Abstract

The hybrid-Trefftz displacement element is formulated and applied to the linear analysis of incompressible hydrated tissues. The system of parabolic differential equations is discretized in the time dimension and written in a format suitable for periodic and non-periodic analyses. The solution of the resulting elliptic system of differential equations is based on the direct approximation of the displacements in the solid and fluid phases of the mixture in the domain of the element and of the surface force and pressure fields on its boundary. Performance is illustrated using the tests reported in the literature on hydrated soft tissue modelling.

Keywords: Trefftz elements, incompressibility, poroelasticity, hydrated soft tissues.

1 Introduction

Modelling of the response of hydrated soft tissues is usually based on the biphasic theory proposed and validated experimentally by V.C. Mow [1]. Typically, the numerical models develop from the discretization in the time domain of the governing system of parabolic equations using trapezoidal rules, and the resulting elliptic description of the response is solved using different solution methods for boundary value problems, in particular the penalty, mixed-penalty, mixed and hybrid formulations that have been developed for the analysis of soft tissues [2].

Modelling of incompressibility remains a sensitive issue and justified the decision to assess the analysis of hydrated soft tissues using the Trefftz concept [3,4] formulated for the alternative stress and displacement models, the latter being addressed here. The formulation develops from the discretization in time of the governing system parabolic differential equations using a high-order integration method designed to enhance the application of the Trefftz concept [5]. This method can be applied to both periodic (frequency domain) and non-periodic (time domain) analyses and leads, in either case, to a system of uncoupled elliptic equations.
This system is discretized next in the space domain using the displacement model of the hybrid finite element formulation [6]. The displacements are approximated directly in the domain of the element using a basis constrained to satisfy locally the incompressibility condition. This basis and the associated strain field are used to enforce on (Galerkin) average the equilibrium condition and the constitutive relations, respectively. The forces in the solid and fluid phases are approximated independently on the Dirichlet boundary of the element, and the basis is used to enforce on average the continuity condition on the components of the solid phase displacement and the normal component of the displacement in the fluid phase.

The resulting algebraic system is symmetric, sparse and well-suited to parallel processing and adaptive refinement. In consequence of the Trefftz constraint, its coefficients have boundary integral expressions. However, the main advantage of the Trefftz variant of the hybrid finite element formulation is the stability and the improved rates of convergence resulting from the use of an approximation basis selected from the free-field solutions of the governing elliptic problem.

The testing problems on cartilage specimens are selected to assess the formulation in the solution of one- and two-dimensional responses under both creep and stress relaxation processes. The results are presented in a format that facilitates their direct comparison with those obtained with the alternative stress element [7].

The frequency domain tests range a wide variation in the relation between the typical dimension of the element and the wavelength of the excitation. They are used to analyse the sensitivity of the element to gross shape distortion and to quasi-incompressibility conditions set on each phase of the mixture. The patterns and rates of convergence obtained under both \( p \)- and \( h \)-refinement procedures are also illustrated, as well as the quality of the estimates obtained for the pressure, stress and displacement fields.

The time domain tests show that the combination of the Trefftz approximation in space, implemented on coarse meshes, with a wavelet approximation in time, applied in a single time increment, can capture the main features of the response, at both local and global levels. This is illustrated by coupling the response measured at particular control points of the testing specimens with time frames of the solutions obtained during the loading process.

The information on the statements associated with the formulation, on the conditions for the existence and uniqueness of the solutions and on the numerical implementation can be found in the report that supports this presentation [3].

2 Basic equations

It is assumed that the system of differential equations governing the linear response of an incompressible hydrated soft tissue, defined on a domain \( V \) with boundary \( \Gamma \) and referred to a Cartesian system \( x \), is parabolic. It is further assumed that a time integration procedure is implemented to replace this problem by a sequence of \( N \) uncoupled elliptic problems, each of which is associated with a given forcing frequency, \( \omega_n \), with \( 1 \leq n \leq N \). The \( n \)-th problem in that sequence can be written as follows, where \( i \) is the imaginary unit:
\[
\begin{bmatrix}
\mathbf{D} & \phi_s \nabla^2 \\
\mathbf{O} & \phi_f \nabla
\end{bmatrix}
\begin{bmatrix}
\sigma_{sn} \\
p_s
\end{bmatrix}
+
\begin{bmatrix}
b_{sn} \\
b_{fs}
\end{bmatrix}
= i\omega \zeta
\begin{bmatrix}
u_{sn} - u_{fn} \\
u_{fn} - u_{sn}
\end{bmatrix}
- \begin{bmatrix}
b_{so} \\
b_{fo}
\end{bmatrix}
\text{ in } V
\]

