Formulation and implementation of hybrid-Trefftz stress elements for cohesive fracture

J.A. Teixeira de Freitas
Technical University of Lisbon, Instituto Superior Técnico
Department of Civil Engineering and Architecture
Av. Rovisco Pais, 1049-001, Lisboa, Portugal
email: freitas@civil.ist.utl.pt

Abstract
The hybrid-Trefftz stress finite element model is formulated to implement the fracture analysis of quasi-brittle materials using a cohesive model. The response is assumed to be geometrically linear and the bulk material to respond elastically. A discontinuous cohesive fracture model is used for tension dominant processes. Besides the derivation of the hybrid-Trefftz stress model, the approximation basis is defined and the procedure to detect and to implement the onset and propagation of damage and fracture through the domain of the element is outlined.

1 Introduction
The study reported has been motivated by the application of the extended finite element formulation to the geometrically linear fracture analysis of quasi-brittle materials reported by Mariani and Perego [1]. The bulk material is assumed to respond elastically and the process of progressive damage, leading to fracture, to concentrate along discontinuity surfaces. A mixed-mode cohesive model is used for predominantly tensile states of stress, as it does reproduce neither frictional effects nor non-associative compressive modes of failure.

This problem seems to be well-suited to modelling with hybrid-Trefftz stress elements. The problem is linear, with the exception of the cohesive constitutive relations, which simplifies the derivation of Trefftz approximation bases. The fracture constitutive relations apply on discontinuity boundaries, and the Trefftz method leads, fundamentally, to the weak enforcement of boundary equations, as the basis is so constrained as to satisfy locally all domain conditions of the problem. Hybrid stress elements in general, and their Trefftz variant in particular, can produce accurate stress estimates using relatively coarse finite element meshes, and a central issue in the modelling of the problem being addressed is to capture the stress concentration developing in the vicinity of the tip of cohesive cracks.

The presentation is organised in three parts. The first part addresses the definition of the basic equations of the problem. The definition of the domain and boundary equilibrium and compatibility conditions is used to establish the notation and, especially, to identify the variables used to describe the statics and the kinematics of the (discontinuous) damage processing zone. The constitutive relations adopted in the motivating paper [1] are recalled next, but they are written in a format better suited to stress element modelling. The following two sections address the derivation of the finite model. The approximations in the domain of the element and on its boundary are stated and their dual transformations are used to set up the finite element solving system. Numerical implementation aspects are analysed in the last part, namely the selection and implementation of the stress approximation basis, and the procedure suggested to detect and to implement the onset and propagation of damage and fracture is presented. Non-essential information is collected in supporting appendices to lighten the presentation.
2 Domain equilibrium and compatibility conditions

Let $\Omega$ denote the domain of the element, which may not be convex, simply connected or bounded. It is convenient to distinguish five (complementary) regions on its boundary, $\Gamma$: the inter-element boundary, $\Gamma_e$, the surfaces where force or displacement components are prescribed, $\Gamma_t$ and $\Gamma_u$, the active fracture processing surface, $\Gamma_p$, and the surface along which cracking (material separation) has occurred, $\Gamma_c$.

![Figure 1. Identification and orientation of boundary surfaces.](image)

In the notation used here, $\Delta \lambda$ and $\Delta \lambda_u$, with $\Delta \lambda \Delta \lambda_u = 0$, represent the increments on the parameter selected to describe (alternative) force- and displacement-driven loading programmes, respectively. Moreover, it is assumed that the increment on every variable, say $\Delta v$, used to describe the response of the structure is expanded in a Taylor series on a non-negative (finite, non-infinitesimal) parameter, $\pi$:

$$\Delta v = \sum_{n=1}^{\infty} v^{(n)} \frac{\pi^n}{n!} \quad \text{with } \pi \geq 0 \quad (1)$$

As this response is assumed to be kinematically linear (small strain and small displacement assumptions), the domain conditions for equilibrium and compatibility are written as follows,

$$D \sigma^{(n)} + \lambda^{(n)}_t = 0 \quad \text{in } \Omega \quad (2)$$

$$\varepsilon^{(n)} = D u^{(n)} \quad \text{in } \Omega \quad (3)$$

where vectors $\sigma$ and $\varepsilon$ collect the independent components of the stress and solid strain tensors, $u$ and $b$ are the displacement and body-force vectors, respectively, and the differential equilibrium and compatibility operators, $D$ and $D^*$, are linear and conjugate.

3 Boundary equilibrium and compatibility conditions

The Neumann and Dirichlet conditions are written as follows,

$$N \sigma^{(n)} = \bar{\lambda} \lambda^{(n)}_t \quad \text{on } \Gamma_t \quad (4)$$

$$u^{(n)} = \bar{u} \lambda^{(n)}_u \quad \text{on } \Gamma_u \quad (5)$$

where matrix $N$ collects the components of the unit outward normal vector. The equations above hold for inter-element boundaries, with adequate interpretation of the stipulation terms (the forces or displacements induced by a connecting element). For simplicity of the presentation, it is assumed from this point onwards that the Neumann boundary of the stress element defined below combines its inter-element boundary with the portion of the Neumann boundary of the mesh the element may contain: $\Gamma_t \leftrightarrow \Gamma_t \cup \Gamma_e$.  

Assuming a direct orientation for boundary $\Gamma$ of the element, as shown in Figures 1 and 2a), the force continuity condition and the displacement discontinuity definition on a processing (or open crack) surface, written in the local system of reference $(\xi, \eta)$, are defined as follows:

\[
\begin{align*}
\{t^+\} &= \begin{bmatrix} I \\ I \end{bmatrix} t & \text{on } \Gamma_p \text{ and } \Gamma_c \\
\delta &= -\begin{bmatrix} I & I \end{bmatrix} \begin{bmatrix} u^+ \\ u^- \end{bmatrix} & \text{on } \Gamma_p \text{ and } \Gamma_c
\end{align*}
\]

(6)

(7)

Figure 2. Processing and cracked boundaries.

