Implementation of Hybrid-Trefftz Elements for 3D Acoustics

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1. Introduction
This report summarizes the information necessary to implement the displacement and stress (pressure) models of the hybrid-Trefftz finite element formulation for linear Acoustics presented in [1].

The basic equations of linear acoustics are recalled in Section 2 and their spectral description is presented in Section 3. The governing differential equation is stated in Section 4 to define the homogeneous solution, in Cartesian and spherical co-ordinates, and the particular solution for point sources.

The finite element approximations criteria are stated in Section 5. They are used in Sections 6 and 7 to derive the displacement and stress finite element equations, respectively, where the resulting solving systems are stated and the general definitions for their coefficients are summarized. The constraints set on the domain approximation bases are applied in Section 8 to obtain boundary integral expressions for all coefficients present in the finite element solving systems.

Although no constraints are placed on the topography of the finite elements, for simplicity they are assumed here to be polytopes. The definition of the geometry of the bounding planar surfaces is summarized in Section 9 and the integration rules used to determine the coefficients of the finite element solving system are defined in Section 10. The basic aspects of the numerical implementation of the displacement and stress models for the hybrid-Trefftz finite formulation for Acoustics are presented in Section 11.
2. Basic equations

The equations governing the dynamic response of a linear acoustic system with domain $V$ are stated as follows:

$$
-\nabla p + b = \gamma \mathbf{v} + \rho \mathbf{a} \quad \text{in } V \\
\varepsilon = \nabla^T \mathbf{u} \quad \text{in } V \\
-p = K \varepsilon \quad \text{in } V
$$

In the equilibrium equation (1), $p$ is the pressure (in tension), $b$, $v$ and $a$ are the body-force, velocity and acceleration vectors, $\gamma$ is the volumetric drag, $\rho$ is the fluid density, and,

$$\nabla = \begin{Bmatrix}
\frac{\partial}{\partial x} \\
\frac{\partial}{\partial y} \\
\frac{\partial}{\partial z}
\end{Bmatrix}
$$

the gradient vector. Equation (2) defines the volumetric strain, $\varepsilon$, compatible with the displacement field, $\mathbf{u}$. In the constitutive relation (3), $K$ is the fluid bulk modulus, to yield the following definition for the speed of sound:

$$c = \sqrt{\frac{K}{\rho}}$$

Equations (6) and (7) define the boundary equilibrium (Neumann) and boundary compatibility (Dirichlet) conditions and equation (8) is the impedance boundary condition:

$$p = \bar{p} \quad \text{on } \Gamma_p$$

$$\mathbf{n}^T \mathbf{u} = \bar{\mathbf{w}} \quad \text{on } \Gamma_u$$

$$(k_i^{-1} \dot{p} + c_i^{-1} p) = \mathbf{n}^T \mathbf{v} \quad \text{on } \Gamma_z$$

Hence, in these equations $\bar{p}$ and $\bar{\mathbf{w}}$ are the prescribed pressure and normal velocity, and parameters $k_i$ and $c_i$ are impedance related parameters. In the equations above $\Gamma = \Gamma_p \cup \Gamma_u \cup \Gamma_z$ is the boundary surface of domain $V$ and $\mathbf{n}$ its unit outward normal vector.

The Dirichlet condition (7) can be alternatively expressed in terms of a prescribed velocity field:

$$\mathbf{n}^T \mathbf{v} = \bar{\mathbf{v}} \quad \text{on } \Gamma_u$$

The initial conditions on the displacement and velocity fields,

$$\mathbf{u} = \mathbf{u}_0 \quad \text{in } V$$

$$\mathbf{v} = \mathbf{v}_0 \quad \text{in } V$$

where $\mathbf{u}_0$ and $\mathbf{v}_0$ define the displacement and velocity fields at instant $t = 0$, complete the set of equations governing the dynamic response of the fluid.
3. Integration in time

The common practice is to assume that all fields, say the displacement field $u(x,t)$, are separable in time and space, in form,

$$u(x,t) = \sum_{n=-\infty}^{\infty} T_n(t) u_n(x)$$  \hspace{1cm} (12)

where functions $T_n(t)$ define the terms of approximation basis in the time domain and $u_n(x)$ the corresponding amplitudes in the space domain.

