Hybrid-Trefftz stress elements for incompressible biphasic media

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SUMMARY

A wavelet-based, high-order time integration method is applied to replace the parabolic problem governing the response of incompressible biphasic media by a set of uncoupled Helmholtz problems. Their formal solutions are used to formulate the stress model of the hybrid-Trefftz finite element formulation. The stress, pressure and displacement fields are directly approximated and designed to satisfy locally the equilibrium condition in each phase of the mixture. This basis is used to enforce on average the compatibility conditions and the constitutive relations of the mixture. The displacements in the solid and the normal displacement in the fluid are approximated independently on the boundary of the element and the basis is used to enforce in weak form the boundary equilibrium conditions. The resulting solving system is sparse, well suited to adaptive refinement and parallel processing. The energy statements associated with the formulation are recovered and sufficient conditions for the uniqueness of the finite element solutions are stated. Testing problems reported in the literature are used to illustrate the quality of the pressure, stress, displacement and velocity estimates obtained with the hybrid-Trefftz stress element. Copyright © 2009 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Modelling of equilibrium and, in particular, of incompressibility have been central issues in the applications of the finite element method (FEM) to the theory of mixtures reported in the literature. This compatibility condition was first enforced using the penalty method [1] and a mixed formulation was developed later to enhance numerical performance [2], particularly in terms of sensitivity to mesh distortion. A different approach reported in the literature is the development

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of formulations based on the direct approximation of the displacement (or velocity) and pressure fields, in which the fluid displacement is eliminated as an explicit variable by satisfying locally the fluid equilibrium condition [3, 4]. The approach followed here is closer to the alternative hybrid finite element formulation originally reported by Vermilyea and Spilker [5], based on stress and pressure approximations that satisfy locally the equilibrium condition of the mixture.

The issues raised in the literature motivated a study [6, 7] on the assessment of the application of the Trefftz concept [8] to linear finite element modelling of incompressible biphasic media [9] to support its extension to non-linear analyses in the time domain accounting for strain-dependent permeability and finite deformation [10, 11]. This study addressed first the application of the hybrid-Trefftz finite element formulation to spectral analysis problems, which are most appropriate to assess the quality of the finite element solutions in terms of accuracy, rate of convergence and sensitivity to mesh distortion and to the wavelength of the excitation. The second part of this preparatory study for non-linear modelling of incompressible biphasic media addressed the implementation of the hybrid-Trefftz formulation in the time domain using a wavelet-based time integration procedure designed to enhance the features of the Trefftz approach [12].

This is the focus of the present paper, as applied to the stress model of the hybrid-Trefftz formulation using two-dimensional applications. It is also used to clarify the relation of the hybrid-Trefftz formulation with the weaker (in terms of the constraints placed on the approximation bases) hybrid finite element formulation [13] and to comment on their relative merits now in the context of soft tissue modelling. A parallel research has been launched, in the same context, on the alternative displacement model of the same formulation [14]. Reporting on this research closes with the direct comparison of the alternative stress and displacement models of the hybrid-Trefftz formulation, in both frequency and time domains and applied to axisymmetric problems [15].

2. LAYOUT OF THE PRESENTATION

The paper is organized in three mains parts, namely basic equations and approximation criteria (Sections 3–5), derivation, implementation and qualification of the finite element model (Sections 6 to 9), and assessment of the finite element solutions (Sections 10–12).

It is assumed here that the parabolic system of equations that governs the response of incompressible biphasic media is first discretized in time to obtain an elliptic problem, which is then discretized in space using the Trefftz concept in the context of the FEM. The most commonly used time integration procedures, namely the Fourier spectral decomposition and the trapezoidal rules based on the Euler method, are recalled in Section 3 to establish their relation with the non-periodic, wavelet-based spectral decomposition method used here. The resulting elliptic problem is stated in Section 4 to establish the notation and the modelling assumptions. The space approximation criteria are defined and justified in Section 5, to clarify the distinguishing aspects regarding the penalty, mixed-penalty, mixed and hybrid finite element formulations that have been developed for soft tissue analysis [16].

The formulation of the hybrid stress element is derived in Section 6. It is based on the direct approximation of the stress and pressure fields in the domain of the element, and a dependent approximation of the relative displacement of the solid and fluid phases is used to ensure that the domain equilibrium condition is satisfied locally, a central feature of stress elements. In the present application, this implies a solution that satisfies the equilibrium condition in each phase of the mixture. The hybrid nature of the element is due to the independent approximation of the
solid and fluid displacements on the boundary of the element. This approximation basis is used to enforce, in weak (Galerkin) form, the force and pressure equilibrium conditions on the inter-element and Neumann boundaries of the element. The Trefftz variant of the hybrid stress element is derived in Section 7 by further constraining the domain approximation basis to satisfy locally all domain conditions of the problem. The relative merits of the hybrid and hybrid-Trefftz formulations are assessed in Section 8, where the main options open to their numerical implementation are briefly discussed. As the finite element equations are derived directly from the basic equations of equilibrium, compatibility and elasticity, the associated energy statements are recovered in Section 9 to establish a basis of comparison with alternative derivation techniques. The energy statements are also used to state sufficient conditions for the uniqueness of the finite element solutions.

The benchmark tests taken from the literature on modelling of soft tissues that are used to assess the performance of the hybrid-Trefftz stress element are defined in Section 10. The frequency domain tests presented in Section 11 are designed to illustrate basic performance aspects, namely sensitivity to full incompressibility and shape distortion, patterns of convergence under p- and h-refinement and accuracy of the estimates obtained for the displacement, stress and pressure fields. The time domain tests presented in Section 12 show that adequate levels of accuracy are attained at every instant of the testing period using relatively coarse, unbiased finite element meshes.

3. DISCRETIZATION IN TIME

Parabolic problems are usually solved by separation of variables. The sequence followed in the implementation of each semi-discretization, in space and in time, is irrelevant when the approximation bases are chosen to be problem-independent. This is typically the case of most of the applications that have been reported, based on the different methods of space discretization used presently. However, when one of the bases is chosen to be problem-dependent, the first discretization should be implemented on the problem-independent basis.

A typical illustration is the modal decomposition approach, which chooses a problem-dependent time approximation basis: discretization in space has to be implemented first to set up the supporting time-dependent system. If modal decomposition in the space dimension of the problem is to be avoided, it becomes necessary to discretize first the time dimension of the hyperbolic problem. The resulting elliptic problem is then discretized in space and its formal solutions are used to set up the approximation basis (thus its problem-dependent nature). This is the option followed here to derive the Trefftz variant of the FEM.

3.1. Non-periodic spectral decomposition

Let \( x \) and \( t \) define the Cartesian system of reference and the time frame, respectively, and assume that every variable and its time derivative, say \( u(x, t) \) and \( v(x, t) = \dot{u}(x, t) \), are approximated independently in the time domain in form

\[
\begin{align*}
\mathbf{u}(x, t) & = \sum_{n=1}^{N} T_n(t) \mathbf{u}_n(x) \quad (1) \\
\mathbf{v}(x, t) & = \sum_{n=1}^{N} T_n(t) \mathbf{v}_n(x) \quad (2)
\end{align*}
\]
where functions \( T_n(t) \) define the time approximation basis, with dimension \( N \) and support \((0; \Delta t)\). Except for completeness, no constraints are set a priori on the time basis.

The non-periodic spectral decomposition method presented in [12] leads to the integration rule

\[
\mathbf{v}_n(x) = i\omega_n \mathbf{u}_n(x) - i\omega_0^0 \mathbf{u}_0(x)
\]

where \( i \) is the imaginary unit, and \( \omega_n \) and \( \omega_0^0 \) are (basis-dependent) equivalent forcing frequencies, the latter being associated with the initial condition of the problem. These algorithmic forcing frequencies are inversely proportional to the time step, \( \Delta t \), and complex, in general, and induce thus intrinsic numerical damping. The stability and convergence properties of this unconditionally stable time integration rule are discussed in detail in [12].

Low-degree polynomial bases are adequate to implement non-linear problems, which require relatively small time increments. However, it is convenient to exploit naturally hierarchical bases in the solution of linear problems using large time increments. As high-degree polynomial bases are hindered by ill-conditioning, the linear applications reported below are implemented on a (non-periodic) wavelet basis with compact support [17, 18] and dimension \( N = 256 \) (family 3, refinement 7). This high order but rather stable basis is implemented on a single time step, as the support of the basis, \( \Delta t \), is identified with the full duration of the test.

3.2. Periodic spectral decomposition

The procedure presented in [12] can be applied to periodic or periodically extended problems using periodic bases, namely polynomial and wavelet systems. However, the usual option is to select the orthonormal Fourier approximation, \( T_n(t) = \exp(\pm i\omega_n t) \), with real forcing frequencies

\[
\omega_n = \frac{2\pi n}{\Delta t}
\]

As periodic problems are independent of the initial conditions, the corresponding term in the integration rule (3) is null, \( \omega_0^0 = 0 \).

3.3. Trapezoidal rules

Although alternative procedures can be used to discretize the time dimension of parabolic problems [19], trapezoidal rules are still frequently adopted due to their simplicity and reliability. They are defined in the general form as

\[
\mathbf{u} = \mathbf{u}_0 + z_0 \Delta t \mathbf{v}_0 + z \Delta t \mathbf{v}
\]

where \( \mathbf{u}_0 \) and \( \mathbf{v}_0 \) represent the value of the variable and its time derivative at instant \( t = 0 \) (start of time step), \( \mathbf{u} \) and \( \mathbf{v} \) the values they take at \( t = \Delta t \) (end of time step), and \( z \) and \( z_0 \) are (real) time integration factors.