\[
\begin{bmatrix}
\varepsilon_{sn} \\
\gamma_s = 0
\end{bmatrix}
= \begin{bmatrix}
\mathbf{D}^* & \mathbf{O} \\
\phi_s \nabla^2 & \phi_f \nabla^2
\end{bmatrix}
\begin{bmatrix}
u_{sn} \\
u_{fn}
\end{bmatrix}
\text{ in } V
\]

\[
\sigma_{sn} = k \varepsilon_{sn} \text{ in } V
\]

\[
N \sigma_{sn} + \phi_f n p_n = \mathbf{t}_n \text{ on } \Gamma_t
\]

\[
\phi_f p_n = \mathbf{p}_n \text{ on } \Gamma_p
\]

\[
u_{sn} = \mathbf{u}_n \text{ on } \Gamma_u
\]

\[
\mathbf{n}^T \mathbf{u}_{fn} = \mathbf{w}_n \text{ on } \Gamma_w
\]

In the equilibrium condition (1), where \(\zeta\) is the diffusive drag coefficient and \(\phi_s\) and \(\phi_f = 1 - \phi_s\) are the solid and fluid fraction ratios, vector \(\sigma\) defines the independent components of the solid stress tensor, \(p\) is the pressure (in tension), \(u_s\) and \(u_f\) are the solid and fluid displacement vectors and \(b_s\) and \(b_f\) are the body force terms. Vectors \(b_{so}\) and \(b_{fo}\) define equivalent body force terms associated with the initial condition.

In the compatibility condition (2), vector \(\varepsilon\) collects the independent components of the solid strain tensor and \(\gamma\) is the volumetric change of the mixture. The differential operators, the divergence matrix \(\mathbf{D}\) and the gradient vector \(\nabla\) and their conjugates, \(\mathbf{D}^*\) and \(\nabla^*\), are assumed to be linear. In the association condition (3), which can be extended to include viscoelastic terms, \(k\) is the (symmetric) local stiffness matrix.

In the Neumann conditions (4) and (5), \(\mathbf{t}\) is the prescribed force vector in the solid phase, \(\mathbf{p}\) is the prescribed force on the fluid phase and matrix \(N\) defines the components of the unit outward normal, \(n\). In the Dirichlet conditions (6) and (7), vector \(\mathbf{u}\) defines the displacements prescribed on the solid matrix and \(\mathbf{w}\) is the prescribed outward normal component of the fluid displacement. It is assumed that equations (4) to (7) are so written as to account for mixed boundary conditions.

### 3 Discretization in time

System (1)-(7) can be derived using alternative time integration procedures [8], namely the trapezoidal rules often applied in the modelling of soft tissues, in form,

\[
u = \mathbf{u}_o + \alpha_o \Delta t \mathbf{v}_o + \alpha \Delta t \mathbf{v}
\]

where \(\mathbf{u}_o\) and \(\mathbf{v}_o\) represent the variable and its time derivative at instant \(t=0\), \(\mathbf{u}\) and \(\mathbf{v}\) define the values they take at \(t=\Delta t\), and \(\alpha\) and \(\alpha_o\) are the time integration factors. The application of this procedure yields a single system (1)-(7), \(N=1\), with frequency \(\omega = (i\Delta t)^{-1}\) and initial condition term:

\[
b_{so} = -b_{fo} = i \omega \zeta \left[ (\mathbf{u}_{so} - \mathbf{u}_{fo}) - \alpha_o \Delta t (\mathbf{v}_{so} - \mathbf{v}_{fo}) \right]
\]
System (1)-(7) is also recovered using the non-periodic, high-order spectral decomposition method detailed in [5]. Every variable and its time derivative in the parabolic problem are approximated independently in the time domain,

\[ u(x, t) = \sum_{n=1}^{N} T_n(t) u_n(x) \]  

\[ v(x, t) = \sum_{n=1}^{N} T_n(t) v_n(x) \]

where functions \( T_n \) define the approximation basis. This decomposition leads to integration rules and initial condition terms of the form,

\[ v_n(x) = i \omega_n u_n(x) - i \omega_n^0 u_n(x) \]  

\[ b_w = -b_{jo} = i \omega_n \zeta (u_n - u_{jo}) \]

and is implemented in the tests reported below using a (non-periodic) wavelet basis with compact support [9,10]. Except for completeness, no constraints are set \textit{a priori} on the time basis for non-periodic approximations.

