k_p \eta_p 
\end{array} \right\} e^{-ik_{rs}x} X_p + \left\{ \begin{array}{c}
  -2 \mu k_s \eta_s \\
  \mu (k_s^2 - \eta_s^2) 
\end{array} \right\} e^{-ik_{rs}x} X_s = \{0\}
\]

(79)

The free-surface boundary condition (60) simplifies to form,

\[
\left\{ \begin{array}{c}
  \sigma_{yy} \\
  \sigma_{xy} 
\end{array} \right\} = \left\{ \begin{array}{c}
  -\lambda \beta_p^2 - 2 \mu \eta_p^2 \\
  2 \mu k_p \eta_p 
\end{array} \right\} e^{-ik_{rs}x} X_p + \left\{ \begin{array}{c}
  -2 \mu k_s \eta_s \\
  \mu (k_s^2 - \eta_s^2) 
\end{array} \right\} e^{-ik_{rs}x} X_s = \{0\}
\]

(80)

and the decay condition (61) still applies for each type of wave.

Condition (80) holds for every point on the surface by setting,

\[k_p = k_S = \beta_S k\]

(81)

to yield to the following eigen problem:

\[
\left[ \begin{array}{cc}
  \lambda \mu^{-1} \beta_p^2 + 2 \eta_p^2 & 2 \eta_s \beta_S k \\
  2 k \beta_S \eta_p & \beta_S^2 k^2 - \eta_S^2 
\end{array} \right] \left\{ \begin{array}{c}
  X_p \\
  X_s 
\end{array} \right\} = \{0\}
\]

(82)

The eigen-values, \(k\), are the roots of the discriminant,

\[
(\lambda \mu^{-1} \beta_p^2 + 2 \eta_p^2) (\beta_S^2 k^2 - \eta_S^2) - 4 \eta_p \eta_s \beta_S^2 k^2 = 0
\]

(83)

where, according to definitions (46) and condition (81),

\[
\eta_p = \pm \beta_S \sqrt{\beta^2 - k^2}
\]

(84)

\[
\eta_s = \pm \beta_S \sqrt{1 - k^2}
\]

(85)

and, according to definitions (75) and (77):

\[
\beta^2 = \frac{\beta_S^2}{\lambda + 2 \mu}
\]

(86)

The eigen-vector can be determined by either of the alternative solutions:

\[
\left\{ \begin{array}{c}
  X_p \\
  X_s 
\end{array} \right\} = \left\{ \begin{array}{c}
  -2 \eta_s \beta_S k \\
  \lambda \mu^{-1} \beta_p^2 + 2 \eta_p^2 
\end{array} \right\} X
\]

(87)

\[
\left\{ \begin{array}{c}
  X_p \\
  X_s 
\end{array} \right\} = \left\{ \begin{array}{c}
  \beta_S^2 k^2 - \eta_S^2 \\
  -2 \eta_p \beta_S k 
\end{array} \right\} X
\]

(88)

6.4 Solution procedure
Results (84) to (86) can be used to express equation (83) and definitions (87) and (88) in non-dimensional form:

\[
\left(2k^2 - 1\right)^2 + 4k^2 \left(\pm \sqrt{1-k^2}\right)\left(\pm \sqrt{\beta^2 - k^2}\right) = 0
\]  
(89)

\[
\begin{cases}
X_p \\
X_s
\end{cases} = \begin{cases}
\frac{\pm 2k \sqrt{1-k^2}}{2k^2 - 1} \\
X
\end{cases}
\]  
(90)

\[
\begin{cases}
X_p \\
X_s
\end{cases} = \left[1 + 4k^2 (\beta^2 - 1)\right]^{1/2} \begin{cases}
\frac{2k^2 - 1}{\mp 2k \sqrt{\beta^2 - k^2}} \\
X
\end{cases}
\]  
(91)

Equation (89) shows that the Rayleigh wave eigen-values depend on the elastic constants of the single-phase medium, through parameter (86), and are independent of the forcing frequency, \(\omega\). The solution procedure used to define this type of plane wave can be summarized as follows:

1. Select a pair (the signs) of the wave number definitions (84) and (85);
2. Solve the fourth-order equation (89) for \(k^2 = 1\),

\[
f(k) = \left(2k^2 - 1\right)^2 + 4k^2 \left(\pm \sqrt{1-k^2}\right)\left(\pm \sqrt{\beta^2 - k^2}\right) = 0
\]  
(92)

and determine \(k = \pm \sqrt{k}\);
3. Compute the wave numbers using definitions (81), (84) and (85);
4. Compute the eigen-vectors of the Rayleigh wave displacement and stress solutions (78) and (79) using the alternative definitions (90) or (91);
5. Eliminate the solutions that violate the decay condition (61).