The application of this discretization procedure to parabolic systems of equations yields a single elliptic problem with the following equivalent imaginary forcing frequency:

\[
\omega = -i(\varepsilon \Delta t)^{-1}
\]
4. BASIC EQUATIONS

Implementation of the alternative time discretization criteria summarized above to the system of equations that governs the response of incompressible biphasic media generates \([9, 20, 21]\) a system of \(N\) (the dimension of the time approximation basis) fully uncoupled equivalent elliptic problems. The typical, \(n\)th-order problem is written as follows \([7, 12]\):

\[
\begin{bmatrix}
\mathcal{D} & \phi_s \nabla \\
\nabla & \phi_f \nabla
\end{bmatrix}
\begin{bmatrix}
\sigma_{sn} \\
p_n
\end{bmatrix}
+ 
\begin{bmatrix}
b_{sn} \\
b_{fn}
\end{bmatrix}
= i\omega_n \zeta 
\begin{bmatrix}
u_{sn} - u_{fn} \\
u_{fn} - u_{sn}
\end{bmatrix}
\quad \text{in } V \tag{5}
\]

\[
\begin{bmatrix}
\varepsilon_{sn} \\
e_n = 0
\end{bmatrix}
= \begin{bmatrix}
\mathcal{D}^* & \nabla \\
\phi_s \nabla^* & \phi_f \nabla^*
\end{bmatrix}
\begin{bmatrix}
u_{sn} \\
u_{fn}
\end{bmatrix}
\quad \text{in } V \tag{6}
\]

\[
\varepsilon_{sn} = \mathbf{f} \sigma_{sn} \quad \text{in } V \tag{7}
\]

\[
\mathbf{N} \sigma_{sn} + \phi_s \mathbf{n} p_n = \mathbf{t}_n \quad \text{on } \Gamma_t \tag{8}
\]

\[
\phi_f p_n = \tilde{p}_n \quad \text{on } \Gamma_p \tag{9}
\]

\[
\mathbf{u}_{sn} = \mathbf{u}_n \quad \text{on } \Gamma_u \tag{10}
\]

\[
\mathbf{n}^T \mathbf{u}_{fn} = \mathbf{w}_n \quad \text{on } \Gamma_w \tag{11}
\]

In the domain equilibrium and compatibility conditions (5) and (6), vectors \(\mathbf{\sigma}_s\) and \(\varepsilon_s\) collect the independent components of the effective stress and solid strain tensors, respectively, \(\mathbf{u}_s\) and \(\mathbf{u}_f\) are the solid and fluid displacement vectors, \(\mathbf{b}_s\) and \(\mathbf{b}_f\) are the body force vectors, \(p\) and \(e\) are the pressure (measured in tension) in the fluid and the volumetric change of the mixture, \(\zeta\) is the diffusive drag coefficient, and \(\phi_s\) and \(\phi_f\) are the solid and fluid fraction ratios, with \(\phi_s + \phi_f = 1\).

The differential operators, namely the divergence matrix \(\mathcal{D}\) and the gradient vector \(\nabla\) and their conjugates, \(\mathcal{D}^*\) and \(\nabla^*\), are assumed to be linear. The constitutive relations (7) for the solid phase, where the local flexibility matrix, \(\mathbf{f}\), is assumed to be symmetric, can be extended to include direct creep and stress relaxation terms.

Four (complementary) regions are distinguished on the boundary of the element, namely the regions whereon forces or displacements are prescribed on the solid phase, \(\Gamma_t\) and \(\Gamma_u\), and the regions whereon the pressure or the outward normal displacement is prescribed on the fluid phase, \(\Gamma_p\) and \(\Gamma_w\), respectively. In the Neumann conditions (8) and (9), \(\mathbf{t}\) is the prescribed force vector in the solid phase, \(\tilde{p}\) is the prescribed force on the fluid phases and matrix \(\mathbf{N}\) collects the adequate components of the unit outward normal vector, \(\mathbf{n}\). In the Dirichlet conditions (10) and (11), which can be alternatively expressed in terms of velocity, vector \(\mathbf{u}\) defines the displacements prescribed on the solid matrix and \(\mathbf{w}\) is the outward normal component of the displacement prescribed on the fluid. It is assumed that the equations above are written to account for mixed conditions.

System (5)–(11) is independent of the initial conditions of the problem when it is derived for a periodic (or periodically extended) response of the medium, in which case the forcing frequency is real: \(\text{Im}(\omega_n) = 0\). It is assumed that, under non-periodic conditions, the effect of the initial conditions is included in the extended definition of the body-force vectors, \(\mathbf{b}_s\) and \(\mathbf{b}_f\). The forcing frequency is imaginary, \(\text{Re}(\omega_n) = 0\), when system (5)–(11) is derived from trapezoidal rules, and complex when non-periodic spectral decomposition methods are used instead.

As the application of trapezoidal integration rules leads to a single elliptic problem (5)–(11), with \( n = N = 1 \), the approximation basis is significantly weakened \[22\] when the Trefftz method is applied to discretize the space dimension of the problem, simply because the basis is restricted to the single frequency spectrum defined by Equation (4). The wide frequency range offered by periodic and non-periodic spectral decomposition methods is the main justification of the high-performance features reported here for the hybrid-Trefftz stress element. Its ability to capture adequately acceleration peaks, and high gradients in general, using elements with typical dimensions one order larger than the forcing wavelength, is a direct consequence of the richness of the space approximation basis offered by the wide frequency spectrum time basis.

5. FINITE ELEMENT APPROXIMATIONS

This section addresses the criteria used in the discretization of the boundary value problem (5)–(11). As the approximation bases are non-nodal and naturally hierarchical, no constraints need to be set on the topology of the element, which may not be convex, simply connected or bounded.

The Neumann boundary of a stress element is defined here as the part of its boundary whereon the displacements are not known \textit{a priori}. It is necessarily non-empty, as it combines its inter-element boundary, \( \Gamma_i \), with the Neumann boundary of the mesh it may share, \( \Gamma_i \cup \Gamma_p \). The complementary Dirichlet boundary, \( \Gamma_u \cup \Gamma_w \), of a stress element may be empty, as it is defined as the part of the Dirichlet boundary of the mesh that the element may share.

Therefore, the continuity conditions (10) and (11) hold for elements that share the Dirichlet boundary of the mesh, while the right-hand side of conditions (8) and (9) defines either prescribed forces and pressures on the mesh Neumann boundary or the reactions on the boundary of a connecting element. To lighten the notation, subscript \( n \) is omitted from this point onwards.

5.1. Primary approximation in the domain

The stress model develops from the direct approximation of the stress and pressure fields in the domain of the element. This is stated in form

\[
\begin{bmatrix}
\sigma_s \\
p
\end{bmatrix} =
\begin{bmatrix}
S & 0 \\
P & \tilde{P}
\end{bmatrix}
\begin{bmatrix}
x_s \\
x_p
\end{bmatrix} +
\begin{bmatrix}
\sigma_s^0 \\
p^0
\end{bmatrix} \quad \text{in } V
\]

(12)

where matrices \( S \) and \( P \) collect approximation functions associated with non-null stress fields, weighed by the generalized stress vector, \( x_s \), \( \tilde{P} \) is the constant pressure mode, with (scalar) weight \( x_p \), and \( \sigma_s^0 \) and \( p^0 \) represent particular solution terms. The boundary forces equilibrated by approximation (12) are not constrained to satisfy locally the Neumann conditions (8) and (9).

The dual transformation of approximation (12) defines generalized strains that ensure the invariance of the inner-product in the finite element mapping

\[
\begin{bmatrix}
a_s \\
a_e = 0
\end{bmatrix} =
\int \begin{bmatrix}
S^* & P^*
\end{bmatrix}
\begin{bmatrix}
\varepsilon_s \\
\varepsilon_e
\end{bmatrix} dV
\]

(13)

\[
x_s^a a_e + x_p^a a_e = \int (\sigma_s - \sigma_s^0) \varepsilon_s dV + \int (p - p^0) \varepsilon_e dV
\]

(14)
They are used in Sections 6.1 and 6.3 to define the weak form of the compatibility (6) and elasticity (7) conditions, respectively. In the notation used here \( x^* \) is the conjugate transpose of array \( x \): \( x^* = \hat{x}^T \).

### 5.2. Dependent approximation in the domain

The hybrid variant of the stress element is constrained to satisfy locally the domain equilibrium condition of the problem [13]. In the present context, this condition is met by coupling the stress approximation (12) with a dependent displacement approximation extracted from the domain equilibrium condition (5). This approximation is written as follows:

\[
\begin{bmatrix}
\{ u_s \\
\{ u_r \\
\end{bmatrix} = \begin{bmatrix} U & R \\ W & R \end{bmatrix} \begin{bmatrix} \{ x_0 \} \\
\{ x_r \} \\
\end{bmatrix} + \begin{bmatrix} \{ u_s^0 \} \\
\{ u_r^0 \} \\
\end{bmatrix} \quad \text{in } V
\]

(15)

where the rigid-body mode, \( R \), is inserted to show that its weight, \( x_r \), remains indeterminate in the implementation of the hybrid stress element (see Section 8).

Substitution of approximations (12) and (15) in Equation (5) yields the following equilibrium constraint on the complementary solution of the dependent displacement approximation:

\[
\begin{bmatrix} \mathcal{D} & \phi_s \mathbf{V} \\ \phi_t \mathbf{V} & \mathbf{P} \end{bmatrix} \begin{bmatrix} \mathbf{S} & 0 \\ 0 & \bar{\mathbf{P}} \end{bmatrix} = \mathbf{i} \omega \begin{bmatrix} \mathbf{U} - \mathbf{W} & 0 \\ 0 & \mathbf{W} - \mathbf{U} \end{bmatrix}
\]

(16)

The corresponding constraint on the particular solution term is recalled in Appendix B.

In order to emphasize its equilibrating role, it is stressed that the dependent displacement approximation (15) is not constrained \textit{a priori} to satisfy the incompressibility condition (6) and that it may not satisfy locally the Dirichlet conditions (10) and (11) or the inter-element continuity conditions. Moreover, it may be associated with a compatible strain field (6) inconsistent with the elastic field (7) caused by the stress approximation (12).