The free-surface condition holds on the open crack portion of the boundary,

\[
t = 0 \quad \text{on } \Gamma_c
\]

(8)

and the work invariance condition is stated on the processing zone,

\[
t^T \delta = \tau w \quad \text{on } \Gamma_p
\]

(9)

where $\tau$ and $w$ are the effective force and displacement used to describe cohesive constitutive relations. In the stress element option followed here, the effective traction is explicitly defined in form:

\[
\tau = \sqrt{t^T R t}
\]

(10)

As the definition of the displacement discontinuity vector should be independent of the cohesive fracture constitutive relations, this vector is expressed in terms of two parameters, namely the effective displacement used to model the constitutive relation, $w$, and a non-dissipative displacement mode, $w_0$:

\[
\delta = n \cdot w + m \cdot w_0
\]

(11)

This definition is constrained to be non-singular,

\[
\delta = (n^T v) (m^T v) - (n^T m) v^2 
eq 0
\]

(12)

as it must be possible to describe the kinematics of the processing zone using the displacement discontinuity vector, and consistent with the work invariance condition (9):

\[
n^T t = \tau
\]

(13)

\[
m^T t = 0 \quad \text{if } w_0 \neq 0
\]

(14)

Definitions (10) and (11) play the role of (discontinuity surface) static and kinematic conditions, respectively, as they are independent of the constitutive relations used to model the cohesive fracture response. As shown in Appendix 1, the displacement discontinuity modes $n$ and $m$ are so defined as to satisfy conditions (12) and (13). The Taylor series expansion of equations (10), (14) and (11),
\[
\begin{align*}
\{ \tau_0^{(n)} = 0 \} &= \begin{bmatrix} n,^T \\ m,^T \end{bmatrix} t^{(n)} + \begin{bmatrix} R_m \\ 0 \end{bmatrix} \quad \text{if} \; w_0 \neq 0 \text{ or } w_0^{(n)} \neq 0 \\
\delta^{(n)} &= \begin{bmatrix} n, \\ m, \end{bmatrix} \begin{bmatrix} w_0^{(n)} \\ w_0^{(n)} \end{bmatrix} + g, t^{(n)} + R_{s_0}
\end{align*}
\]

is also presented in Appendix 1. It is noted that the static and kinematic transformations (15) and (16) are dual, symmetric (\( g, = g,**T** \)) and recursive (the \( n \)-th order residual term depends on variables of order \( n-1 \)).

### 4 Constitutive relations

It is assumed that the undamaged material behaves elastically,

\[
\varepsilon^{(n)} = f \sigma^{(n)} + R_{s_0} \quad \text{in} \; \Omega \setminus \Gamma_p
\]

where the (symmetric) flexibility matrix \( f \) collects the (bulk) mechanical parameters and the residual term is used to model eventually non-linear responses (\( R_{s_0} = 0 \) for linear elasticity). It may be used to model, also, forcing stress or strain (e.g. thermal) fields.

Although non-linear relations can be easily incorporated in the incremental description used here, it is assumed for simplicity of the presentation that the causality relation on the damage surface \( \Gamma_p \) is linear, as illustrated in Figure 2b), where \( \tau_M \) is the limit effective traction and \( w_i \) is the limit displacement beyond which the process zone is completely damaged. The incremental association condition for the softening and the unstressing modes (see Appendix 2),

\[
\tau^{(n)} - k, w^{(n)} = R_{p_m} \quad \text{on} \; \Gamma_p
\]

where \( k, \) defines the mode stiffness and the residual term \( R_{p_m} \) is null for (piecewise) linear modes. Its (local) implementation is discussed in Appendix 3.

### 5 Finite element approximations

The finite element model used here develops from the direct approximation of the stress field,

\[
\sigma^{(n)} = S X^{(n)} + s, \lambda^{(n)} \quad \text{in} \; \Omega
\]

where \( X \) is a (generalised stress) weighting vector, matrix \( S \) contains the stress approximation modes, \( s, \) is a particular stress solution vector, and \( \lambda \) is the increment control parameter. Letting \( \pi = \Delta \lambda \) in the series expansion (1), the following identifications hold for force- and displacement-driven loading programmes, where \( \delta_{mn} \) denotes the Kronecker symbol:

\[
\begin{cases}
\lambda_i^{(n)} = \lambda_i^{(n)} = 0 \\
\lambda_i^{(n)} = \lambda_i^{(n)} = 0
\end{cases}
\]

Typical of the hybrid stress model used here is to constrain approximation (19) to satisfy locally the domain equilibrium condition (2):

\[
DS = 0 \quad \text{in} \; \Omega
\]

\[
Ds, + b = 0 \quad \text{in} \; \Omega \text{ if } \Delta \lambda_s = 0
\]

\[
Ds, = 0 \quad \text{in} \; \Omega \text{ if } \Delta \lambda_s = 0
\]

A Trefftz basis is used in the implementation of the stress approximation (19), which satisfies locally conditions (21) to (23).

The generalised strains defined by the dual transformation of approximation (19),
\[ e^{(n)} = \int S^T \varepsilon^{(n)} \, d\Omega \]  
(24)

which ensure the invariance of the inner-product in the finite element mapping, are used to enforce weakly (and independently) the compatibility and the elasticity equations (3) and (17):

\[ e^{(n)} = \int S^T (D^{*} u^{(n)}) \, d\Omega \]  
(25)

\[ e^{(n)} = \int S^T \left( f_n \sigma^{(n)} + R_{in} \right) \, d\Omega \]  
(26)

The average compatibility condition (25) is integrated by parts to force the emergence of boundary terms and incorporate thus the Dirichlet condition (5) and the displacement discontinuities along the damage processing and open crack surfaces the element may contain, as described by conditions (6) and (7).