To simplify the presentation, it is assumed that the body is initially at rest, \( b_{wo} = b_{jo} = 0 \) in equation (1). The subscript identifying the order of the approximation, \( n \), is omitted from this point onwards. In addition, it is assumed that \( V \) and \( \Gamma \) in system (1)-(7) define the domain and the boundary of a typical element of the mesh.

## 4 Discretization in space

The hybrid-Trefftz displacement formulation is based on the independent approximation of the displacement field in the domain of the element and of the flux field on its Dirichlet boundary (extended to include its inter-element boundary):

\[
\begin{align*}
\begin{bmatrix}
  u_i \\
  u_f
\end{bmatrix} &= \begin{bmatrix} U & R \\ W & R \end{bmatrix} \begin{bmatrix} x_i \\
  x_f
\end{bmatrix} \quad \text{in } V \\
\end{align*}
\]

\[ t = Z_t y_i \quad \text{on } \Gamma_u \]  

\[ \phi_f p = Z_p y_p \quad \text{on } \Gamma_w \]

Matrices \( Z_t \) and \( Z_p \) in the boundary approximations (13) and (14) and the weighting vectors \( y_i \) and \( y_p \) define generalized (non-nodal) surface forces. They are assigned to the elements that share the same boundary to establish the inter-element flux continuity condition.

Matrices \( U \) and \( W \) in the domain approximation (12) collect the functions associated with non-trivial deformation modes, weighted by the generalized displacement vector \( x_i \), matrix \( R \) defines the rigid-body modes, with amplitudes \( x_r \). This approximation will not satisfy, in general, the Dirichlet conditions (6) and (7). However, approximation (12) is assumed to be associated with a strain field that satisfies locally the compatibility condition (2):

\[ \varepsilon = \nabla u \]

\[ \sigma = D : \varepsilon \]

\[ \sigma = 0 \quad \text{on } \Gamma_u \]  

\[ \sigma = \tau \quad \text{on } \Gamma_w \]
\[
\begin{bmatrix}
\varepsilon_x \\
\gamma = 0
\end{bmatrix} = \begin{bmatrix}
E \\
G
\end{bmatrix} x_e \tag{15}
\]

\[
\begin{bmatrix}
E & O \\
G & 0
\end{bmatrix} = \begin{bmatrix}
D^* & 0 \\
\phi_r \nabla^* & \phi_y \nabla^*
\end{bmatrix} \begin{bmatrix}
U \\
R
\end{bmatrix} \tag{16}
\]

The dual transformations of approximations (12) and (15) define (prescribed) generalised body forces and (free) generalised stresses,

\[
\begin{bmatrix}
\overline{\epsilon}_w \\
\phi_w
\end{bmatrix} = \int \begin{bmatrix}
U^* \\
R^*
\end{bmatrix} \begin{bmatrix}
\sigma^* \\
p
\end{bmatrix} dV \tag{17}
\]

\[
s = \int \begin{bmatrix}
E^* \\
G^*
\end{bmatrix} \begin{bmatrix}
\sigma^* \\
p
\end{bmatrix} dV \tag{18}
\]

and are used to enforce on (Galerkin) average the equilibrium and elasticity conditions (1) and (3), respectively (\(x^*\) denotes the conjugate transpose of array \(x\)). The dual transformations of the boundary approximations (13) and (14) define (prescribed) generalized boundary displacements on the solid and fluid phases,

\[
\bar{x}_u = \int \bar{Z}_u^* \bar{u} d\Gamma_u \tag{19}
\]

\[
\bar{x}_w = \int \bar{Z}_p^* \bar{w} d\Gamma_w \tag{20}
\]

that are used to enforce on average the element (and inter-element) Dirichlet conditions (6) and (7) for the assumed displacements (12).

The auxiliary variables defined above ensure the invariance of the inner-product in the finite element mapping, under the incompressibility conditions (15) and (16): \(x^*_e \overline{\epsilon}_w + x^*_r \overline{\phi}_w = \int (u^*_i b_s + u^*_j b_f) dV\)

\[
x^*_e s = \int \varepsilon^*_s \sigma^* dV
\]

\[
y^*_r \bar{u}_u = \int t^* \bar{u} d\Gamma_u
\]

\[
y^*_p \bar{w}_w = \int \phi^*_f p^* \bar{w} d\Gamma_w
\]