### 5.3. Approximations on the boundary

The domain approximation (12) is complemented with independent approximations of the displacements on the solid phase displacements and of the normal fluid phase displacement on the (extended) Neumann boundary

\[
\begin{align*}
\mathbf{u}_s &= \mathbf{Z}_u \mathbf{y}_u \quad \text{on } \Gamma_f \\
\mathbf{w} &= \mathbf{Z}_w \mathbf{y}_w \quad \text{on } \Gamma_p
\end{align*}
\]

(17)

(18)

where matrix \( \mathbf{Z}_u \) and (row-) vector \( \mathbf{Z}_w \) collect as columns the boundary approximation functions, typically complete Legendre or Chebyshev polynomials on finite boundaries. The weighting vectors \( \mathbf{y}_u \) and \( \mathbf{y}_w \) define generalized (non-nodal) displacements.

Approximations (17) and (18) are often implemented ensuring displacement continuity at the intersection points of element boundaries. This form of continuity is easily met using nodal bases, as it is the common practice in the implementation of hybrid finite elements [23, 24], at the cost of weakening the enforcement of the inter-element flux continuity conditions [25]. In the approach followed here, approximations (17) and (18) are independently enforced on each Neumann boundary of the mesh. They are uniquely defined on boundaries shared by (at most) two
connecting elements, and implement thus the inter-element continuity condition of hybrid stress elements. The advantage of implementing the boundary approximations independently on each side, and per component, is to induce a stronger enforcement of the element (and inter-element) Neumann continuity conditions (8) and (9).

As shown in Section 6.2, that is, in essence, the roles played by the generalized boundary forces on the solid phase and by the pressure forces on the fluid phase defined by the dual transformations of the boundary displacement approximations (17) and (18) are

\[ \bar{b}_t = \int Z_{ut}^* \dd{\Gamma}_t \]  
\[ \bar{b}_p = \int Z_{wp}^* \dd{\Gamma}_p \]  

which ensure, also, the invariance of the inner-product in the boundary finite element mapping

\[ y_u^* \bar{b}_t + y_w^* \bar{b}_p = \int u_t^* \dd{\Gamma}_t + \int w^* \dd{\Gamma}_p \]  

6. HYBRID STRESS ELEMENT

The governing equations of the hybrid stress element are derived below directly from system (5)–(11) using the dual variables defined in Section 5. The direct relation of the duality and Galerkin weighted residual approaches is implicit in the derivation.

6.1. Compatibility condition

The compatibility condition combines the kinematic admissibility conditions (6), (10) and (11),

\[ \left\{ \begin{array}{c} a_e \\ \dot{a}_e = 0 \end{array} \right\} = \begin{bmatrix} B_u & B_w \\ b_u & b_w \end{bmatrix} \begin{bmatrix} y_u \\ y_w \end{bmatrix} - (i\omega)^* \begin{bmatrix} C \\ 0 \end{bmatrix} x_\sigma + \begin{bmatrix} \dot{a}_e \\ \dot{\bar{a}}_e \end{bmatrix} - (i\omega)^* \begin{bmatrix} c^0 \\ 0 \end{bmatrix} \]  

and is obtained as follows: the local condition (6) is inserted in definition (13) for the generalized strains and the resulting expression is integrated by parts to retrieve a boundary term; the dependent displacement approximation (15), under constraint (16), is inserted in the domain term of the resulting expression; the boundary term is uncoupled into the Neumann and Dirichlet parts to enforce approximations (17) and (18) and conditions (10) and (11).

The expressions thus found for the (Hermitian) damping and boundary compatibility matrices, and for the particular solution and prescribed displacements terms are

\[ C = \int (U - W)^* \zeta (U - W) \dd{V} \]  

\[ \begin{bmatrix} B_u & B_w \\ b_u & b_w \end{bmatrix} = \begin{bmatrix} \int (NS + \phi_n \mathbf{nP})^* Z_{ut} \dd{\Gamma}_t & \int (\phi_i \mathbf{nP})^* Z_{ut} \dd{\Gamma}_p \\ \int (\phi_i \mathbf{nP})^* Z_{wp} \dd{\Gamma}_t & \int (\phi_i \mathbf{nP})^* Z_{wp} \dd{\Gamma}_p \end{bmatrix} \]
\[ c^0 = \int (U - W)^* \zeta (u_s^0 - u_f^0) \, dV \] (25)

\[
\left\{ \begin{array}{l}
\bar{a}_e \\
\bar{a}_e^0
\end{array} \right\} = \int \left[ \begin{array}{l}
(\mathbf{NS} + \phi_s \mathbf{nP})^* \\
(\phi_s \mathbf{nP})^*
\end{array} \right] \bar{u} \, d\Gamma_u + \int \left[ \begin{array}{l}
\mathbf{P}^*
\end{array} \right] \phi_f \bar{w} \, d\Gamma_w
\]

6.2. Equilibrium condition

As approximations (12) and (15) satisfy locally the domain equilibrium condition (5), the static admissibility condition reduces to the weak enforcement of the Neumann (and inter-element flux continuity) conditions (8) and (9). They are implemented using the generalized boundary force definitions (19) and (20) for the assumed stress and pressure fields (12), and define the dual transformation of the kinematic admissibility condition (22), as result (24) holds:

\[
\left[ \begin{array}{l}
\mathbf{B}_u^* \\
\mathbf{b}_u^0 \\
\mathbf{B}_w^* \\
\mathbf{b}_w^0
\end{array} \right] \left\{ \begin{array}{l}
\mathbf{x}_\sigma \\
x_p
\end{array} \right\} = \left\{ \begin{array}{l}
\bar{b}_\sigma - \bar{b}_\sigma^0 \\
\bar{b}_p - \bar{b}_p^0
\end{array} \right\}
\]

\[
\bar{b}_\sigma^0 = \int Z_u^*(\mathbf{No}_s^0 + \phi_s \mathbf{nP}_s^0) \, d\Gamma_t
\]

\[
\bar{b}_p^0 = \int Z_w^* \phi_f \mathbf{P}_p^0 \, d\Gamma_p
\] (26)

6.3. Indeterminacy numbers

The kinematic admissibility condition (22) shows that the element is kinematically indeterminate, as the number of unknowns exceeds the number of equations. On the other hand, the static admissibility condition (26) shows that the (linearly independent) domain and boundary bases must be so balanced as to ensure a non-negative indeterminacy number,

\[
\alpha = N_x - N_y \geq 0
\] (27)

where \( N_x \) is the dimension of the stress and pressure approximation basis (12) and \( N_y \) is the dimension of the boundary displacement bases (17) and (18). This limit situation corresponds to an element with an empty Dirichlet boundary. Equation (27) shows that the smaller the indeterminacy number, the stronger the enforcement of the element force continuity conditions.

6.4. Elasticity condition

The finite element elasticity condition is obtained inserting the constitutive relations (7) and the stress approximation (12) in definition (13) for the generalized strain vector, and enforcing the mixture incompressibility condition (6), to yield

\[
a_e = \mathbf{F} \mathbf{x}_\sigma + \mathbf{a}_e^0
\] (28)
where the (Hermitian) flexibility matrix and the vector associated with the particular solution are defined by

\[
F = \int S^* f S \, dV \tag{29}
\]

\[
a^0_\varepsilon = \int S^* f \sigma^0_\varepsilon \, dV \tag{30}
\]

6.5. Solving system

The solving system for the hybrid stress finite element model is obtained equating the compatibility and elasticity equations (22) and (28), to eliminate the generalized strains as independent variables, and adding the boundary equilibrium equation (26)

\[
\begin{bmatrix}
D & O & -B_u & -B_w \\
O & O & -b_u & -b_w \\
-B_u^* & -b_u^* & O & O \\
-B_w^* & -b_w^* & O & O \\
\end{bmatrix}
\begin{bmatrix}
x_\sigma \\
x_\rho \\
y_u \\
y_w \\
\end{bmatrix}
=
\begin{bmatrix}
\tilde{\varepsilon}_e - \tilde{\varepsilon}^0_e \\
\tilde{\varepsilon}_e \\
\tilde{\varepsilon}^0_t - \tilde{\varepsilon}_t \\
\tilde{\varepsilon}^0_p - \tilde{\varepsilon}_p \\
\end{bmatrix}
\tag{31}
\]

It can be verified that the only terms that do not present boundary integral expressions are those dependent on material properties, namely damping and constitutive coefficients

\[
D = F + (i\omega)^* C \tag{32}
\]

\[
\tilde{\varepsilon}^0_e = a^0_\varepsilon + (i\omega)^* c^0 \tag{33}
\]

7. HYBRID-TREFFTZ STRESS ELEMENT

The two main approaches that have been used in the application of the Trefftz method in computational mechanics are strongly influenced by the competing FEM and boundary element method. In the latter case, the (singular) fundamental solutions are selected to form the approximation basis and the boundary conditions are typically enforced by collocation. In the alternative finite element approach, pioneered by Jirousek \[23\], a regular basis is used and the boundary conditions are enforced in a weak form.

This approach, rooted in the work of Pian \[24\] on hybrid elements, places unnecessary constraints (conformity) on the boundary approximation basis and calls upon the condensation of the elementary solving system on its boundary degrees-of-freedom (DOF) to emulate the conventional finite element formulation. These constraints are abandoned in the approach followed here \[26\], as the Trefftz finite element formulation is simply understood as a particular case of the hybrid formulation presented in Section 6 that is obtained by forcing the stress approximation (12) and the dependent displacement approximation (15) to satisfy locally all domain conditions of the problem, the so-called Trefftz constraint.
7.1. Trefftz constraint

The Trefftz constraint consists of identifying the domain approximation basis with the formal solution of the governing system of differential equations, that is, the system obtained combining the domain conditions on equilibrium (5), compatibility (6) and elasticity (7),

\[
k_p^{-2} \nabla (\nabla^* u_s) + k_s^{-2} \tilde{\nabla} (\tilde{\nabla}^* u_s) + \phi_I(u_s - u_f) = -i k_p^{-2}(b_s + b_f)
\]

\[
\nabla p = i k_p^2 \phi_I(u_f - u_s) - \phi_I^{-1} b_f
\]

\[
\nabla^* (\phi_s u_s + \phi_I u_f) = 0
\]

where \( k^2 = \zeta \omega \phi_I^{-2} \), \( k_p \) and \( k_s \) are the wavenumbers for \( P \)- and \( S \)-waves, respectively, and \( \tilde{\nabla} \) is the anti-gradient vector and \( \tilde{\nabla}^* \) its conjugate (see Appendices A and C).