When the response is assumed to be periodic, with period $T$, or is periodically extended in that fictitious period,

$$u(x,t + T) = u(x,t)$$  \hspace{1cm} (13)

a Fourier approximation (or transformation) is usually assumed, to yield,

$$T_n(t) = \exp(j \omega_n t)$$  \hspace{1cm} (14)

where $j$ is the imaginary unit and $\omega_n$ is the forcing frequency, defined as follows for a Fourier expansion:

$$\omega_n = \frac{2\pi}{T}n$$  \hspace{1cm} (15)

The velocity and acceleration integration rules simplify to the following forms,

$$v_n = j\omega_n u_n$$  \hspace{1cm} (16)

$$a_n = -\omega_n^2 u_n$$ \hspace{1cm} (17)

to yield, using different techniques, the following frequency domain description of the domain conditions (1) to (3) and of the boundary conditions (6) to (8):

$$-\nabla p_n + b_n + (\omega_n^2 \rho - j\omega_n \gamma) u_n = 0 \text{ in } V$$  \hspace{1cm} (18)

$$\varepsilon_n = \nabla^\top u_n \text{ in } V$$  \hspace{1cm} (19)

$$-p_n = K \varepsilon_n \text{ in } V$$  \hspace{1cm} (20)

$$p_n = \bar{p}_n \text{ on } \Gamma_p$$  \hspace{1cm} (21)

$$n^\top u_n = \bar{w}_n \text{ on } \Gamma_u$$  \hspace{1cm} (22)

$$p_n = j\omega_n \bar{Z}_n n^\top u_n \text{ on } \Gamma_z$$  \hspace{1cm} (23)

In this equation, $\bar{Z}_n^{-1} = c_i^{-1} + j\omega_n k_i^{-1}$ represents the admittance coefficient.

The spectral problem (18) to (23) is independent of the initial conditions (10) and (11), in consequence of periodicity, and equation (22) holds for the Dirichlet condition (9) in terms of the
velocity field by letting \( \vec{w}_n \propto -j\omega_n^{-1} \vec{v}_n \). The prescribed terms in system (18)-(23), say \( b_n \), are defined as follows, where \( \hat{T}_n(t) \) is the complex conjugate of \( T_n(t) \):

\[
b_n(x) = \frac{1}{T} \int_0^T \hat{T}_n(t) b(x, t) \, dt
\]

Although the results presented below are obtained for system (18)-(23), they can be readily adapted to non-periodic time integration methods [3]. These results are stated omitting the order of the forcing frequency, \( n \), to lighten the notation.

4. Formal solutions of the wave equation

Combination of the domain conditions (18) to (20) leads to a Beltrami-type solving system in the form of the inhomogeneous Helmholtz equation,

\[
(\nabla^2 + k^2) p + \nabla^T b = 0 \quad \text{in } V
\]

where \( \nabla^2 \) is the Laplacian and \( k \) the wave number:

\[
\nabla^2 p = \nabla^T \nabla p = (\partial_{xx} + \partial_{yy} + \partial_{zz}) p
\]

\[
k^2 = (\omega^2 \rho - j\omega \gamma)/K
\]

The general solution of the homogeneous Helmholtz equation (25) and the particular solution for a point source are presented below for Cartesian and spherical co-ordinate systems. The first is best suited to model acoustics in regular polytopes and the latter to implement the model using irregular elements.

The homogeneous solution of the Helmholtz equation (25),

\[
(\nabla^2 + k^2) P_{mn} = 0 \quad \text{in } V
\]

is separable in a number of coordinate systems. The usual practice is to adopt Fourier expansions in Cartesian co-ordinates. However, the resulting finite element approximations are known to be sensitive to distortion of the shape of the finite elements, in which case it is convenient to implement an approximation basis implemented on a spherical co-ordinate system. This is the option followed here to model the effect of a point radiator using the particular solution term.

4.1 Homogeneous solution in Cartesian co-ordinates

The general solution of the homogeneous differential equation (28) in Cartesian co-ordinates is,

\[
P_{mn}(x, y, z) = s_{mn}^{-1} \cdot \exp(\pm jk_{x_{mn}} x / L_x) \cdot \exp(\pm jk_{y_{mn}} y / L_y) \cdot \exp(\pm jk_{z_{mn}} z / L_z) \quad \text{in } V
\]

under the following constraint,

\[
(k_{x_{mn}} / L_x)^2 + (k_{y_{mn}} / L_y)^2 + (k_{z_{mn}} / L_z)^2 = k^2
\]
where $L_x$, $L_y$ and $L_z$ are the dimensions of the polytope and $s_{mn}$ is a scaling factor.

A complete Fourier approximation should be used whenever the acoustic field is periodic in space. As this is not the case in most situations, half-range series are used instead, for instance the following cosine approximation in the $(x,y)$ plane, which is written assuming that the origin of the co-ordinate system is placed at the geometric centre of the polytope,

$$P_{mn}(x,y,z) = s_{mn}^{-1} \cdot \cos[k_{xmn} \left(\frac{1}{2}x/L_x\right)] \cdot \cos[k_{ymn} \left(\frac{1}{2}y/L_y\right)] \cdot \exp[\pm j k_{zmn} \left(\frac{1}{2}z/L_z\right)]$$ \hspace{1cm} (31)

under the Fourier conditions,

$$k_{xmn} = m \pi$$ \hspace{1cm} (32)

$$k_{ymn} = n \pi$$ \hspace{1cm} (33)

$$k_{zmn} = \begin{cases} k & \text{if } \bar{k}^2 > 0 \\ j|\bar{k}| & \text{if } \bar{k}^2 < 0 \end{cases}$$ \hspace{1cm} (34)

and the Helmholtz equation condition (30):

$$\bar{k}^2 = (k L_z)^2 - (k_{xmn} L_x/L_z)^2 - (k_{ymn} L_y/L_z)^2$$ \hspace{1cm} (35)