7. RAYLEIGH WAVES IN SATURATED POROUS MEDIA

The free-surface condition (60) becomes independent of the surface point position by setting:

\[
k_j = \beta_j k, \quad j = 1, 2, 3
\]  
(93)

Definition (46) is now written as follows,

\[
\eta_j = \pm \beta_j \sqrt{\beta_j^2 - k^2}, \quad j = 1, 2, 3
\]  
(94)

with \(\beta_j^2 = 1\) and,

\[
\beta_j^2 = \frac{\beta_j^2}{\beta_j^2} = \frac{\mu/M}{\kappa + \alpha \gamma_j} = \frac{\mu/M}{\alpha + \gamma_j}\]  
(95)

according to definitions (36), (41) and (42), with:

\[
\bar{\rho} = \rho/\rho_w
\]  
(96)
7.1 Eigen value problem

Definition (25) and results (93) to (95) can be used to express the eigen-problem (60) in the following form,

\[
\begin{bmatrix}
E_{11} & E_{12} & E_{13} \\
E_{21} & E_{22} & E_{23} \\
E_{31} & E_{32} & 0
\end{bmatrix}
\begin{bmatrix}
X_1 \\
X_2 \\
X_3
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

(97)

where, for \( j = 1, 2 \):

\[
E_{ij} = 2k^2 - \frac{\bar{p} + \gamma_j}{\bar{p} + \gamma_3}
\]

(98)

\[
E_{2j} = 2k \left( \pm \sqrt{\bar{\beta}_j^2 - k^2} \right)
\]

(99)

\[
E_{3j} = -\frac{1 - \gamma_j}{\bar{p} + \gamma_3}
\]

(100)

\[
E_{13} = -2k \left( \pm \sqrt{1 - k^2} \right)
\]

(101)

\[
E_{23} = 2k^2 - 1
\]

(102)

7.2 Solution

The last equation of system (97) can be solved in the weighting terms of the \(P\)-wave solutions to yield the following result, according to definition (100):

\[
\begin{bmatrix}
X_1 \\
X_2 \\
X_3
\end{bmatrix} =
\begin{bmatrix}-(1 - \gamma_2 / \gamma_3) \\
+(1 - \gamma_1 / \gamma_3)
\end{bmatrix} X_0
\]

(103)

Substitution of this result in the first two equations of system (97) leads to a second-order eigen-value problem,

\[
\begin{bmatrix}
E_{10} & E_{13} \\
E_{20} & E_{23}
\end{bmatrix}
\begin{bmatrix}
X_0 \\
X_3
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

(104)

where, according to results (98) and (99):

\[
E_{10} = (1 - 2k^2) \frac{\gamma_1 - \gamma_2}{\gamma_3}
\]

(105)

\[
E_{20} = -2k \left[ (1 - \gamma_2 / \gamma_3) \left( \pm \sqrt{\bar{\beta}_j^2 - k^2} \right) - (1 - \gamma_1 / \gamma_3) \left( \pm \sqrt{\bar{\beta}_j^2 - k^2} \right) \right]
\]

(106)

The alternative expressions for the eigen-vector of system (97) are determined combining definition (103) with the alternative expressions of the eigen-vector of system (104):
\[
\begin{aligned}
\begin{bmatrix}
X_1 \\
X_2 \\
X_3
\end{bmatrix} &= \begin{cases}
+2k \left(1 - \gamma_2 / \gamma_1 \right) \left( \pm \sqrt{1 - k^2} \right) X \\
-2k \left(1 - \gamma_1 / \gamma_3 \right) \left( \pm \sqrt{1 - k^2} \right) X \\
(\gamma_1 / \gamma_3 - \gamma_2 / \gamma_3) (2k^2 - I)
\end{cases}
\end{aligned}
\] (107)

\[
\begin{aligned}
\begin{bmatrix}
X_1 \\
X_2 \\
X_3
\end{bmatrix} &= \begin{cases}
-(1 - \gamma_2 / \gamma_3) E_{23} \\
+(1 - \gamma_1 / \gamma_3) E_{23} \\
-E_{20}
\end{cases} X
\end{aligned}
\] (108)

The eigen-values, \( k \), are the roots of the discriminant of system (104):

\[
(2k^2 - I)^2 + 4k^2 \left( \pm \sqrt{1 - k^2} \right) \left[ \frac{\gamma_3 - \gamma_2}{\gamma_1 - \gamma_2} \left( \pm \sqrt{\beta_2^2 - k^2} \right) - \frac{\gamma_3 - \gamma_1}{\gamma_1 - \gamma_2} \left( \pm \sqrt{\beta_3^2 - k^2} \right) \right] = 0
\] (109)

The solution procedure summarized above for single-phase media can be adapted to the analysis of saturated porous media. The essential difference is that equation (109) is complex, in general. Moreover, definitions (37), (42) and (95) indicate that the eigen-values of Rayleigh waves propagating in saturated porous media depend both on the material properties and on the forcing frequency.