Therefore, in the present application the Trefftz constraint implies that, besides the equilibrium constraint (16), the stress and displacement approximations (12) and (15) are associated with a deformation field that satisfies locally the compatibility and elasticity relations

\[
\begin{align*}
\varepsilon_s & = 0 \\
\sigma & = \begin{bmatrix} E & 0 \\ G & 0 \end{bmatrix} x + \begin{bmatrix} \varepsilon_s^0 & 0 \\ 0 & \varepsilon_f^0 \end{bmatrix}
\end{align*}
\]

\[
\begin{bmatrix} E & 0 \\ G & 0 \end{bmatrix} = \begin{bmatrix} \phi^* & \phi^* \\ \phi^* & \phi^* \end{bmatrix} \begin{bmatrix} U & R \\ W & R \end{bmatrix}
\]

\[
E = fS
\]

The corresponding constraints on the particular solution are summarized in Appendix B.

The displacement potentials used to obtain the formal solutions of the homogeneous form of system (34) are defined in Appendix C, and the Trefftz basis used in the implementation of the numerical tests presented below is presented in Appendix D [6]. The basis contains a frequency-independent family of frozen mixture modes, \( u_s = u_f \), which combines a constant pressure mode, a rigid-body mode and a family of static, incompressible modes, and two families of unfrozen mixture modes, \( u_s \neq u_f \), namely a \( P \)-wave mode associated with an Helmholtz pressure field and an \( S \)-wave mode associated with an harmonic pressure field.

7.2. Finite element equations

The compatibility, equilibrium and elasticity equations (22), (26) and (28), and, consequently, the solving system (31) obtained for the hybrid stress element still hold for the Trefftz variant. The fundamental difference is that the domain approximation is subject to the additional compatibility (36) and elasticity (37) constraints.

When these constraints are enforced in the flexibility matrix definition (29), and the resulting expression is integrated by parts to enforce the equilibrium constraint (16) on equilibrium while recalling definition (23) for the damping matrix, the following boundary integral expression for
the dynamic matrix (32) is found:

\[ D = \int (NS)^*U d\Gamma + \int (nP)^*(\phi_n U + \phi_l W) d\Gamma \] (38)

The boundary integral expression for the particular solution term (33) is obtained applying the same procedure to definition (30), and recalling now result (25)

\[ \bar{a}_x^0 = \int (NS)^*u_x^0 d\Gamma + \int (nP)^*(\phi_n u_x^0 + \phi_l u_l^0) d\Gamma \] (39)

8. NUMERICAL IMPLEMENTATION

The operations involved in the computation of the coefficients of the solving system (31) and in the assemblage for the finite element mesh are identical to those involved in elastostatic analysis [25], when the dynamic matrix (32) coalesces to the flexibility matrix \[(afii9853 = 0)\]. Therefore, the comments below focus on the relative merits of the alternative hybrid and hybrid-Trefftz variants and on the options adopted in the manipulation of the finite element solving system.

8.1. Relative merits

It is straightforward to set up the stress and pressure approximation basis (12) for the hybrid stress element. Master elements and nodal polynomial bases can be used, or complete and naturally hierarchical polynomial bases can be defined and implemented on elements with arbitrary geometry [27]. Although the dependent displacement approximation is defined in form (15), the equations above show that only the relative displacement is involved in the hybrid element formulation. Its definition is determined by condition (16), which suggests the derivation of the pressure and relative displacement fields from the direct approximation of the stress field.

Besides completeness, the major problem in the derivation of this basis is the control of spurious modes induced by linear dependency. They can be removed \(a priori\), through appropriate testing, or detected and solved in the process of solution of the finite element solving system [26, 27]. The Trefftz constraint, when adequately enforced in the determination of the domain approximation basis, ensures completeness and filters out linear dependence. However, neither this feature nor the boundary integral description found for all structural coefficients justifies the option followed here in adopting the Trefftz variant of the hybrid stress element.

The major strength of this element is the use of an approximation basis that embodies the physics of the problem, as this has a direct impact in the quality of the finite element solutions and on their rate of convergence. The price paid is relatively high, because manipulation of the approximation functions can be costly, as is the case of the Helmholtz solution in the present application, and, mainly, because the derivation of the Trefftz basis is problem-dependent.

It is easy to foresee situations where the derivation of Trefftz bases is too involved or simply impossible, as is the case of most non-linear applications. However, it is still advantageous in such instances to construct approximation bases on approximate (linearized) Trefftz solutions, at the marginal cost of losing the boundary integral expressions (38) and (39).
8.2. Emulation of displacement elements

System (31) can be condensed, at element level, on the boundary displacements, extended to include the constant pressure mode (playing the role of a null energy mode), to obtain the governing system that emulates the conventional (conform) displacement element

\[ Kq = Q \]  

This system is implemented in a commercial finite element code using a (polynomial) nodal basis to define the boundary displacement approximations (17) and (18), leading to elements with three DOF per node, the displacements in the solid phase and the normal displacement on the fluid phase, with vector \( Q \) representing the corresponding nodal forces. System (40) would then be assembled ensuring the equilibrium of the nodal forces due to the nodal displacements \( (K_{ij}) \), and to the prescribed nodal forces \( (Q_i) \), which, in the present context, include the effect of the initial condition and particular solution terms.

8.3. Enhancement of features

The emulation procedure described above typifies the development of hybrid and hybrid-Trefftz stress elements since their origin \([23, 24]\). This option offers the possibility of using the available finite element codes to implement high-performance elements at the cost, in the present context, of lessening the strongest features of system (31) in terms of numerical implementation.

Assembly of the explicit form of system (31) consists simply in assigning the same approximations (17) and (18) to elements that share the same boundary. This leads to the direct allocation of the (block-diagonal) dynamic and equilibrium/compatibility matrices, without the nodal summation operations that typify the assemblage of conventional elements.

The assembled system is highly sparse, with the same structure of the elementary system (31). The implementation of adaptive refinement procedures is rather simple \([28]\) because the bases are naturally hierarchical. As the domain variables, \( x_r \) and \( x_p \), are strictly element-dependent and the boundary variables, \( y_u \) and \( y_w \), are shared by, at most, two connecting elements, the system is well-suited to parallelization \([29]\). These features are not exploited here due to the simplicity of the benchmark tests used in soft tissue modelling.

8.4. Post-processing

Under the conditions stated in Section 9, system (31) yields unique estimates for the stress, pressure and strain fields in each element, as determined from Equations (12) and (35). In the Trefftz variant, the incompressibility of the mixture is ensured exactly and respected on (weighted) average in the hybrid formulation. In either case, system (31) yields unique estimates for the solid displacement components and for the outward normal component of the fluid displacement on the internal and external boundaries of the finite element mesh. They are determined from approximations (17) and (18) on the Neumann and internal boundaries of the mesh and defined by their prescribed values on the Dirichlet boundary.

System (31) shows that the rigid-body mode in approximation (15) remains undetermined. This indeterminacy can be solved (in a non-unique way) by average fitting of the domain and boundary displacement approximations. The remaining displacement components are uniquely determined in the solid and fluid phases when the Trefftz element is used. As the hybrid element can be
implemented approximating the relative displacement, only this information would be available in the representation of the motion of points in the domain of the mixture. In time domain analyses, the solution at every instant of the test is obtained implementing approximations (1) and (2) using the fast Fourier transform extended to systems of wavelets.

9. ENERGY STATEMENTS AND UNIQUENESS CONDITIONS

As the hybrid and hybrid-Trefftz stress elements are derived from first principles, it is convenient to recover the energy statements they are associated with. These statements are used next to establish sufficient conditions for uniqueness. Two distinct cases are considered, depending on whether the time integration procedure yields Hermitian or non-Hermitian solving systems (31), namely for imaginary and real or complex forcing frequencies, respectively, according to definition (32) for the dynamic matrix.

9.1. Definitions

Finite element formulations are usually derived from alternative energy statements, namely on the mechanical energy, and, in the solution of Hermitian problems, on stationary conditions on the potential energy and on the complementary potential energy

\[ \mathcal{M} = 2\mathcal{E} + 2\mathcal{C} - \mathcal{W} - \mathcal{W}^* = 0 \]  

\[ \mathcal{P} = \mathcal{E} + \mathcal{C} - \mathcal{W} \]  

\[ \mathcal{P}_s = \mathcal{E} + \mathcal{C} - \mathcal{W}^* \]  

In the context of spectral decomposition, and under the incompressibility condition, \( e = 0 \), in the equations above \( \mathcal{E} \) and \( \mathcal{C} \) are the strain and damping components of the energy, and \( \mathcal{W}^* \) and \( \mathcal{W} \) define the external work associated with prescribed displacements and forces, respectively

\[ \mathcal{E} = \frac{1}{2} \int (e^s_s + e^s_p) dV \]  

\[ \mathcal{C} = \frac{1}{2} i\omega \int (u^s_s - u^s_0)^* (u^s_0 - u^s_0) dV \]  

\[ \mathcal{W}^* = \int u^s_t d\Gamma_u + \int w^s p d\Gamma_w \]  

\[ \mathcal{W} = i\omega \int (u^s_s - u^s_0)^* (u^s_0 - u^s_0) dV + \int (u^s_s b_s + u^s_0 b_0) dV + \int u^s_t d\Gamma_t + \int u^s n^p d\Gamma_p \]

9.2. Duality and virtual work

Consequent upon duality, it can be readily confirmed that the inner product of the compatibility and equilibrium conditions (22) and (26) recovers definition (1) for the mechanical energy

\[ \mathcal{M} = (a_e - i\omega c^0)^* x_\sigma + i\omega x^*_\sigma C x_\sigma - \tilde{a}^*_k x_k - \tilde{y}^*_u (\tilde{b}_t - \tilde{b}_0) - y^*_u (\tilde{b}_p - \tilde{b}_0^0) = 0 \]
As the equilibrium and compatibility conditions are independent of the constitutive relations of the mixture, the equality above states the virtual work equation. This result is recovered recalling relations (14) and (21), using the definitions for the equilibrium and compatibility terms, and implementing the domain and boundary approximations (12), (15), (17) and (18), under the equilibrium constraint (16) and definitions (44) to (47).