As it is shown below, this operation leads to the direct approximation of the displacements on the Neumann boundary of the element (which contains the inter-element portion of the element boundary),

\[ u^{(n)} = Z_0 q_0^{(n)} \quad \text{on } \Gamma_i \]  
(27)

the displacement discontinuity on the open crack boundary the element may contain,

\[ \delta^{(n)} = Z_c q_c^{(n)} \quad \text{on } \Gamma_c \]  
(28)

as well as the so-called effective displacements on the damage processing boundary:

\[ w^{(n)} = Z_p q_p^{(n)} \quad \text{on } \Gamma_p \]  
(29)

\[ w_0^{(n)} = Z_0 q_0^{(n)} \quad \text{on } \Gamma_p \]  
(30)

According to definitions (15) and (16), \[ Z_0 \] in approximation (30) is the Dirac function. The remaining boundary displacement approximations are regular and, by option, polynomial. Chebyshev polynomials are used to in the approximation of the displacements on the Neumann boundary (27) and of the relative displacements on (open) crack boundaries (28). The effective displacements on damage processing boundaries (29) are approximated with Johnson (non-negative) polynomials.

The dual transformations of the boundary approximations define (generalized) forces on the Neumann and crack boundaries of the element and (generalized) effective tractions on the damage processing surface:

\[ \tilde{Q}_i = \int Z_i^T \tau \, d\Gamma_i \]  
(31)

\[ \tilde{Q}_c^{(n)} = \int Z_c^T t^{(n)} \, d\Gamma_c \]  
(32)

\[ \tilde{Q}_w^{(n)} = \int Z_w^T \varepsilon^{(n)} \, d\Gamma_p \]  
(33)

\[ \tilde{Q}_0^{(n)} = \int Z_0^T \varepsilon^{(n)} \, d\Gamma_p \]  
(34)

They are used to enforce weakly the Neumann (and inter-element) flux condition (4), the crack surface equilibrium condition (8), and the effective traction definitions (15),

\[ \tilde{Q}_i = \int Z_i^T t^{(n)} \, d\Gamma_i \]  
(35)

\[ \tilde{Q}_c^{(n)} = \int Z_c^T t^{(n)} \, d\Gamma_c = 0 \]  
(36)

\[ \tilde{Q}_w^{(n)} = \int Z_w^T \left( n_i^T t^{(n)} + R_{in} \right) \, d\Gamma_p \]  
(37)

\[ \tilde{Q}_0^{(n)} = \int Z_0^T \left( m_p^T t^{(n)} \right) \, d\Gamma_p = 0 \]  
(38)

for the boundary force field equilibrated by the stress approximation (19):

\[ t^{(n)} = T X^{(n)} + t_h \lambda^{(n)} \quad \text{on } \Gamma \]  
(39)

\[ T = NS \]  
(40)
The generalized traction definition (33) is also used to enforce weakly the cohesive fracture association condition (18):

\[ Q^w = \int Z^T (k_m u'^n + R_{m}) \, d\Gamma_p \]  

\[ \lambda = Ns_\lambda \]  

\[ (41) \]

\[ (42) \]

### 6 Finite element equations

As the domain equilibrium condition (2) is locally satisfied by the assumed stress field (19), consequent upon constraints (21) to (23), the finite element static admissibility condition is limited to the boundary conditions (35) to (38). The finite element kinematic admissibility condition reduces to (the boundary integral) description of equation (25), as it combines the weak enforcement of the domain and boundary compatibility conditions (3), (5) and (7). The finite element constitutive relations are stated by the bulk elasticity equation (26), for the assumed stress field (19), and by the cohesive fracture association condition (42), under the same approximation. To lighten the presentation, the finite element equations are derived assuming (piecewise) linear association conditions (17) and (18): \( R_{en} = \mathbf{0} \) and \( R_{pm} = \mathbf{0} \).

To obtain the finite element kinematic admissibility condition, definition (25) is written for each sub-domain where the displacements are continuous (they are bounded by damage processing and open crack discontinuity surfaces \( \Gamma_p \) and \( \Gamma_c \) that the element may contain, see Figure 3), and integrated by parts:

\[ \varepsilon^{(n)} = \sum_i \int S^T \left( D^T u'^n \right) \, d\Omega_i \]  

\[ e^{(n)} = -\sum_i \int (DS)^T u'^n \, d\Omega_i + \sum_i \int (NS)^T u'^n \, d\Gamma_i \]

\[ (43) \]

\[ (44) \]

Figure 3. Domain decomposition along displacement discontinuity surfaces.

As the complementary solution of the stress approximation is self-equilibrated, as stated by condition (21), and is constrained to produce continuous force fields on the internal boundaries of the element, as required by condition (6), enforcement of the Dirichlet and displacement discontinuity conditions (5) and (7) yields the following expression, after reassembling the element sub-domains and using result (40):

\[ e^{(n)} = \int T^T u'^n \, d\Gamma_1 - \int T^T \delta^n \, d\Gamma_1 - \int T^T \delta^n \, d\Gamma_p + \int T^T \bar{u} \lambda_{n} \, d\Gamma_u \]

\[ (45) \]

The finite element kinematic admissibility condition is obtained substituting definitions (16), (27) to (30) and (39) in equation (45),

\[ e^{(n)} = Aq^{(n)} - F, X^{(n)} - f, \lambda^{(n)} + \bar{e} u \lambda_{n} - R_{en} \]

\[ (46) \]

where vector \( q \) collects the boundary displacement degrees of freedom and \( A \) is the finite element compatibility matrix defined in Appendix 4, where the definitions for the remaining terms in equation (46) are also presented.

The dual of transformation (46), where \( F \) plays the role of a geometric flexibility matrix, defines the finite element static admissibility condition:
\[ A^T X^{(n)} = Q_n - a^T \lambda^{(n)} \]  

(47)

To recover the explicit form of this equation, presented in Appendix 4, it suffices to enforce the boundary force approximation (39) in conditions (35) to (38).