## 5 Finite element equations

The weak form of the equilibrium condition combines the local domain and boundary equilibrium conditions (1), (4) and (5),

\[
\begin{bmatrix}
s \\
0
\end{bmatrix} = \begin{bmatrix}
B_l & B_p \\
b_l & b_p
\end{bmatrix} \begin{bmatrix}
y_l \\
y_p
\end{bmatrix} - i \omega \begin{bmatrix}
C \\
O
\end{bmatrix} x_e + \begin{bmatrix}
\overline{\epsilon}_w + \overline{\phi}_w \\
\overline{\epsilon}_r + \overline{\phi}_r
\end{bmatrix} \tag{21}
\]

and is obtained inserting the local condition (1) in definition (17) for the generalized body forces, and integrating by parts the resulting equation to retrieve the boundary.
term. This term is uncoupled in its Neumann and Dirichlet parts to enforce conditions (4) and (5) and approximations (13) and (14). Result (21) is recovered enforcing the displacement approximation (12), under constraint (16) and definition (18). The expressions found for the boundary equilibrium matrices, the (Hermitian) damping matrix and the terms associated with the prescribed surface forces are:

\[
\begin{bmatrix}
B_t & B_p \\
b_t & b_p
\end{bmatrix}
= 
\begin{bmatrix}
\int U^* Z_t d\Gamma_u & \int W^* n Z_p d\Gamma_w \\
\int R^* Z_t d\Gamma_u & \int R^* n Z_p d\Gamma_w
\end{bmatrix}
\]

(22)

\[C = \int (U - W)^* \zeta (U - W) dV\]

(23)

\[
\begin{bmatrix}
\bar{y}_e \\
\bar{y}_r
\end{bmatrix}
= \int \begin{bmatrix}
U^* \\
R^*
\end{bmatrix}
\bar{t} d\Gamma_r + \int \begin{bmatrix}
W^* \\
R^*
\end{bmatrix}
\bar{n} \bar{p} d\Gamma_p
\]

As condition (2) is locally satisfied, the compatibility condition reduces to the weak enforcement of the Dirichlet (and inter-element continuity) conditions (6) and (7). It is stated using the boundary displacement definitions (19) and (20) for the assumed displacement field (12), to yield the dual of the equilibrium condition (21):

\[
\begin{bmatrix}
B_t^* & b_t^* \\
B_p^* & b_p^*
\end{bmatrix}
\begin{bmatrix}
x_e \\
x_r
\end{bmatrix}
= 
\begin{bmatrix}
\bar{y}_u \\
\bar{y}_w
\end{bmatrix}
\]

(24)

Equation (21) shows that the element is statically indeterminate, as the number of unknowns exceeds the number of equations, and equation (24) shows that the (linearly independent) domain and boundary bases must be so balanced as to ensure a non-negative kinematic indeterminacy number, \(\beta = N_u + N_r - N_y \geq 0\), where \(N_u + N_r\) is the dimension of the displacement approximation basis (12) and \(N_y\) is the dimension of the boundary force basis, defined by equations (13) and (14). Therefore, the smaller the indeterminacy number of the element the stronger the enforcement of the element displacement continuity conditions (24).

The weak form of the elasticity condition is stated inserting the local constitutive relation (3) in definition (18) for the generalised stress vector, using the dependent strain approximation (15) and accounting for the incompressibility condition implied by equations (15) and (16), to yield the Hermitian form:

\[s = K x_e\]

(25)

\[K = \int E^* k E dV\]

(26)

In the resulting solving system, obtained combining the equilibrium (21), compatibility (24) and elasticity (25) equations,

\[
\begin{bmatrix}
D & O & -B_t & -B_p \\
O & O & -b_t & -b_p \\
-B_t^* & -b_t^* & O & O \\
-B_p^* & -b_p^* & O & O
\end{bmatrix}
\begin{bmatrix}
x_e \\
x_r \\
y_e \\
y_p
\end{bmatrix}
= 
\begin{bmatrix}
\bar{y}_e + \bar{y}_e^{co} \\
\bar{y}_r + \bar{y}_r^{co} \\
-\bar{x}_e \\
-\bar{x}_p
\end{bmatrix}
\]

(27)
all terms have boundary integral expressions, except for the body force terms and the element dynamic matrix:

$$D = K + i \omega C$$

System (27) holds for the Trefftz variant of the hybrid displacement element, with the advantage that boundary integral expressions can be found for these arrays. For the present application, the Trefftz constraint consists in assuming that the primary displacement approximation (12) satisfies locally the local compatibility condition (2), as implied by the strain estimate (15), and is associated with stress and pressure fields that satisfy locally the elasticity condition (3)

\[
\begin{bmatrix}
\sigma_x \\
p
\end{bmatrix} = \begin{bmatrix} S & 0 \\ P & P \end{bmatrix} \begin{bmatrix} x_x \\ x_p \end{bmatrix} \quad \text{in } V
\]