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REFERENCES

APPENDIX A: P-WAVES IN TWO-PHASE MEDIA

According to approximation (32), the \( P \)-wave displacement solution is:

\[
\mathbf{u}_s = \begin{bmatrix} \mathbf{u}_x \\ \mathbf{u}_y \end{bmatrix} = \begin{bmatrix} \partial_x \\ \partial_y \end{bmatrix} \Phi_j
\]  

(110)

\[
\mathbf{w} = \begin{bmatrix} w_x \\ w_y \end{bmatrix} = \gamma_j \begin{bmatrix} \partial_x \\ \partial_y \end{bmatrix} \Phi_j
\]  

(111)

The strains are computed from the compatibility condition (10), under definitions (17) and (21), with \( \mathbf{D}^* = \mathbf{D}^T \), where \( \nabla^2 = \partial_{xx} + \partial_{yy} \) is the Laplacian:

\[
\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \\ \varepsilon_{yx} \end{bmatrix} = \begin{bmatrix} \partial_{xx} \\ \partial_{yy} \\ \frac{1}{2} \partial_{xy} \\ \frac{1}{2} \partial_{yx} \end{bmatrix} \Phi_j
\]  

(112)

The stresses are computed from the elasticity condition (11), under definition (22):

\[
\mathbf{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \tau \end{bmatrix} = \begin{bmatrix} (\kappa + \alpha \gamma_j) M \nabla^2 - 2\mu \partial_{yy} \\ (\kappa + \alpha \gamma_j) M \nabla^2 - 2\mu \partial_{xx} \\ 2\mu \partial_{xy} \\ (\alpha + \gamma_j) M \nabla^2 \end{bmatrix} \Phi_j
\]  

(113)

The following relation is obtained inserting results (110), (111) and (113) in the equilibrium condition (9), under definitions (21), (23) and (24),

\[
\left\{ \begin{array}{c}
\partial_x \\
\partial_y
\end{array} \right\} \left[ (\kappa + \gamma_j \alpha) M \nabla^2 \Phi_j + \omega^2 (\rho + \gamma_j \rho_w) \Phi_j \right] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

\[
\left\{ \begin{array}{c}
\partial_x \\
\partial_y
\end{array} \right\} \left[ (\alpha + \gamma_j) M \nabla^2 \Phi_j + \omega^2 (\rho_w + \gamma_j \rho_w^2) \Phi_j \right] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]  

(114)

recovering thus the system of differential equations (34).
APPENDIX B: S-WAVES IN TWO-PHASE MEDIA

According to approximation (38), the $S$-wave displacement solution is:

$$
\mathbf{u}_s = \begin{cases}
    u_x \\
    u_y
\end{cases} = \begin{cases}
    +\partial_y \\
    -\partial_x
\end{cases} \Phi_3
$$

(115)

$$
\mathbf{w} = \begin{cases}
    w_x \\
    w_y
\end{cases} = \gamma_j \begin{cases}
    +\partial_y \\
    -\partial_x
\end{cases} \Phi_j
$$

(116)

The strains are computed from the compatibility condition (10), under definitions (17) and (21), with $\mathbf{D}^* = \mathbf{D}^T$:

$$
\mathbf{e} = \begin{cases}
    \varepsilon_{xx} \\
    \varepsilon_{yy} \\
    \gamma_{xy} \\
    \varsigma
\end{cases} = \begin{cases}
    \partial_{xy} \\
    -\partial_{yy} \\
    \partial_{yy} - \partial_{xx} \\
    0
\end{cases} \Phi_3
$$

(117)

The stresses are computed from the elasticity condition (11), under definition (22):

$$
\mathbf{\sigma} = \begin{cases}
    \sigma_{xx} \\
    \sigma_{yy} \\
    \sigma_{xy} \\
    \pi
\end{cases} = \begin{cases}
    2\mu \partial_{xy} \\
    -2\mu \partial_{xy} \\
    \mu (\partial_{yy} - \partial_{xx}) \\
    0
\end{cases} \Phi_3
$$

(118)

The following relation is obtained inserting results (110), (111) and (113) in the equilibrium condition (9), under definitions (21), (23), (24) and (35),

$$
\left\{ \begin{array}{c}
+\partial_y \\
-\partial_x
\end{array} \right\} \left[ \mu \nabla^2 \Phi_3 + \omega^2 (\rho_3 + \gamma_3 \rho_w) \Phi_3 \right] = \begin{cases}
0 \\
0
\end{cases}
$$

(119)

recovering the system of differential equations (40).

APPENDIX C: MOLSAND SOIL

$$
\rho_w = 1000 \text{ kg} / \text{m}^3
$$

$$
\rho = 2650 \text{ kg} / \text{m}^3
$$

$$
M = 5.67 \times 10^4 \text{ N} / \text{m}^2
$$

$$
E = 0.298 \times 10^6 \text{ N} / \text{m}^2
$$

$$
\nu = 0.333
$$
\[ \lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \]

\[ \mu = G = \frac{E}{2(1 + \nu)} \]

\[ \alpha = 1.0 \]

\[ n^w = 0.388 \]

\[ k = 0.01 \text{ m/s} \]

\[ g = 9.81 \text{ m/s}^2 \]

\[ \xi = \frac{\rho^w g}{k} n^{w2} \]