The finite element constitutive relations are obtained inserting the stress approximation (19) in the weak enforcement of the local (bulk) elasticity condition (26) for the generalized strain \( \varepsilon \), and the effective displacement approximation (29) in the weak enforcement of cohesive fracture condition (42) for the generalized effective traction, to yield the following results (for piecewise linear constitutive laws),

\[ e^{(n)} = F X^{(n)} + f_{\lambda} \lambda^{(n)} \]  

(48)

\[ Q^{(n)}_w = K\lambda q^{(n)}_w \]  

(49)

where the elastic flexibility matrices and the cohesive fracture stiffness matrix are defined as follows:

\[ F = \int S^T f S d\Omega \]  

(50)

\[ f_{\lambda} = \int S^T f \lambda d\Omega \]  

(51)

\[ K\lambda = \int Z^T \kappa, Z d\Gamma_p \]  

(52)

The finite element governing system is obtained combining the kinematic and static admissibility conditions (46) and (47), and using the constitutive relations (48) and (49) to eliminate the generalized strains and effective forces as explicit variables, under the loading parameter identification (20). The resulting system is highly sparse, symmetric, linear and recursive, and all coefficients have boundary definitions with the exception of the elastic flexibility matrices (see Appendix 4):

\[
\begin{bmatrix}
F + F_s & -A_t & +A_c & +A_w & +A_0 \\
-A_t^T & 0 & 0 & 0 & 0 \\
+A_c^T & 0 & 0 & 0 & 0 \\
+A_w^T & 0 & 0 & -K \lambda & 0 \\
+A_0^T & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
X^{(n)} \\
q_s \\
q_c \\
q_w \\
q_0 \\
\end{bmatrix}
= 
\begin{bmatrix}
\varepsilon_{\lambda} \lambda^{(n)} - R_{\varepsilon n} \\
0 \\
R_{\varepsilon n} \\
0 \\
\end{bmatrix}
+ 
\begin{bmatrix}
f_{\lambda} + f_s \\
-a_t^T \\
+a_c^T \\
+a_w^T \\
+a_0^T \\
\end{bmatrix}
\lambda^{(n)}
\]

(53)

7 Numerical implementation

Mesh assemblage of hybrid elements does not involve the double summation typical of conventional (conform) finite element models. It consists simply in collecting the elementary systems (53) while assigning the same boundary displacements \( q \), to the boundaries shared by (at most, two) connecting elements, as well as the same displacement loading parameter, \( \lambda \), to the elements sharing boundaries subject to prescribed displacements. Consequently, besides preserving the structure and the properties of the elementary governing system (53), the assembled finite element solving system is particularly well suited to parallel processing.

The (generalized) effective displacements can be eliminated in system (53) using the weak enforcement of the cohesive fracture constitutive relation (18):

\[ A^T_w X^{(n)} + a^T_w \lambda^{(n)} - K\lambda q^{(n)}_w + R_{\varepsilon n} = 0 \]  

(54)

This operation, which is equivalent to eliminate the displacement discontinuity directly in the weak enforcement of the kinematic admissibility condition (45) using definitions (16) and (18), does not affect structure of the solving system and influences marginally its sparsity.

A technique commonly used since hybrid stress elements were first suggested by T.H.H. Pian [4], and adopted by J. Jiřiousek in Trefftz element modelling [5], is to condense, at element level, the governing system (53) on the boundary displacement degrees of freedom. The use of this technique is advisable when the degree of the stress approximation is low, as it enables the emulation of conventional
displacements elements and, consequently, the implementation of hybrid stress elements in the libraries of the available finite element computer codes. However, the implementation of this technique affects strongly both the sparsity and the natural suitability for parallel processing, in particular when the option is to base the analysis on (relatively) coarse methods of high-degree elements.

Basic aspects of numerical implementation of hybrid-Trefftz elements, such as element design, assemblage of elementary governing systems, manipulation of sparse solving systems and adaptive solution procedures, as presented, for instance, in Ref. [6, 7]. Three main implementation aspects are addressed here, namely the selection of the stress approximation basis, the computation of finite element arrays and the implementation of the operations designed to detect and implement the onset and propagation of damage and fracture.

8 Trefftz approximation bases

The method proposed by E. Trefftz [8] to solve boundary value problems consists, in essence, in constraining the selected approximation basis to satisfy locally all domain conditions of the problem. In the present context, this implies to extend the equilibrium constraints (21) to (23) on the stress approximation (19) to the existence of associate strain and displacement fields,

\[ \varepsilon^{(n)} = EX^{(n)} + e_\lambda \lambda^{(n)} \quad \text{in } V \quad (55) \]
\[ u^{(n)} = UX^{(n)} + u_\lambda \lambda^{(n)} \quad \text{in } V \quad (56) \]

so constrained as to satisfy locally the compatibility condition (3),

\[ E = DU \quad (57) \]
\[ e_\lambda = D u_\lambda \quad (58) \]

and the (bulk) elasticity condition (17), with \( R_{eh} = 0 \), in the present formulation:

\[ E = f S \quad (59) \]
\[ e_\lambda = f s_\lambda \quad (60) \]

This basis can be derived from displacement potentials that solve the Navier equation obtained combining the equilibrium (2), compatibility (3) and elasticity (17) conditions. The major advantage of using a Trefftz stress basis, that is, the basis first proposed by J.H. Michell [9] in the present context of linear elastostatics, is to implement the analysis on an approximation basis that embodies the physics of the bulk material, so weighted as to satisfy on average, in the sense of Galerkin, the Neumann boundary conditions and the cohesive fracture constitutive relations.