\((S = k E)\) and the equilibrium condition (1), under the assumption that the effect of the body forces are negligible (see [3] for initial condition and body force effects):

\[
\begin{bmatrix} D & \phi_f \nabla \\ 0 & \phi_j \nabla \end{bmatrix} \begin{bmatrix} S & 0 \\ P & P \end{bmatrix} = i \omega \zeta \begin{bmatrix} U - W & 0 \\ W - U & 0 \end{bmatrix}
\]

The constant pressure mode, \(P\), is included in equation (29) to stress the fact that the corresponding weight, defined by (scalar) vector \(x_p\), remains indeterminate in the implementation of the displacement model for incompressible media (it is computed \textit{a posteriori}, enforcing the flux continuity conditions). The resulting Trefftz approximation basis combines frozen mixture, P- and S-wave solution modes [3].

The procedure described above to formulate the displacement model of the hybrid displacement element is not affected by the additional constraints (29) and (30) on the domain approximation basis (12), meaning that system (27) holds for the Trefftz variant. However, the approximation basis is now fundamentally different, as the Trefftz constraint implies an approximation that embodies the physics of the problem. In addition, when these constraints are used to integrate by parts and simplify definition (28), using definitions (23), (26) and (30), the following boundary integral expression is found for the dynamic matrix:

\[
D = \int\int (\phi^*_f U + \phi^*_j W) n P d\Gamma
\]

6 Test problems

The unconfined indentation and the compression of cartilage specimens under confined and unconfined conditions are often used as modelling tests of incompressible biphasic media. The tests are defined in Figure 1 and the loading programmes used in time domain problems are defined in Figure 2, where \(\bar{u}\) and \(\bar{r}\) are the prescribed displacement and force, respectively. The force-driven, step-load programme is used only in the indentation test. All tests are solved in a single time step \((\Delta t = t_{\text{max}} = 10^3 \text{s})\). The displacement, stress and pressure fields are determined from equations (12) and (29) for a given frequency, which are combined through
approximation (8) to define their variation in time. Equation (9) and the integration rule (10) are used to determine velocity fields. No smoothing is used in the representation of the finite element solutions.

\[ h \]

\[ x \]

\[ w \]

\[ y \]

\[ h \]

\[ x \]

\[ w \]

\[ y \]

Incompressible cartilage sample

Modelled quadrant

Figure 1: Indentation, confined and unconfined compression (plane strain) tests.

\[ 0t \]

\[ 0t \]

\[ t_{\text{max}} \]

\[ t_{\text{max}} \]

\[ t_0 \]

\[ t_0 \]

\[ \pi \]

\[ \pi \]

\[ \sigma \]

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\[ \tau \]

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\[ \sigma_{xy} \]

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The two-dimensional problems defined by the unconfined compression tests,

\[
\sigma_{xx}^* = \sigma_{yy}^* = p = 0 \text{ on } x = \pm \frac{1}{2} w
\]  

are solved for two boundary conditions, namely a prescribed displacement applied to impermeable, adhesive end-platens, and a prescribed pressure applied to permeable, lubricated end-platens:

\[
u^* = 0 \text{ and } u^* = u^* = \pm \bar{u} \text{ on } y = \mp \frac{1}{2} h
\]  

\[
\sigma_{yy}^* = -\tau \text{ and } p = \sigma_{yy}^* = 0 \text{ on } y = \pm \frac{1}{2} h
\]

Single-element, \(2 \times 2\) and \(4 \times 4\)-element meshes are used to implement the frequency domain tests. They are used to discretize the specimen of the confined compression test shown in Figure 1 and a quadrant of the specimen subject to unconfined compression shown in the same figure. All tests are implemented using the same order of approximation in each domain element, \(dV\), and boundary element, \(d\Gamma\), of a given mesh. The resulting dimension, \(N\), of the solving system in the explicit (non-condensed) form (27) is defined for each application.

The tests are solved for three values of the forcing frequency, \(0.01\), \(0.10\) and \(1.00\) rad/s, so selected to ensure a range in the variation of the ratio of the characteristic length of the element, \(L_{FE}\), to the wavelength of the excitation, \(\lambda\), that far exceeds the limits that are usually recommended: \(0.25 < \frac{L_{FE}}{\lambda} < 10\).