9.3. Non-Hermitian problems

When periodic (Fourier) and non-periodic time integration procedures are used to establish system (5)–(11), the resulting forcing frequency is real and complex, respectively. Under these conditions, the dynamic matrix (32), defined by Equation (38) for the Trefftz variant, is non-Hermitian, and the same property extends to the solving system (31), which is written in form

\[
\begin{bmatrix}
A & -B \\
-B^* & O
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
=
\begin{bmatrix}
a \\
b
\end{bmatrix}
\tag{48}
\]

for simplicity, with \(x = (x_\sigma, x_p)\) and \(y = (y_u, y_w)\). It can be shown that the mathematical program associated with this system is the following [30]:

\[
\text{Min } z = \text{Re}(x^*Ax - x^*a - y^*b) \text{ subject to system (48)} \tag{49}
\]

Using a procedure similar to that briefly described above to recover the virtual work equation, but enforcing now also the constitutive relations (32) and (33) of the mixture, it is found that the objective function represents the real part of the mechanical energy (1) as

\[
z = \text{Re}[x_\sigma^*Dx_\sigma - x_\sigma^*(\bar{a}_e - \bar{a}_e^0) - x_p^*\bar{a}_e - y_u^*(\bar{b}_f - \bar{b}_f^0) - y_w^*(\bar{b}_p - \bar{b}_p^0)] = \text{Re}(\Ref) = 0
\]

Recalling that the two groups of equations in system (31) enforce (weak) kinematic and static admissibility conditions, the optimal solutions of program (49) are (weak) statically and kinematically admissible and the Hermitian part of the mechanical energy is null at optimality.

Mathematical program (49) is useful to establish the following multiplicity condition:

**Multiple optimal solutions (S1):** If \((x, y)\) is an optimal solution to program (49), then the feasible solution \((x, y) + z(\Delta x, \Delta y)\) is also an optimal solution if

\[
\text{Re}(\Delta x^*A\Delta x) = \text{Re}(\Delta x_\sigma^*[D + D^*] \Delta x_\sigma) = 0 \tag{50}
\]

\[
\begin{bmatrix}
A & -B \\
-B^* & O
\end{bmatrix}
\begin{bmatrix}
\Delta x \\
\Delta y
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\tag{51}
\]

The following statement results from definition (32) and condition (50):

**Stress field estimate (S2):** The stress estimate is uniquely determined if the Hermitian part of the dynamic matrix is positive definite,

\[
\frac{1}{2} \Delta x_\sigma^*[D + D^*] \Delta x_\sigma = \Delta x_\sigma^*[F - \Im(\omega)C] \Delta x_\sigma > 0
\]

and multiple if \(\Im(\omega)\) is an eigenvalue of the Hermitian part of the dynamic matrix.

For Trefftz bases, the flexibility matrix is positive definite, \(\Delta x_\sigma^*[F] \Delta x_\sigma > 0\), and the damping matrix is positive semi-definite, \(\Delta x_\sigma^*[C] \Delta x_\sigma \geq 0\), due to the frozen mixture modes, for which \(W = U\) (see Appendix D). Moreover, the frequency in periodic problems is real, \(\Im(\omega) = 0\).
The following conditions result from the homogeneous form (51) for a unique stress solution:

**Constant pressure field estimate (S3):** If the stress field estimate is unique, \(\Delta x_\sigma = 0\), and the boundary pressure equilibrium matrix \([b_u \ b_w]^*\) is full rank, the constant pressure field estimate is uniquely determined, \(\Delta x_p = 0\).

**Boundary displacement estimate (S4):** If the boundary compatibility matrix (24) is full rank, and the approximation in the domain is unique, \(\Delta x_\sigma = 0\) and \(\Delta x_p = 0\), the boundary displacement estimate is uniquely determined, \(\Delta y_u = 0\) and \(\Delta y_w = 0\).

### 9.4. Hermitian problems

As the flexibility and damping matrices are Hermitian, definition (32) shows that system (31) is Hermitian only for imaginary forcing frequencies, \(\text{Re}(\omega) = 0\). This is the result obtained when trapezoidal time integration rules are used to derive system (5)–(11). Program (49) and statement (S1) still hold, the latter under the additional condition [31]:

\[
a^*\Delta x - b^*\Delta y = 0
\]

Statements (S2) and (S3) remain valid and the boundary compatibility matrix in statement (S4) is extended to include the condition defined above with \(\Delta x_\sigma = 0\), as \(\Delta x_\sigma = 0\) and \(\Delta x_p = 0\).

It can be shown [30] that system (31), under notation (8), is now associated with the following pair of quadratic programs:

\[
\begin{align*}
\text{Min } z &= \frac{1}{2}x^*Ax - \text{Re}(y^*b) \quad \text{subject to: } Ax - By = a \\
\text{Min } z &= \frac{1}{2}x^*Ax - \text{Re}(x^*a) \quad \text{subject to: } B^*x = b
\end{align*}
\]

The (weak) static and kinematic admissibility conditions of system (31) are thus uncoupled, meaning that the programs above recover the theorems on minimum potential energy and complementary energy, respectively, as the following identifications hold, under definitions (42) and (43), and noting that the constant terms do not affect the minimization:

\[
\begin{align*}
z &= \frac{1}{2}x^*Dx_\sigma - \text{Re}[y_u^*(\tilde{b}_t - \tilde{b}_t^0) + y_w^*(\tilde{b}_p - \tilde{b}_p^0)] = \text{Re}(\mathcal{P}) + \text{constant} \\
z &= \frac{1}{2}x^*Dx_\sigma - \text{Re}[x_u^*(\tilde{a}_t - \tilde{a}_t^0) + x_p^*\tilde{a}_p] = \text{Re}(\mathcal{P}_s) - \text{constant}
\end{align*}
\]

System (48) is still associated with the following pair of quadratic programs:

\[
\begin{align*}
\text{Min } z &= -\frac{1}{2}x^*Ax + \text{Re}(y^*B^*x + x^*a - y^*b) \\
\text{Min } z &= -\frac{1}{2}x^*Ax + \text{Re}(y^*B^*x) \quad \text{subject to system (48)}
\end{align*}
\]

the first of which is unconstrained, meaning that a stationary value for its functional [30] recovers the solution set of system (31). The definition of the objective function of the unconstrained program recovers the Hellinger–Reissner functional for domain approximations that satisfy locally the equilibrium condition, as stated by Equation (16) and its equivalent to the particular solution terms:

\[
z = -\text{Re}\left[\mathcal{P}_s + \int u_s^*n(\tilde{t}) \, d\Gamma_t + \int u_t^*n(\tilde{p} - \phi_t p) \, d\Gamma_p\right] + \text{constant}
\]
10. TESTING PROBLEMS

The unconfined indentation test and the compression of cartilage specimens under confined and unconfined conditions shown in Figure 1 are frequently used in the literature on numerical modelling of incompressible biphasic media. They are solved here under both frequency and time domain conditions.

The spectral tests are implemented for forcing frequencies that lead to wavelengths, $\lambda$, that range from a tenth to five times the characteristic length of the elements, $L_{FE}$, a value well beyond the maximum length recommended in the implementation of conventional finite elements. The length ratios summarized in Table I are defined as follows, where Re$(k_p)$ is the real part of the $P$-wave number associated with the forcing frequency:

$$r = \frac{L_{FE}}{\lambda} = \frac{\text{Re}(k_p L_{FE})}{2\pi}$$ (52)

The time domain analysis tests are implemented for the loading programmes described by the ramp function shown in Figure 1. Both displacement-driven and force-driven loading programmes are implemented, with prescribed displacements and forces $\bar{u}$ and $\bar{\sigma}$, respectively.

![Figure 1. Testing problems (plane strain) and loading programmes.](image)

Table I. Length ratios, $r$, for testing meshes and frequencies.

<table>
<thead>
<tr>
<th></th>
<th>Confined compression</th>
<th>Mesh</th>
<th>Unconfined compression</th>
</tr>
</thead>
<tbody>
<tr>
<td>Confined compression</td>
<td>0.975 3.083 9.749</td>
<td>1 × 1</td>
<td>0.879 2.779 8.787</td>
</tr>
<tr>
<td></td>
<td>0.487 1.541 4.874</td>
<td>2 × 2</td>
<td>0.439 1.389 4.394</td>
</tr>
<tr>
<td></td>
<td>0.244 0.771 2.437</td>
<td>4 × 4</td>
<td>0.220 0.695 2.197</td>
</tr>
<tr>
<td></td>
<td>0.01 0.10 1.00  $\omega$(rad/s)</td>
<td>0.01 0.10 1.00</td>
<td></td>
</tr>
</tbody>
</table>
The conditioning, the sparsity and the dimension of the solving system (31) obtained for the testing problems render irrelevant the choice of solver, direct (as used) or iterative. The stress, pressure, displacement and velocity estimates are determined as stated in Section 8.4. No smoothing is used in the representation of the (double precision) finite element solutions.

11. FREQUENCY DOMAIN TESTS

The confined and unconfined compression tests are solved for three forcing frequencies, namely 0.01, 0.10 and 1.00 rad/s. The data used are for human cartilage specimens [9]: width \( w = 3.0 \) mm, height \( h = 2.0 \) mm, modulus of elasticity \( E = 0.442855 \) MPa, Poisson’s ratio \( \nu = 0.126 \), permeability \( \kappa = 1.16 \times 10^{-15} \) m\(^4\)N\(^{-1}\)s\(^{-1}\) and fluid fraction \( \phi_f = 0.8 \).

The aspects that are assessed are the sensitivity to full incompressibility and shape distortion, the convergence patterns of the finite element solutions under \( p \)- and \( h \)-refinement and the quality of the stress, pressure and displacement solutions. The boundary conditions and the discretization used are presented first to clarify the testing conditions (see Table II).