8.5 Trefftz approximation bases

The method proposed by E. Trefftz [8] to solve boundary value problems consists, in essence, in constraining the selected approximation basis to satisfy locally all domain conditions of the problem. In the present context, this implies to extend the equilibrium constraints (21) to (23) on the stress approximation (19) to the existence of associate strain and displacement fields,

\[ \varepsilon^{(n)} = EX^{(n)} + e_\lambda \lambda^{(n)} \quad \text{in } V \quad (55) \]
\[ u^{(n)} = UX^{(n)} + u_\lambda \lambda^{(n)} \quad \text{in } V \quad (56) \]

so constrained as to satisfy locally the compatibility condition (3),

\[ E = DU \quad (57) \]
\[ e_\lambda = D u_\lambda \quad (58) \]

and the (bulk) elasticity condition (17), with \( R_{eh} = 0 \), in the present formulation:

\[ E = f S \quad (59) \]
\[ e_\lambda = f s_\lambda \quad (60) \]

This basis can be derived from displacement potentials that solve the Navier equation obtained combining the equilibrium (2), compatibility (3) and elasticity (17) conditions. The major advantage of using a Trefftz stress basis, that is, the basis first proposed by J.H. Michell [9] in the present context of linear elastostatics, is to implement the analysis on an approximation basis that embodies the physics of the bulk material, so weighted as to satisfy on average, in the sense of Galerkin, the Neumann boundary conditions and the cohesive fracture constitutive relations.
To ensure completeness, it suffices to build the stress approximation basis on regular (polynomial) solutions. However, it is convenient to enrich the basis with solutions that model local effects that condition the rate of convergence, namely the stress concentrations developing in the neighbourhood of wedge points \( W \) and \( W' \), of crack tip points \( C \) and of support points \( A \) and \( A' \) in the typical fracture modelling tests illustrated in Figure 4. When the cause of these effects is directly controlled, as in the case of the point load \( P \) in the simply supported beam test, the stress concentration is modelled with the particular solution term. The use of singular stress modes is not consistent with the local enforcement of the cohesive fracture constitutive relations, unless these relations are enforced in weak form, under adequate integrability conditions. As it is shown below, this justifies the identification of the non-dissipative displacement mode present in definition (11).

9 Implementation of Trefftz bases

The Trefftz constraints (21), (57) and (59) can be used to replace definitions (50) and (51) by the equivalent boundary integral expressions, a feature typical of all solution methods relying on approximation bases satisfying locally all domain conditions of the partial differential equation problem:

\[
F = \int_T T^T U \, d\Gamma \quad (61)
\]

\[
f_{i} = \int_T T^T u_{i} \, d\Gamma \quad (62)
\]

This section addresses the implementation of these definitions, written for each boundary side of the element with a distinct analytical description, e.g.:

\[
F = \sum_i \int_T T^T U \, d\Gamma_i \quad (63)
\]

No particular reference is made to the computation of the compatibility matrices present in the solving system (53), as their definitions are similar to expression (63), with the boundary displacement bases, \( Z \), replacing the displacement modes, \( U \), associated with the direct stress approximation.

The elements are here classified as regular elements, singular elements and crack elements. Elements implemented on polynomial stress approximation bases (19), associated with displacement fields (56) with similar properties in the domain of the element are termed here regular elements. They are termed singular elements when this basis is enriched to model singular local effects, namely those induced by geometry discontinuities and by point loads. A crack element is a singular element with embedded displacement discontinuity boundary designed to model the onset and the propagation of cohesive damage and fracture.

9.1 Regular elements

The stress approximation functions and the associate displacement fields can be found, for instance, in Ref. [10]. They are expressed in a local polar co-ordinate system, associated with the principal directions of the element, with origin at its barycentre. They define polynomial approximations in interior problems and combine polynomial and rational solutions in the modelling of exterior problems. In the latter case, the basis may contain singular or hyper-singular terms. However, in such situations the source point is not contained in the domain of the element, as it is typically the case of exterior problem solutions.

As the domain of a regular element may not be convex or simply connected, as illustrated in Figure 5a), and the unit outward normal vector, \( n \), may not be uniquely defined at boundary points, equations (61) and (62) take the general forms defined below, and are integrated numerically using Gaussian quadrature rules, with the sufficient number of polynomial points in the implementation of polynomial bases.

\[
F_{kr} = \sum_i \int_T T^T_k U_k \, d\Gamma_i \quad (64)
\]

\[
f_{kr} = \sum_i \int_T T^T_k u_k \, d\Gamma_i \quad (65)
\]
9.2 Singular elements

In order to simplify numerical implementation, it is convenient to assume that the design of the mesh is so constrained as to ensure that each singular element contains a single singularity point, necessarily placed on its boundary. The origin of the local system of reference is assigned to this point, and its orientation is dictated by the system used to define the pertinent stress and displacement fields.

Two situations are illustrated in Figure 5b), namely weakly singular stress fields associated with a free wedge and strongly singular stress fields caused by point loads, as defined, for instance, in Ref. [10], where the local systems of reference are assumed to be placed symmetrically to the wedge solid angle. This condition and the positioning of the singularity point on the boundary of the element, ensures that the displacement field associated with the stress approximation is continuous in the domain of the element.

The boundary definition of the flexibility terms, similar to equation (63), is integrated numerically, with the exception of the terms associated with the sides of the element aligned with the sides adjacent to singularity point, $\theta = \pm \frac{1}{2} \omega$, which are integrated analytically, as shown in Appendix 5.

9.3 Crack elements

Crack elements are designed to model the onset and propagation of damage and fracture. They are implemented on a polynomial Trefftz basis, with origin at the geometric centre of the element, enriched with moving crack solutions. These solutions are defined letting $\omega = 2\pi$ in the free wedge stress and displacement solutions, e.g. Ref. [10], which are now discontinuous along crack direction.

Therefore, besides accounting for the (weak) singularity, the implementation of definition (63) has to be extended to include the effect of the displacement discontinuity within the element and on the external boundary sides intersected by the discontinuity line, as shown in Figure 6.

The implementation of the flexibility matrix is summarized in Appendix 6. As before, those results can be easily adapted to the computation of equilibrium and compatibility terms.