### 7.1 Incompressibility tests

The enforcement of the mixture incompressibility condition (15) can be tested on the confined compression test modelling the confining chamber and the loading platen as rigid, impermeable and lubricated. When a prescribed displacement is enforced, system (27) exposes an ill-posed problem, as the equation associated with the prescribed displacement on the fluid is rendered linearly dependent and inconsistent.

The following tests model the response of a mixture in which the compressibility of each phase, solid and fluid, tends to zero. As the domain approximation basis used satisfies locally the incompressibility condition of the mixture, full incompressibility is modelled by increasing the Poisson’s ratio of the solid phase \(\nu \to 0.5\). The results obtained are presented in Figure 3, where \(E_{FE}\) is the norm of the internal energy, \(E_{ref}\) is the energy norm obtained with \(\nu = 0.49\) and \(N9\) represents the number of approximating digits, e.g. \(N9 = 3\) for \(\nu = 0.4999\). The graph for the force driven confined compression test (33), shows that the shear stress is null in the specimen and that the load is transferred to the solid skeleton under a vanishing pressure, as Poisson ratio tends to the incompressibility limit. The confinement condition (31) induces vanishing displacements in both phases of the mixture, which produce a null energy at the incompressibility limit. The distinct response found in the unconfined compression test, under the same loading condition (37) is shown in the same figure. As the load is increasingly transferred to the solid, the variation of the internal energy is practically insensitive to the variation of the Poisson’s ratio.
7.2 Mesh distortion tests

The mesh distortion schemes used to assess the sensitivity of the solutions for the displacement driven confined and unconfined compression tests (32) and (36) are shown in Figure 4. Sensitivity is measured on the variation of the energy norm with respect to the value found for the undistorted mesh, $\gamma = 0.5$, using the same order of approximation (eleven in the domain, $d_{V} = 11$, five on the boundary, $d_{\Gamma} = 5$).

![Figure 4: Distortion schemes for the confined and unconfined compression tests.](image)

The results presented in Figure 5 show that the solution is sensitive to gross mesh distortion (leading to collapsing elements) only for the highest frequency, when the typical length of the (initial, undistorted) element is four times larger than the length of the wave. These results reverse the pattern found for the alternative hybrid-Trefftz stress element [7]. This is caused by the boundary conditions of the tests, which are enforced in complementary form, namely explicit enforcement of the Neumann (Dirichlet) condition for the stress (displacement) element.

![Figure 5: Sensitivity to mesh distortion (confined and unconfined tests).](image)
7.3 Convergence tests

The displacement driven loading programme is used in the assessment of the convergence of the finite element solutions for both confined and unconfined compression tests, under boundary conditions (32) and (36). They are solved with the single-element mesh and the (regular, unbiased) $2 \times 2$ - and $4 \times 4$-element meshes for an increasing refinement of the bases, under combination $d_V = 2d_T + 1$. The results obtained are presented in Figure 6, where $N$ defines the dimension of system (27).

The strong convergence under $p$-refinement is typical of hybrid-Trefftz elements and, for coarse meshes, the rate of convergence under $h$-refinement is directly affected by the boundary conditions of the problem, which affects differently the alternative displacement and stress models [11,12]. The rate of convergence is now also affected by the ratio between the typical element length to the wavelength, which in the tests shown varies from 0.7 ($4 \times 4$-element mesh) to 3.1 (single-element mesh). The reference values for the energy norm correspond to stabilized solutions obtained with boundary approximations of order $(d_V; d_T) = (19; 9)$ for both confined and unconfined tests. The confined compression test, under boundary conditions (31) and (32), and the unconfined compression test, under boundary conditions (35) and (36), are solved for the lowest forcing frequency, $\omega = 0.01 \text{rad/s}$. A non-uniform colour scale, defined by the bounds found for each field in each test, is used to enhance the representation of the stress, pressure and displacement estimates shown in Figures 7 and 8. The displacement components are normalized to the prescribed displacement, $\vec{u}$, and the stress and pressure components are normalized to the reference stress $\sigma_w = E \bar{\sigma} / w$.

The solutions presented in Figures 7 and 8 show that the boundary conditions and the inter-element continuity conditions are adequately enforced. The solutions shown in Figure 7 recover the one-dimensional response (34) expected for the confined compression test and expose a boundary layer effect, which is strongly amplified by increasing the frequency [7]. The two-dimensional nature of the response of the unconfined compression test is illustrated in Figure 8. Both tests are run with approximation $(d_V; d_T) = (11; 5)$, which leads to solving systems (27) with a total of 114, 444 and 1,752 degrees-of-freedom for the $1 \times 1$-, $2 \times 2$- and $4 \times 4$-element meshes, respectively, for the confined compression test.
Figure 7: Stress, pressure and displacement fields (confined compression test).