11.1. Boundary conditions

The containing chamber is assumed to be rigid, impermeable and lubricated in all confined compression tests, to yield the following boundary condition:

\[
\begin{align*}
    u_n^s &= u_n^f = 0 \text{ and } \sigma_{xy}^s = 0 \quad \text{on } x = 0, y = 0 \text{ and } x = w
\end{align*}
\] (53)

where subscript \( n \) identifies the displacement component normal to the boundary. Two boundary conditions are considered, namely a prescribed displacement and a prescribed pressure applied to the permeable, lubricated loading platen

\[
\begin{align*}
    u_y^s &= -\bar{u} \quad \text{and} \quad p = \sigma_{xy}^s = 0 \quad \text{on } y = h
\end{align*}
\] (54)

\[
\begin{align*}
    \sigma_{yy}^s &= -\bar{\sigma} \quad \text{and} \quad p = \sigma_{xy}^s = 0 \quad \text{on } y = h
\end{align*}
\] (55)

Under the boundary conditions stated above, the confined compression test yields one-dimensional solutions of the form

\[
\begin{align*}
    u_x^s &= u_x^f = 0 \quad \text{and} \quad \sigma_{xy}^s = 0 \quad \text{in } V
\end{align*}
\] (56)

The two-dimensional problems defined by the unconfined compression tests,

\[
\begin{align*}
    \sigma_{xx}^s = \sigma_{xy}^s = p = 0 \quad \text{on } x = \pm w/2
\end{align*}
\] (57)

<table>
<thead>
<tr>
<th>Test type</th>
<th>Loading platen</th>
<th>Loading programme</th>
</tr>
</thead>
<tbody>
<tr>
<td>Confined compression,</td>
<td>Permeable, lubricated</td>
<td>Displacement-driven,</td>
</tr>
<tr>
<td>Equation (52)</td>
<td>Permeable, lubricated</td>
<td>Equation (53)</td>
</tr>
<tr>
<td>Unconfined compression,</td>
<td>Impermeable, adhesive</td>
<td>Force-driven,</td>
</tr>
<tr>
<td>Equation (56)</td>
<td>Permeable, lubricated</td>
<td>Equation (57)</td>
</tr>
</tbody>
</table>

Table II. Boundary conditions used in frequency domain tests.
are solved for a prescribed displacement applied to impermeable, adhesive end-platens,

\[ u_s^x = 0 \quad \text{and} \quad u_s^y = u_f^y = \pm \tilde{u} \quad \text{on} \quad y = \mp h/2 \quad (58) \]

and for a prescribed pressure applied to permeable, lubricated end-platens,

\[ \sigma_{yy}^s = -\tilde{\sigma} \quad \text{and} \quad p = \sigma_{xy}^s = 0 \quad \text{on} \quad y = \pm h/2 \quad (59) \]

11.2. Discretization

The specimen under confined compression is discretized using single-element, 2×2- and 4×4-element meshes. The same regular, unbiased meshes are applied in the solution of the unconfined compression tests, using the double symmetry of the problem shown in Figure 1. All tests are implemented on uniform approximations, meaning that the same order of approximation is used in each domain element, \( d_V \), and boundary element, \( d_T \). The dimension of a regular Trefftz basis is \( 6d_V + 4 \) per element and the dimension of a polynomial boundary basis is \( d_T + 1 \) per side and per displacement component. In the notation used below, \( N \) defines the total number of DOF, that is, the dimension of the solving system in the explicit form (31).

The typical lengths used in definition (52), \( L_{FE} \), are the height and the diagonal length of the element in the confined and unconfined compression tests, respectively, as they represent one- and two-dimensional problems. In order to stress the effect of the forcing frequency in the response of the incompressible mixture, the confined compression test is solved for boundary condition (54). The variation along the height of the specimen obtained for the vertical component of the displacement in the solid, normalized to the prescribed displacement, \( \delta_y = -u_s^y / \tilde{u} \), is shown in Figure 2 and illustrates the boundary layer effect induced by increasing forcing frequencies. These results are obtained with the regular 4×4-element mesh and approximation \( (d_V; d_T) = (11; 5) \), which yields a solving system with dimension \( N = 1648 \).

![Figure 2. Profile of the vertical displacement in the solid phase.](image)
11.3. Incompressibility tests

The first set of tests checks the formulation for ill-posed problems and to the modelling of a mixture in which each phase approaches full incompressibility. As the mixture incompressibility condition is locally satisfied by the Trefftz basis, Equation (35), it suffices to increase Poisson’s ratio to model the quasi-incompressibility of the solid and fluid phases.

If the confined compression test is implemented for the force-driven test (55), the shear stress is null in the specimen and the load is transferred to the solid skeleton \( (\sigma_{xx}^f, \sigma_{xy}^f \rightarrow -\bar{\sigma}) \) under a vanishing pressure \( (p \rightarrow 0) \), as Poisson’s ratio approaches the incompressibility limit \( (v \rightarrow 0.5) \). Consequent upon the confinement conditions (53), the displacements tend to zero in both phases of the mixture, to yield a null energy at the incompressibility limit. This is the behaviour illustrated in Figure 3, where \( E_{FE} \) is the norm of the internal energy (the norm of the sum of the strain and damping energies), \( E_{ref} \) is the energy norm obtained with a Poisson’s ratio \( v = 0.49 \) and \( N9 \) represents the number of approximating digits, e.g. \( N9 = 3 \) for \( v = 0.4999 \). A distinct response is found in the unconfined compression test, under the same loading condition (59). The load is transferred to the solid skeleton \( (\sigma_{yy}^f \rightarrow -\bar{\sigma} \text{ and } \sigma_{xx}^f, \sigma_{xy}^f, p \rightarrow 0 \text{ as } v \rightarrow 0.5) \) and the energy is practically insensitive to the variation of Poisson’s ratio.

When the confined compression test is solved for a prescribed displacement applied to an impermeable, lubricated platen, system (31) exposes as an ill-posed problem, as the mixture incompressibility condition yields the result \( b_u = 0 \) and \( b_w = 0 \) with \( \bar{a}_e \neq 0 \). It is noted that the solutions of periodic spectral tests are independent of the initial conditions, meaning that the particular solution terms in the stipulation vector are null.

11.4. Mesh distortion tests

The two mesh distortion schemes used to assess the sensitivity of the hybrid-Trefftz finite element solutions for displacement-driven tests, under confined (54) and unconfined (58) compression conditions are represented in Figures 4 and 5. The sensitivity is measured in terms of the variation of the energy norm normalized to the value obtained for the undistorted mesh, \( \gamma = 0.5 \), and the approximation basis \( (d_V; d_T) = (11; 5) \) remains unchanged throughout the tests.
The distortion scheme used in the confined compression test (permeable, lubricated loading platen) is more demanding. Element 1 collapses for $\gamma = 0$ and two sides of the mesh vanish for $\gamma = 1$. The latter effect is captured by the distortion scheme used in the unconfined compression test (impermeable, adhesive platens). The results presented in Figures 4 and 5 show that the solution is sensitive to gross mesh distortion only for the highest forcing frequency, when the typical length of the (undistorted) element is four times larger than the wavelength.

11.5. Convergence tests

The displacement-driven loading programme is used in the assessment of the convergence of the finite element solutions for both compression tests, under boundary conditions (54) and (58) and implemented on the single-element and the (regular, unbiased) $2 \times 2$- and $4 \times 4$-element meshes for an increasing refinement of the approximation bases, under the combination $d_V = 2d_T + 1$.

The results shown in Figure 6, where $N$ is the dimension of the solving system (31), recover the convergence patterns found for the hybrid-Trefftz stress element for elastostatics [25], in particular in what concerns the strong convergence under $p$-refinement. The rate of convergence is now
Figure 6. Convergence patterns for $p$- and $h$-refinement (forcing frequency $\omega=0.10\text{rad/s}$).

Figure 7. Stress, pressure and displacement fields for the confined compression test.

affected by the ratio between the typical length of the element and the wavelength, which in the tests shown varies from 0.7 (4 × 4-element mesh) to 3.1 (single-element mesh). The reference values for the energy norm correspond to stabilized solutions obtained with boundary approximations of order $(d_V; d_T) = (19; 9)$ for both confined and unconfined compression tests.

### 11.6. Stress, pressure and displacement estimates

The confined compression test is solved for boundary condition (54) and the lowest forcing frequency, $\omega=0.01\text{rad/s}$. The solutions shown in Figure 7 recover condition (56) and confirm the one-dimensional nature of the problem. These results show, also, that the inter-element force and displacement continuity conditions are adequately modelled, as well as the Neumann and Dirichlet conditions. They are obtained with basis $(d_V; d_T) = (11; 5)$, which leads to 94 400 and 1648 DOF for the $1 \times 1$-, $2 \times 2$- and $4 \times 4$-element meshes, respectively.

The unconfined compression test (impermeable, adhesive loading platens) is solved for the same frequency and the same prescribed deformation, as stated by condition (58). The two-dimensional nature of the problem is illustrated in Figure 8. These solutions are obtained with the $2 \times 2$-element mesh and approximation $(d^V; d^F) = (11; 5)$, leading to a total of 412 DOF. Although the boundary and inter-element continuity conditions are modelled with sufficient accuracy in the coarse mesh solution, the estimates obtained show that convergence can be substantially improved either by modelling locally the high stress gradients developing at the end-points of the loading platens or by using an adequate, biased discretization.

A non-uniform colour scale, defined by the bounds found for each field, is used to enhance the representation of the solutions shown in Figures 7 and 8. The displacement components are normalized to the prescribed displacement, $\bar{u}$, and the stress and pressure components are normalized to the reference stress $\bar{\sigma} = E \bar{u}/w$, where $E$ is the modulus of elasticity and $w$ is the width of the specimen. The pressure is measured as positive in compression.