Figure 6. Embedded crack elements.
10 Implementation of damage and cracking

This section addresses the implementation of the operations involved in the modelling of damage and fracture, namely the detection of the direction of onset of damage, the propagation of damage and the onset and propagation of fracture. This modelling is based on the assumption that damage and cracking develop along linear segments of fixed (small) length. According to transformation (37), it is controlled using the $Z_w$—average enforcement of the potentials defined in Appendix 2, namely the damage potential, the damage (or crack) activation potential and the unstressing potential:

$$\varphi_d = \int_0^\ell Z_w (\tau + \frac{\tau_m}{w_0} w - \tau_m) \, dr \quad (66)$$

$$\varphi_c = \int_0^\ell Z_w (\tau - \tau_m) \, dr \quad (67)$$

$$\varphi_u = \int_0^\ell Z_w (\tau - \frac{\tau_m}{w_0} w) \, dr \quad (68)$$

The direction of propagation, $\theta$, is determined for the minimum (positive) increment that exposes the activation of damage:

$$\text{Min } \pi(\theta) > 0 : \varphi_c = \int_0^\ell Z_w \left[ (\tau(\pi, \theta) - \tau_m) \right] \, dr = 0 \quad (69)$$

The potential is written in form (1) as follows, where $\varphi_{c,c} < 0$ is the current potential:

$$\varphi_c = \varphi_c^0 + \sum_{n=1}^\infty \varphi_{c,n} \frac{\pi^n}{n!}$$

$$\varphi_{c,c} = \int_0^\ell Z_w \tau^n \, dr \quad (70)$$

$$\varphi_{c,c} = \int_0^\ell Z_w \tau^n \, dr \quad (71)$$

In the implementation of this definition using result (15) for the effective traction, it is convenient to compute separately the regular term of the force vector and its singular term, associated with the crack function placed at the tip of the origin of the segment,

$$t^{(n)} = t^{(n)}_c + r \frac{\pi^n}{n!}$$

(72)

to evaluate the integral using semi-analytical procedures similar to those described in Appendices 5 and 6.

Propagation of damage is implemented using equations (54) and (68), to implement the null variation of potential (66) for active damage modes, and of potential (68) for damage modes under unstressing:

$$\varphi_{d,c} = \int_0^\ell Z_w \left( \tau^n + \frac{\tau_m}{w_0} w^{(n)} \right) \, dr \quad (73)$$

$$\varphi_{u,c} = \int_0^\ell Z_w \left( \tau^n - \frac{\tau_m}{w_0} w^{(n)} \right) \, dr \quad (74)$$

The implementation of these integral terms are based on the extension of expression (72) to take into account the fact that singular crack functions may be active at both ends of the segment. The implementation of these expressions is discussed in Appendix 7.

The step control criteria defined in Appendix 2 for the detection of the limit states of undamaged and unstressed configurations, of immediate and delayed unstressing of active damage modes and of development of cracking are implemented using the average potentials defined above. These criteria are complemented with the control of the accuracy of the (truncated) Taylor series expansion (1) on the cohesive fracture variables, by letting,

$$|v^{(m)}| \frac{Z_w}{m!} \leq \epsilon |v_{\text{max}}|$$

(75)
where $\varepsilon = 10^{-3}$ is the level of numerical precision, and with the control of deactivation of singular crack functions, written in terms of their weights in the stress approximation (19),

$$X_0 + \sum_{n=1}^{m} X^{(n)} \frac{\pi^n}{n!} = 0$$

(76)

which are subsequently removed from the approximation basis. This operation, combined with the initiation of a singularity point at the instant when a new damage mode is activated, models the propagation of the crack function during the damage and fracture process.

11 Closure

The formulation presented here is designed to address the main difficulties reported in the literature on modelling of cohesive fracture with the conventional formulation of the finite element method. The use of a direct, self-equilibrated, high-order stress approximation enhances the implementation of relatively coarse meshes while preserving the level of accuracy in the stress estimates needed to control effectively the onset and propagation of fracture. In addition, the approximation bases are so defined as to allow this propagation to develop through the elements, thus avoiding the need of remeshing. Moreover, the control of the activation of damage, both in terms of direction and extent, is implemented using average criteria consistent with the local criteria, which tend to be highly dependent on the quality of the local estimates for the variables involved.

Consequent upon these modelling options, non-linearity is circumscribed to the macro-element embedding the propagation of fracture, meaning that only the coefficients of the solving system associated with that element need regular updating during the incremental analysis process. The option to use a Taylor expansion method instead of any of the alternative methods suggested in the literature to solve non-linear algebraic systems is dictated by the possibilities it offers in terms of direct control of the level of accuracy and of direct adaptation of the increment to the degree of non-linearity of the structural response.

Acknowledgement

This research has been partially funded by Fundação para a Ciência e Tecnologia, through contract PTDC/ECM/70781/2006.

References

Appendix 1: Cohesive fracture variables

In the implementation of displacement-based formulations [1], it is convenient to control the evolution of damage using an effective opening displacement, which is defined as follows in Ref. [2, 3],

\[ w = \sqrt{\delta_1^2 + \beta^2 \delta_2^2} \]

where \( \delta_1 \) and \( \delta_2 \) define mode I and mode II displacement discontinuities, respectively, and \( \beta \) is a coupling parameter designed to weight the relative influence of each mode [3]. The definition above together with the first variation in displacements of the work invariance condition (9) leads to definition (10) of the effective traction, with:

\[ R = \begin{bmatrix} 1 & 0 \\ 0 & \beta^2 \end{bmatrix} \]

The Taylor series expansion of definition (10) yields result (15) with,

\[ n_\ast = \frac{1}{R} R_0 \]

which satisfies constraint (13). The following expression is found for the (recursive) residual term:

\[ R_{\ast t} = 0; R_{\ast z} = \frac{1}{t} R_0 \]

The orthogonality condition (14) is met by letting:

\[ m_\ast = \frac{1}{R} R_0 \]

Definitions (A1) and (A2) satisfy condition (12), as they yield \( \delta = 1 \). The following results hold in the series expansion (16) of the displacement discontinuity definition (11):

\[ g_\ast = \frac{1}{R} R_0 \]