Figure 8: Stress and displacement fields (unconfined compression test).
8 Time domain tests

The data, used for tests in time domain, is the following [13]: width \( w = 6.35 \text{mm} \), height \( h = 1.78 \text{mm} \), permeability \( \kappa = 7.6 \times 10^{15} \text{m}^4 \text{N}^{-1} \text{s}^{-1} \), fluid fraction \( \phi_f = 0.83 \), Poisson’s ratio \( \nu = 0.125 \) and modulus of elasticity \( E = 0.675 \text{MPa} \). The displacement-driven, ramp-load programme shown in Figure 2 is designed to reach a deformation of 5% at instant \( t_o = 500 \text{s} \), for a prescribed displacement \( \bar{u} = 0.0890 \text{mm} \) in the indentation and confined compression tests, and \( \bar{u} = 0.0445 \text{mm} \) in the unconfined compression test. The force-driven, step-load programme is used also in the implementation of the indentation test, with \( t_o = 10 \text{s} \) and \( \tau = 100 \text{Pa} \).

The confined and unconfined compression tests are modelled with a single-element mesh and a regular, 2x2 element mesh. The approximation \( (dV; d\Gamma) = (11; 5) \) in both instances. The 3x2-element mesh shown in Figure 1, with approximation \( (dV; d\Gamma) = (15; 7) \), is used in the indentation problem. The solutions shown below (no smoothing and non-uniform colour scales, as in Section 7) are frames taken from the animations available in address www.civil.ist.utl.pt/HybridTrefftz under link ‘Animation of the response of hydrated soft tissues’.

8.1 Confined compression test

The confined compression test is implemented with boundary conditions (31) and (32) to recover the one-dimensional response (34) of the cartilage specimen. The solution obtained with the displacement element recovers closely the stress relaxation response obtained with the alternative stress element reported in [7], where five time frames of the variation of the displacement, stress and pressure fields are presented.

The variation in time of the results obtained with a single-element mesh and a regular mesh of 2x2 elements is shown in Figure 9. The co-ordinates of the control point are measured in reference system defined in Figure 1. These results recover the values reported in [13], which are obtained with a mesh with 5x2 pairs of hybrid elements. Time integration is based on a trapezoidal rule, implemented with \( \Delta t = 5 \text{s} \) (200 time step).

![Figure 9: Evolution in time of the confined compression test.](image-url)
8.2 Unconfined compression test

The unconfined compression test represented in Figure 1 is useful to illustrate two distinct forms of response, namely a one-dimensional response, in the x-direction, and a fully two-dimensional response. The displacement-driven, ramp-loading programme defined in Figure 2, with \( u = 0.0445 \text{mm} \), is used in both tests, under the free-surface condition (35). The two-dimensional response is caused by boundary condition (36), to model the action of rigid, impermeable, adhesive loading platens. The one-dimensional response is mobilized using lubricated platens:

\[
 u_x' = u_y' = \pm \bar{u} \quad \text{and} \quad \sigma_{xy}^t = 0 \text{ on } y = \mp h/2
\]  

(38)

The number of functions used in the domain approximation (12) is \( N_x = 72 \) per element, with \( d_y = 11 \). The fifth-degree used in the boundary approximations (13) and (14), \( d_f = 5 \), yields solving systems (27) with dimension \( N = 444 \) for the 2x2-element mesh used in the implementation of the test with adhesive end-platens. The corresponding value for the lubricated variant is \( N = 432 \).

The evolution in time of the results obtained at control point \((x;y) = (w/2;0)\), see Figure 1, is shown in Figures 10 and 11. They are obtained applying the integration method presented in [5] using a single time increment \((\Delta t = 10^3 \text{s})\) and capture adequately the peaks in acceleration occurring at the end-points of the loading phase. The two-dimensional (adhesive) test results recover the values reported [13], using a trapezoidal rule \((\Delta t = 5 \text{s})\) and a regular mesh with \(12x6\) pairs of elements.

Figure 10: Evolution in time of the confined (lubricated) compression test.

Figure 11: Evolution in time of the confined (adhesive) compression test.
Figure 12: Unconfined (lubricated) test solutions at instants \( t = 230 \text{s} \) and \( t = 850 \text{s} \).

The solutions obtained at two instants, in the loading phase and close to the end of the test period, are presented in Figures 12 and 13. They confirm the adequate modelling of the boundary and inter-element displacement and force continuity conditions.