12. TIME DOMAIN TESTS

The data for the time domain tests is [5]: width $w=6.35\,\text{mm}$, height $h=1.78\,\text{mm}$, modulus of elasticity $E=0.675\,\text{MPa}$, Poisson’s ratio $\nu=0.125$, permeability $\kappa=7.6 \times 10^{-15}\,\text{m}^4\text{N}^{-1}\text{s}^{-1}$ and fluid fraction $\phi_f=0.83$. The displacement-driven load programme shown in Figure 1 is designed to reach a deformation of 5% at $t_0=500\,\text{s}$, which corresponds to a prescribed displacement $\bar{u} = 0.0890\,\text{mm}$ in the indentation and confined compression tests, and $\bar{u} = 0.0445\,\text{mm}$ in the unconfined compression test. All tests are solved in a single time step: $\Delta t = t_{\text{max}} = 10^3\,\text{s}$.

The confined and unconfined compression tests are solved with a single-element and a $2 \times 2$-element mesh. The approximation used is $(d^V; d^F) = (11; 5)$ in both instances. The $3 \times 2$-element mesh shown in Figure 1, with approximation $(d^V; d^F) = (15; 7)$, is used in the indentation test. The alternative boundary conditions that are tested are summarized in Table III. The solutions shown below (no smoothing and non-uniform, scaled colour scales, as in the previous section) are frames extracted from the animations that can be accessed using the address www.civil.ist.utl.pt/HybridTrefftz and link ‘Animation of the response of hydrated soft tissues’.

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### Table III. Boundary conditions used in time domain tests.

<table>
<thead>
<tr>
<th>Test type</th>
<th>Loading platen</th>
<th>Loading programme</th>
</tr>
</thead>
<tbody>
<tr>
<td>Confined compression, Equation (52)</td>
<td>Permeable, lubricated</td>
<td>Displacement-driven, Equation (53)</td>
</tr>
<tr>
<td>Unconfined compression, Equation (56)</td>
<td>Impermeable, adhesive</td>
<td>Displacement-driven, Equation (57)</td>
</tr>
<tr>
<td>Unconfined compression, Equation (56)</td>
<td>Impermeable, lubricated</td>
<td>Displacement-driven, Equation (59)</td>
</tr>
<tr>
<td>Indentation, Equation (60)</td>
<td>Permeable, lubricated</td>
<td>Displacement-driven, Equation (61)</td>
</tr>
</tbody>
</table>

![Figure 9. Loading and unloading phases.](image)

![Figure 10. Solution at instant \( t_A = 230 \text{s} \).](image)

12.1. **Confined compression test**

The modelling of the response of the cartilage specimen under the confined compression loading defined by conditions (53) and (54), see Table III, is illustrated at the three instants of the loading process identified in Figure 9.

The solutions presented in Figures 10–12 illustrate the one-dimensional nature of the problem, as stated by Equation (56), and confirm the adequate enforcement of the boundary and inter-element continuity conditions, both in terms of displacement and flux. The build-up in-stress and pressure is reached at the end of the loading phase, \( t_B = 500 \text{s} \), when the still entrapped fluid shares a significant portion of the load. The relaxation phase is well captured in the illustrations presented in Figures 11 and 12. As the fluid leaves the cartilage, the load is transferred to the solid matrix, leading to uniform stress fields and a vanishing pressure.

The variation in time of the results obtained with a single-element mesh, and with a regular mesh of \( 2 \times 2 \) elements, is shown in Figure 13. They recover the (graph estimated) values obtained...
12.2. Unconfined compression test

The unconfined compression test illustrates two forms of response, namely a one-dimensional response, now in the $x$-direction, and a fully two-dimensional response. The displacement-driven loading programme is used in both tests, with $\bar{u} = 0.0445$ mm, under condition (57). The two-dimensional response is caused by boundary condition (58), modelling rigid, impermeable, adhesive loading platens. The one-dimensional response is obtained with lubricated platens. The one-dimensional response is obtained with lubricated platens:

$$u'_y = u'_f = \pm \bar{u} \quad \text{and} \quad \sigma^o_{x'y} = 0 \quad \text{on} \quad y = \mp h/2$$ (60)
The approximation \((dV/dt) = (11.5)\) leads to systems with 100 and 412 DOF for the single- and 2 × 2-element meshes in the test with adhesive end-platens. The corresponding values for the lubricated test are marginally higher, 106 and 424, respectively.

The results obtained at control point \(x = w/2\) and \(y = 0\), see Figure 1, with the two meshes, in a single time step, \(\Delta t = 10^3\) s, is shown in Figures 14 and 15. Similarly to the confined compression test solutions, presented in Figure 13, the single-element solutions obtained for the alternative unconfined compression tests capture the peaks in acceleration occurring at the end-points of the loading phase. The two-dimensional results (adhesive platens) obtained with the single-element mesh recover the values reported by Vermilyea and Spilker [5] using a trapezoidal rule with time step \(\Delta t = 5\) s and a regular mesh of 12 × 6 pairs of elements.

The solutions obtained at two instants, in the loading phase and close to the end of the test period, are presented in Figure 16 for the two-dimensional test. The relatively weak quality of these solutions results from the discretization in a single-element mesh. The corresponding

---

**Figure 14.** Evolution in time of the unconfined (lubricated) compression test solution.

**Figure 15.** Evolution in time of the unconfined (adhesive) compression test solution.
Figure 16. Unconfined (adhesive) test solutions at instants $t_A = 230\text{s}$ and $t_E = 850\text{s}$. 
one-dimensional solutions obtained with the $2 \times 2$-element mesh recover closely the results obtained with the alternative displacement model of the hybrid-Trefftz finite element formulation [7].

12.3. Indentation test

The loading platen is placed symmetrically in the indentation test represented in Figure 1, and its length is a third of the width of the specimen. The reaction platen is modelled as rigid, impermeable and adhesive, and the loading platen as permeable and lubricated

\begin{equation}
\begin{align*}
    u_x^\infty &= u_y^\infty = u_y^L = 0 \quad \text{on } y = 0 \\
    u_y^\infty &= -\bar{u} \quad \text{and} \quad \sigma_{xy}^\infty = p = 0 \quad \text{on } y = h
\end{align*}
\end{equation}

The solutions found for the stress and pressure fields at two instants of the loading programme are presented in Figure 17. The same regular basis is used, meaning that it is not enriched locally to model the stress concentration and the tips of the loading platen. The evolution in time of the values obtained for the vertical component of the velocity in the solid and fluid phases, for the stress components and for the pressure obtained at control point $x = w/2$ and $y = 0.9h$ is shown in Figure 18. All results are measured in the system of reference defined in Figure 1 and are obtained in a single time step, $\Delta t = 10^3$ s, using the $3 \times 2$-element mesh shown in the same figure.

![Figure 17. Indentation (permeable) test solutions at instants $t_A = 230$ s and $t_E = 850$ s.](image-url)

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13. CLOSURE

The stress element for incompressible biphasic media presented here is based on high-order approximations in time and in space, and is so formulated as to enhance adaptive refinement and parallelization. The hybrid formulation relates directly with an earlier proposition by Vermilyea and Spilker [5]. The main difference is that the domain equilibrium condition is satisfied in each phase of the medium, instead of ensuring only the equilibrium of the mixture. However, the derivation of the hybrid formulation is justified here mainly by the objective of stressing its direct relation with the Trefftz variant and to support the formal assessment of its relative merits.

The alternative stress models for incompressible biphasic media are derived from first principles, namely the equilibrium, compatibility and elasticity relations of the mixture, in order to state explicitly the role of each finite element equation. This option facilitates, also, the subsequent extension of the formulation to physically or geometrically non-linear modelling. The associated energy statements are recovered and sufficient conditions for the existence of unique solutions are stated for Hermitian and non-Hermitian applications.

The frequency domain tests show that the element is basically insensitive to gross distortion in shape. This sensitivity is noticeable, but remains marginal when the typical dimension of the element is one-order higher than the length of the excitation wave. The tests also show that the element remains stable when each phase of the medium approaches the incompressibility limit. Moreover, they confirm that the element developed to model the dynamic response of incompressible biphasic media preserves the high rates of convergence that had been identified earlier in its application to the analysis of homogeneous, isotropic elastostatic problems.

As the approximation basis satisfies the domain conditions of the problem, the accuracy of the solution depends strongly on the enforcement of the inter-element and boundary conditions. The quality of this enforcement is illustrated with results obtained with relatively coarse and unbiased meshes. It is shown that these levels of performance can be extended into the time domain analyses combining the Trefftz approximation in space with a wavelet approximation in time.

The animations in time of the numerical applications reported here show that the criteria used in the approximations in space and time model adequately solutions that oscillate throughout the entire domain of the problem. This is usually attained using small time-stepping and highly refined
meshes, constrained by the leading wavelength of the excitation. It is thought that the option followed here compares well with the techniques that have been proposed more recently to handle this issue, namely wave basis and discontinuous enrichment methods, e.g. [32–34].