Appendix 2: Control of cohesive fracture relations

The relations stated below are written for the rigid-damage law illustrated in Figure 2, assuming that fracture is induced by tension \(( t_2 \geq 0 \) \). They are expressed using auxiliary (potential) variables associated with the damage processing mode, with the fully damaged (or crack) mode, and with the unstressing (or unloading) mode:

\[ \varphi_d = \tau + \frac{\delta}{w_0} w - \tau_M \]

\[ \varphi_i = \tau - \tau_M \]

\[ \varphi_u = \tau - \frac{\delta}{w_0} w \]

The conditions that define the alternative configurations are:

Undamaged:

\[ \begin{cases} -\tau_M \leq \varphi \leq 0 \\ w = 0 \end{cases} \quad \text{and} \quad \begin{cases} -\tau_M \leq \varphi + \Delta \varphi \leq 0 \\ \Delta w = 0 \end{cases} \]

Evolution of damage:

\[ \begin{cases} \varphi_d = 0 \\ w \geq 0 \end{cases} \quad \text{and} \quad \begin{cases} \varphi_d^{\text{un}} = 0 \\ w > 0 \end{cases} \]
Damage unstressing:
\[
\begin{align*}
\varphi_u &= 0 \\
\omega > 0 \quad &\text{and} \quad \varphi_u^{(n)} = 0 \\
\omega &\ll 0
\end{align*}
\] (A5)

Full damage:
\[
\begin{align*}
\varphi &= -\tau_m \\
\omega &\geq \omega_t, \quad &\text{and} \quad \varphi^{(n)} = 0 \\
\omega &> 0
\end{align*}
\]

Appendix 3: Control of step increment

The implementation of the cohesive-fracture relations hinges on the control of capacities and of monotonicity. Assuming that the incremental loading parameter is so chosen as to be non-negative, \( \pi \geq 0 \), the increment that exposes the limit value of a given variable, say \( v \),
\[
v + \Delta v = v_{\lim}
\]
is the smallest non-negative root of its series expansion (1):
\[
\min \pi \geq 0 : \sum_{n=1}^{m} v^{(n)} \frac{\pi^n}{n!} = -v + v_{\lim}
\]

The immediate and delayed changes in monotonicity are exposed by conditions:
\[
\frac{\partial \Delta v}{\partial \pi} = 0 \quad \text{if} \quad v \dot{v} < 0 \\
\frac{\partial^2 \Delta v}{\partial \pi^2} \bigg|_\pi < 0
\]
\[
\frac{\partial^{(k)} \Delta v}{\partial \pi^{(k)}} = \sum_{n-k}^{m} v^{(n)} \frac{\pi^{n-k}}{(n-k)!}
\]

Thus, according to result (A3), the step increments that expose the limit points of the undamaged configuration are:
\[
\min \pi \geq 0 : \sum_{n=1}^{m} \varphi^{(n)} \frac{\pi^n}{n!} = -\varphi \\
\min \pi \geq 0 : \sum_{n=1}^{m} \varphi^{(n)} \frac{\pi^n}{n!} = -\varphi - \tau_m
\] (A6)

Evolution of damage is defined by condition (A4), with:
\[
\varphi^{(n)} = \tau^{(n)} + \frac{\tau_{\omega}}{\omega_0} w^{(n)}
\]

Immediate unstressing of a damage processing mode is exposed by,
\[
\pi = 0 \quad \text{if} \quad \dot{\omega} < 0
\]
and its (eventually) delayed unstressing by:
\[
\min \pi \geq 0 : \sum_{n=2}^{m} w^{(n)} \frac{\pi^{n-1}}{(n-1)!} = -\dot{\omega} \quad \text{with} \quad \dot{\omega} \geq 0 \quad \text{and} \quad \sum_{n=2}^{m} w^{(n)} \frac{\pi^{n-2}}{(n-2)!} < 0
\]

The increment that exposes a fully damaged configuration can be determined by condition (A6). Evolution of unstressing is defined by condition (A5), with,
\[
\varphi^{(n)} = \tau^{(n)} - \frac{\tau_{\omega}}{\omega_0} w^{(n)}
\]
and the step increments that expose the limit configurations are defined by condition (A6) and:
\[ \min \pi \geq 0 : \sum_{n=1}^{m} \frac{\varphi_d^{(n)}}{n!} \pi^n = -\varphi_d \]

Appendix 4: Finite element arrays

The explicit form of the finite element kinematic admissibility condition (46) is obtained letting:

\[ q = \{ q, q_c, q_u, q_0 \} \]

\[ A = \begin{bmatrix} +A_t & -A_c & -A_u & -A_0 \end{bmatrix} \]

\[ A_d = \int T^T Z_d \, d\Gamma_a \quad \text{with} \quad \alpha = t, c; \quad A_u = \int (n_i^T \mathbf{T})^T Z_u \, d\Gamma_p; \quad A_0 = \int (m_i^T \mathbf{T})^T Z_0 \, d\Gamma_p \quad \text{(A7)} \]

\[ F = \int T^T g, T \, d\Gamma_p; \quad f_c = \int T^T g_c, T \, d\Gamma_p; \quad \mathbf{e}_u = \int T^T \mathbf{u}, \mathbf{u} \, d\Gamma_p; \quad R_{cn} = \int T^T R_{cn}, d\Gamma_p \quad \text{(A8)} \]

The explicit form of the finite element static admissibility condition (47) is obtained letting:

\[ Q = \{ \bar{Q}_t, \lambda^{(n)}_t, 0, R_{cn}, -Q_w^{(s)} - 0 \} \]

\[ a = \begin{bmatrix} +a_t & -a_c & -a_u & -a_0 \end{bmatrix} \]

\[ a_d = \int \bar{t}_d^T Z_d \, d\Gamma_a \quad \text{with} \quad \alpha = t, c; \quad a_u = \int (n_i^T \bar{t}_d)^T Z_u \, d\Gamma_p; \quad a_0 = \int (m_i^T \bar{t}_d)^T Z_0 \, d\Gamma_p; \quad R_{cn} = \int Z^T R_{cn}, d\Gamma_p \]