### 8.3 Indentation test

In the indentation test represented in Figure 1, the loading platen is placed symmetrically and its length is a third of the width of the specimen. The reaction platen is modelled as rigid, impermeable and adhesive:

\[
\begin{align*}
  u_x^0 &= u_y^0 = u_x^f = 0 & \text{on } y &= 0
\end{align*}
\]  

(39)
Figure 13: Unconfined (adhesive) test solutions at instants $t=230s$ and $t=850s$.

Besides the free surface conditions, the boundary conditions are modelled with a permeable, lubricated platen and a displacement-driven ($\bar{u} = 0.0890\text{mm}$) ramp-loading, and the same platen subject to an impact load ($t_o=10s$ and $\tau = 100\text{Pa}$):

$$u^i_y = -\bar{u} \quad \text{and} \quad p = \sigma^i_{xy} = 0 \quad \text{on} \quad y = h$$  \hspace{1cm} (40)

$$\sigma^i_{xy} = -\tau \quad \text{and} \quad \sigma^i_{xy} = \pi = 0 \quad \text{on} \quad y = h$$  \hspace{1cm} (41)
The stress, pressure and displacement fields obtained with the permeable, lubricated platen subjected to a displacement-driven ramp-loading at two instants, in the loading phase and close to the end of the test period, are shown in Figure 14. The evolution in time of velocities in solid and fluid phases and stresses and pressure obtained at control point \((x,y) = (w/2; 0.9h)\), see Figure 1, with the \(3\times2\)-element mesh, in a single time step, \(\Delta t = 10^3\) s, is shown in Figures 15 and 16 for the two boundary and loading conditions tested.

Figure 14: Indentation (permeable) test solutions at instants \(t=230\) s and \(t=850\) s.
9 Closure

The displacement model of the alternative hybrid and hybrid-Trefftz finite element formulation of parabolic, incompressible biphasic problems is derived from the basic conditions on equilibrium, compatibility and elasticity relations. This facilitates the interpretation of the role played by each finite equation of the formulation and its subsequent extension to account for physically or geometrically non-linear effects. The associated energy statements are recovered and sufficient conditions for the existence of unique solutions are stated for Hermitian and non-Hermitian applications.

The work presented here is part of a wider study on the application of the alternative stress and displacement models of the hybrid-Trefftz finite element formulation to the linear analysis of two-dimensional and axisymmetric incompressible biphasic media [7,14]. This preliminary study has been developed to assess the relative merits of the Trefftz formulation, as compared with alternative hybrid and mixed finite element formulations, and gain the insight necessary to support a decision to extend the formulation to the modelling of the non-linear response of incompressible soft tissues. The non-linear extension of this work depends on the assessment of performance of the element under linear conditions, as there are no published results on the finite element analysis of the response of soft tissues using the hybrid-Trefftz displacement model.

The spectral formulation is used to illustrate the relatively high rates of convergence this model can attain and to show that the solutions it produces are only
marginally sensitive to full incompressibility, shape distortion and wavelength excitation. The time domain applications also show that relatively high quality results are obtained using coarse meshes of high-order elements. They illustrate well the mechanisms of wave propagation and reflexion under both creep and stress relation conditions and recover the results published in the literature when the time integration procedure recalled here is implemented in a single-time step, using a high-order, wavelet approximation basis.

The main strength of this approach is to combine features that typify the finite element and the boundary element methods, namely domain decomposition and symmetric (when applicable) and sparse solving systems with coefficients described by boundary integral expressions. In addition, the approximation basis is regular, avoiding thus the difficulties associated with singular fundamental solutions used in the implementation of the boundary element method. Moreover, the hybrid formulation that is used and the direct implementation of the naturally hierarchical basis yield a solving system that is well-suited to implementations that exploit parallel processing and adaptive refinement.

The major weakness of the approach reported here is that the space approximation basis is strictly problem-dependent. The weakness centres mainly on the necessity of deriving complete approximation bases for each class of problem, a task that tends to increase in difficulty with the complexity of the problem. It is noted, however, that the approach is not limited to linear modelling. The usual practice of linearization can be followed, using basis taken from the linearized problem, at the relatively marginal cost of deriving algebraic solving systems that are no longer fully described by boundary integral expressions. In what regards the time integration procedure, the non-periodic spectral decomposition method can still be applied, using now weaker (low-degree polynomial) approximation bases, as the limitations induced by non-linearity no longer justify the use of strong wavelet approximations designed to implement large time steps.

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References