APPENDIX A: BASIC EQUATIONS FOR TWO-DIMENSIONAL PROBLEMS

The explicit, Cartesian description of Equations (5)–(11) is the following for plane stress problems, where the stiffness format of the constitutive relations is used for convenience:

\[
\begin{bmatrix}
\partial_x & 0 & \phi_s \partial_x \\
0 & \partial_y & \phi_s \partial_y \\
0 & 0 & 0 & \phi_f \partial_x \\
0 & 0 & 0 & \phi_f \partial_y
\end{bmatrix}
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{bmatrix}
+ \begin{bmatrix}
b_x^s \\
b_y^s \\
b_x^f \\
b_y^f
\end{bmatrix}
= \iota \omega \zeta
\begin{bmatrix}
u^x - u^f_x \\
u^y - u^f_y \\
u^x - u^f_x \\
u^y - u^f_y
\end{bmatrix}
\]

\[
\begin{bmatrix}
e_{xx} \\
e_{yy} \\
\gamma_{xy}
\end{bmatrix}
= \begin{bmatrix}
\partial_x & 0 & 0 & 0 \\
0 & \partial_y & 0 & 0 \\
\phi_s \partial_x & \phi_s \partial_y & \phi_f \partial_x & \phi_f \partial_y
\end{bmatrix}
\begin{bmatrix}
u^x \\
u^y \\
\nu^x \\
\nu^y
\end{bmatrix}
\]

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{bmatrix}
= \frac{\mu}{1-\nu}
\begin{bmatrix}
2 & 2\nu & 0 \\
2\nu & 2 & 0 \\
0 & 0 & 1-\nu
\end{bmatrix}
\begin{bmatrix}
e_{xx} \\
e_{yy} \\
\gamma_{xy}
\end{bmatrix}
\]

The definitions above, with the equivalent Poisson ratio \(\nu<\nu/(1-\nu)\) and the effective shear modulus \(\mu\), hold for plane strain applications. The following identifications and relations hold for the gradient and rotation operators and their conjugates:

\[
\nabla = \begin{bmatrix}
\partial_x \\
\partial_y
\end{bmatrix}, \qquad \nabla^* = \begin{bmatrix}
+\partial_y \\
-\partial_x
\end{bmatrix}
\]

\[
\nabla^* \nabla (\phi) = \nabla^* \nabla (\phi) = \nabla^2 (\phi)
\]

\[
\nabla^* \nabla (\phi) = \nabla^* \nabla (\phi) = 0
\]

APPENDIX B: CONSTRAINTS ON PARTICULAR SOLUTION TERMS

The equilibrium constraint on the particular solution terms present in the stress and displacement approximations (12) and (15) is

\[
\begin{bmatrix}
\nabla - \phi_s \nabla \\
\phi_f \nabla
\end{bmatrix}
\begin{bmatrix}
\sigma_s^0 \\
p^0
\end{bmatrix}
+ \begin{bmatrix}
b_s \\
b_f
\end{bmatrix}
= \iota \omega \zeta
\begin{bmatrix}
u^0_s - u^0_f \\
u^0_f - u^0_f
\end{bmatrix}
\]
The elasticity and compatibility conditions on the strain particular solution term are the following, when the Trefftz constraint is enforced:

\[
\varepsilon_s^0 = 0
\]

\[
\begin{bmatrix}
\varepsilon_s^0 \\
e^0 = 0
\end{bmatrix} = \begin{bmatrix}
\mathcal{D}^* & \mathcal{O} \\
\phi_s \nabla^* & \phi_t \nabla^*
\end{bmatrix} \begin{bmatrix}
u_s^0 \\
u_t^0
\end{bmatrix}
\]

APPENDIX C: TREFFTZ POTENTIALS FOR TWO-DIMENSIONAL PROBLEMS

The expressions for the \(P\) - and \(S\)-wavenumbers present in the system of differential equations (34) are \(k_p^2 = \frac{1}{2}(1 - \nu)k_s^2\) and \(k_s^2 = -ik^2/\mu\) with \(k^2 = \omega \kappa \phi_t^{-2} = \omega / \kappa\), where \(\kappa\) is the permeability.

The regular solutions defined below are referred to a local Cartesian system, which is centred on the source of the singularity in the implementation of singular solution modes. It is convenient to express these functions in polar coordinates \(r^2 = x^2 + y^2\) and \(\tan \theta = y/x\).

The homogeneous system (34) has three sets of solutions, namely constant pressure, harmonic pressure and Helmholtz pressure solutions:

\[
\nabla p = 0 \quad \text{with} \quad u_s = \nabla \varphi, \quad u_t = u_s \quad \text{and} \quad \nabla^2 \varphi = 0 \quad (C1)
\]

\[
\nabla^2 p = 0 \quad \text{with} \quad u_s = \nabla \varphi, \quad u_t = u_s - i k^{-2} \phi_t^{-1} \nabla p \quad \text{and} \quad \mu \nabla (\nabla^2 \varphi) = -\nabla p \quad (C2)
\]

\[
\nabla^2 p + k_p^2 p = 0 \quad \text{with} \quad u_s = i k^{-2} \nabla p \quad \text{and} \quad u_t = -\phi_s \phi_t^{-1} u_s \quad (C3)
\]

The general solution for the harmonic potential present in definition (C1),

\[
\varphi = r^m \exp(in\theta) \quad \text{with} \quad n = \pm m
\]

generates polynomial modes in \((x, y)\) for \(m > 1\) and singular solutions for \(m < 1\). Alternative singular and discontinuous solutions are found for potentials \(\ln(r)\), \(\theta\) and \(\theta \ln(r)\).

The displacement potential associated with the harmonic pressure field solution (C2),

\[
p = -4m(m + 1)\mu r^m \exp(in\theta) \quad \text{with} \quad n = \pm m
\]

is defined by

\[
\varphi = inr^{m+2} \exp(in\theta) \quad \text{with} \quad n = \pm m
\]

This solution generates polynomial and singular pressure fields for \(m > 0\) and \(m < 0\), respectively. The three rigid-body displacement modes, associated with null pressure fields, and the constant pressure mode, associated with null displacement fields, are particular cases of the constant and harmonic pressure solutions (C1) and (C2) with \(m = 1\) and \(m = 0\), respectively.

The Helmholtz pressure field solution (C3) is defined by

\[
p = -W_m(k_p r) \exp(in\theta) \quad \text{with} \quad n = \pm m
\]

where the \(m\)th-order Bessel function \(W_m \equiv J_m\) \((W_m \equiv Y_m)\) is of the first (second) kind for regular (singular) solutions. Incoming and outgoing waves are modelled by Bessel functions of the third kind, the Hankel functions \(W_m^{(1)} \equiv J_m + iY_m\) and \(W_m^{(2)} \equiv J_m - iY_m\), respectively.
APPENDIX D: TREFFTZ SOLUTIONS FOR TWO-DIMENSIONAL PROBLEMS

The regular Trefftz solutions for two-dimensional problems are defined in polar coordinates, e.g. \( \mathbf{S} = \{ S_{rr}, S_{\theta\theta}, S_{r\theta} \} \) and \( \mathbf{U} = \{ U_r, U_\theta \} \) for stress and displacement modes in the solid phase. The geometry of the element is characterized by its largest dimension, \( 2\ell \), and by its area, \( \Omega \). To improve conditioning, the point co-ordinates are defined by the non-dimensional radius, \( \rho = r/\ell \), and the angle, \( \theta \), measured in the element local system of reference, with origin at its baricentre and determined by its principal directions. In addition, the scaling parameters \( s_1 \) and \( s_2 \) are chosen to yield diagonal coefficients of the elementary dynamic matrices with norms close to unity. In the expressions summarized below, \( m \) is a non-negative integer, \( n = \pm m \) and \( \delta = n/m = \pm 1 \), with \( \delta = 0 \) for \( m = n = 0 \).

Besides the (nontrivial) constant pressure field and the rigid-body displacement mode,

\[
P = \bar{P} = 1, \quad \mathbf{S} = 0, \quad \mathbf{W} = \mathbf{U} = 0
\]

\[
P = 0, \quad \mathbf{S} = 0, \quad \mathbf{W} = \mathbf{U} = \mathbf{R} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \end{bmatrix}
\]

the regular Trefftz solutions are the following:

**Constant pressure solution:** \( m > 1, s_1^2 = 2(m - 1)\mu/\ell, s_2^2 = 2\Omega/\ell \)

\[
P = 0, \quad \mathbf{W} = \mathbf{U}
\]

\[
\mathbf{S} = \frac{s_1}{s_2} \begin{bmatrix} +1 \\ -1 \\ i\delta \end{bmatrix} \rho^{m-2} \exp(i\theta), \quad \mathbf{U} = \frac{1}{s_1 s_2} \begin{bmatrix} +1 \\ i\delta \\ \end{bmatrix} \rho^{m-1} \exp(i\theta)
\]

**Harmonic pressure solution:** \( m > 0, s_1^2 = 2(m + 1)\mu/\ell, s_2^2 = 2\pi\Omega/\ell \)

\[
P = -\frac{2s_1}{ms_2} \rho^m \exp(i\theta), \quad \mathbf{W} = \mathbf{U} + \frac{4(m + 1)}{\phi_s(k_s\ell)^2 s_1 s_2} \begin{bmatrix} +1 \\ i\delta \\ \end{bmatrix} \rho^{m-1} \exp(i\theta)
\]

\[
\mathbf{S} = \frac{s_1}{s_2} \begin{bmatrix} +1 \\ -1 \\ i\delta \end{bmatrix} \rho^m \exp(i\theta), \quad \mathbf{U} = \frac{1}{ns_1 s_2} \begin{bmatrix} +n \\ i(m + 2) \end{bmatrix} \rho^{m+1} \exp(i\theta)
\]

**Helmholtz pressure solution:** \( m \geq 0, s_1^2 = k_p \mu/(1 - \nu), s_2^2 = 4\pi\Omega/\ell \), with \( \mathbf{W} = -\phi_s \phi_f^{-1} \mathbf{U} \)

\[
P = -4\frac{s_1}{s_2} W_m \exp(i\theta), \quad \mathbf{W} = -\phi_s \phi_f^{-1} \mathbf{U}
\]

\[
\mathbf{S} = \frac{s_1}{s_2} \begin{bmatrix} 2(1 + \nu)W_m - (1 - \nu)(W_{m+2} + W_{m-2}) \\ 2(1 + \nu)W_m + (1 - \nu)(W_{m+2} + W_{m-2}) \end{bmatrix} \exp(i\theta)
\]

\[
\mathbf{U} = \frac{1}{s_1 s_2} \begin{bmatrix} W_{m+1} - W_{m-1} \\ -i\delta(W_{m+1} + W_{m-1}) \end{bmatrix} \exp(i\theta)
\]
Parameter \( \tau \) is defined as follows the harmonic and Helmholtz pressure solutions:

\[
\begin{align*}
\tau & = \left\| 1 - \frac{4(m+1)}{m\Phi_1(k_s\ell)^2} \right\| \\
\tau & = \left\| (\hat{W}_{m+1} - \hat{W}_{m-1}) W_m + \frac{1-v}{z} [(m+1)\hat{W}_{m-1} W_{m+1} - (m-1)\hat{W}_{m+1} W_{m-1}] \right\|_{z=k_p\ell}
\end{align*}
\]

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