The following are the definitions for the (bulk elasticity) flexibility and (cohesive fracture) stiffness matrices present in the equations (48) and (49):

\[ F = \int S^T f S \, d\Omega; \quad f_c = \int S^T f_c d\Omega; \quad K_s = \int Z^T k, Z_u \, d\Gamma_p \]

Appendix 5: Integration in singular elements

The coefficient of the flexibility matrix (63) associated with the \( m \)-th stress mode and \( n \)-th regular displacement mode is written as follows, where \( \ell \) is the side length, \( 0 \leq \xi \leq 1 \) the side co-ordinate:

\[ F_{mn} = \sum \ell \int_0^1 \left( N S_m \right)^T U_n \, d\xi \quad \text{(A9)} \]

It is convenient to isolate in the expression above the terms associated with the (linear) sides adjacent to the origin of the local system of reference:

\[ F_{mn} = \sum \ell \int_0^1 \left( N S_m \right)^T U_n \, d\xi + < F_{mn} > \quad \text{(A10)} \]

\[ < F_{mn} > = \ell \int_0^1 \left( (N S_m)^T U_n \right) \, d\xi + \ell' \int_0^1 \left( (N S_m)^T U_{n+1} \right) \, d\xi + \ell'' \int_0^1 \left( (N S_m)^T U_{n+2} \right) \, d\xi \]

Letting \( \lambda_m \) and \( \lambda_n \) denote the powers of the radial component of the regular stress and displacement fields, the analytical expression of the corresponding term in definition (A9) is,

\[ < F_{mn}^R > = \frac{\ell}{\lambda_m} \left[ (N S_m^R)^T U_n^R \right]_{\theta=\pm \pi/2}^{\pi/2} \]

\[ < F_{mn}^R > = \frac{\ell}{\lambda_n} \left[ (N S_m^R)^T U_n^R \right]_{\theta=\pm \pi/2}^{\pi/2} \]

where \( \lambda_m = \lambda_n + I > 0 \) and where it is assumed that the stress and displacement fields are computed at the end-point of the segment.

Results (A10) and (A11) hold for both regular and wedge solution modes, as well as their combinations, since \( \lambda_m \geq -0.5 \) and \( \lambda_n \geq +0.5 \) (equalities hold only for crack functions). Moreover, by definition.
\[
\left[ N S_m^w \right]_{\theta = \pm \alpha/2} = 0
\]  
(A12)

In the implementation of definition (50) for point loads, the infinitesimal neighbourhood of the singularity point is isolated and integration by parts is applied to the complementary domain, where the stress and displacement are regular and continuous. As result (A12) holds also for the point load solutions, the resulting expression is similar to (A10), where now,

\[
<F_{mn}> = \lim_{\epsilon \to 0} \left\{ \int_{-\alpha/2}^{\alpha/2} \int_0^\epsilon S_m^T f S_n r \, dr \, d\theta + \left[ \int_{-\alpha/2}^{\alpha/2} (NS_m)^T U_n r \, dr \right]_{r=\epsilon} \right\}
\]

and used to derive analytical solutions for singular and hyper-singular terms, Ref. [10].

Appendix 6: Integration in crack elements

The results summarized in Appendix 5 are directly applicable to the combination of regular stress fields with regular displacement fields, as stated by equation (A9), written for the external sides of the element. It applies, also, to the combination of crack stress fields (\( S_m = S_m^w \) in Figure 6) with regular displacement fields (\( U_m = U_m^R \) in the same figure), provided that the discontinuity in the stress field is accounted for in the external side of the mesh intersected by the crack function direction.

Definition (A10) is applicable to the combination of regular or crack stress fields (\( S_m = S_m^R \) or \( S_m = S_m^w \) in Figure 6) with crack displacement fields (\( U_n = U_n^w \)) under the following provisions: the implementation of the first term accounts for the external sides of the elements; the integration on external sides intersected by the direction of the crack functions accounts for the discontinuity in the displacement field and, eventually, in the stress field; the second term accounts for the displacement field discontinuity, in form:

\[
<F_{mn}> = 2\ell_n \left[ \int_0^{\epsilon} (NS_m)^T U_n^w d\xi \right]_{\theta = \pm \pi}
\]

When the systems of reference share the same origin, but not necessarily the same orientation, the definition above can be integrated analytically, to yield:

\[
<F_{mn}> = \frac{2\ell_n}{\lambda_{mn}} \left[ (NS_m)^T U_n^w \right]_{\theta = \pm \pi}
\]

This analytical result can be easily extended to the case of crack functions with distinct origins but sharing the same direction. It is replaced by the following semi-analytical result when neither the origin nor the direction are shared, where \( S_m^0 \) defines the value of the stress field computed at \( r_n = 0 \):

\[
<F_{mn}> = 2\ell_n \left[ \int_0^{\epsilon} \left[ \nabla (S_m - S_m^0) \right]^T U_n^w d\xi \right]_{\theta = \pm \pi} + \frac{2\ell_n}{\lambda_{mn}} \left[ (NS_m^0)^T U_n^w \right]_{\theta = \pm \pi}
\]

Appendix 7: Integration of cohesive fracture relations

The numerical implementation of the definitions (A7) and (A8) for the compatibility and geometric flexibility matrices, \( A_w \) and \( F_* \), typify the operations involved in the modelling of cohesive fracture.

Definition (10) for the effective traction and expression (72) for the force vector can be used to show that the order of the inverse of the effective traction is \( r^0 \) for regular stress fields and \( r^{0.5} \) for singular crack fields. Consequently, the order of vectors \( *n \) and \( *m \), defined by equations (A1) and (A2), respectively, is \( r^0 \) in both situations.

These properties ensure that \( A_w \)-type integrals are bounded, as the order of the force approximation basis, \( T \), is \( r^{0.5} \), as stated by equation (72). The second condition in equation (15), together with the regularity of vector \( m_* \), is essential to ensure that \( F_* \)-type integrals are also bounded.