This set of approximation functions is repeated for the $(x,z)$ and $(y,z)$ planes to generate a basis with dimension,

$$\text{dim } (P) = 6 \left( N + 1 \right)^2$$ \hspace{1cm} (36)

where $N$ is highest-order in the approximation: $N = m_{\text{max}} = n_{\text{max}}$.

According to equation (20), the associated volumetric strain is defined by,

$$E_{mn} = -K^{-1} P_{mn}$$ \hspace{1cm} (37)

Using equation (18), with $b = \theta$, the wave number definition (27), the following expression is found for the displacement vector in Cartesian coordinates, where $\nabla$ is the gradient vector (4):

$$U_{mn} = (k^2 K)^{\frac{1}{2}} \nabla P_{mn}$$ \hspace{1cm} (38)

4.2 Homogeneous solution in spherical co-ordinates

The homogeneous solution of the Helmholtz equation (28) is separable also in spherical co-ordinates, under the notation defined in Figure 1. The solution thus found for equation (28), using the Laplacian definition given in Appendix 1, is, for interior problems,

$$P_{mn} = s_{mn}^{-1} (kr)^{\frac{1}{2}} J_{n+\frac{1}{2}}(kr) \cdot P_n^m(\xi) \cdot \exp(\pm jm\theta) \hspace{1cm} \text{with } n = 0, 1, \cdots, N \text{ and } m \leq n$$ \hspace{1cm} (39)

where $J_{n+\frac{1}{2}}(kr)$ is the Bessel function of the first kind and (fractional) order $n + \frac{1}{2}$, with $0 \leq r \leq L$.
\[ P_n^m(\xi) = (-1)^m (1 - \xi^2)^{m/2} \frac{d^m P_n(\xi)}{d\xi^m} \]  

is the associated Legendre function of degree \( n \) and order \( m \), with \( \xi = \cos(\phi) \), with \( 0 \leq \phi \leq \pi \), \( \exp(\pm jm\theta) \) is the complex description of the sinusoid, with \( -\pi \leq \theta \leq +\pi \), and \( \alpha \) is a scaling parameter (see Section 5). It can be verified that the dimension of this basis is:

\[ \dim(P) = (N + 1)^2 \]  

As definition (39) has to extended to include spherical coordinates on the rotates axis, and each of the resulting families involves the same radial solution, defined for \( m = n = 0 \), the dimension of the spherical basis is:

\[ \dim(P) = 3N(N + 1) + 1 \]  

Figure 1: Notation for spherical coordinates, Ref [2]

The associated volumetric strain is still defined by equation (37). The expression found for the displacement vector in Cartesian coordinates is,

\[ U_{mn} = (k^2 K)^{-1} M \nabla P_{mn} \]  

where \( M \) is the orthogonal matrix, \( M^T M = I \), associated with the transformation of coordinates,

\[ M = \begin{bmatrix} \cos \theta \cdot \sin \phi & -\sin \theta & \cos \theta \cdot \cos \phi \\ \sin \theta \cdot \sin \phi & \cos \theta & \sin \theta \cdot \cos \phi \\ \cos \phi & 0 & -\sin \phi \end{bmatrix} \]  

and \( \nabla^T = \{ \partial_r, r^{-1} \sin^{-1} \phi \partial_\phi, r^{-1} \partial_\phi \} \) is the gradient vector in spherical co-ordinates, to yield:

\[ \nabla P_{mn} = \alpha k \frac{\exp(\pm jm\theta)}{(2n + 1)\sqrt{k r}} \begin{cases} [n J_{n+\frac{1}{2}} - (n + 1)J_{n+\frac{3}{2}}] P_n^m \\ \pm jm \left( J_{n+\frac{1}{2}} + J_{n+\frac{3}{2}} \right) \frac{P_n^m}{\sin \phi} \\ \left( J_{n+\frac{1}{2}} + J_{n+\frac{3}{2}} \right) \left( P_{n+1}^{m+1} + m \cot \phi P_n^m \right) \end{cases} \]
The following results are useful in the computation of the fields defined above and show that the approximation functions are regular:

\[
\lim_{x \to 0^+} \frac{J_{n+\frac{1}{2}}(x)}{\sqrt{x}} = \begin{cases} 
\sqrt{\frac{2}{\pi}} & \text{if } n = 0 \\
0 & \text{if } n > 0 
\end{cases} \tag{46}
\]

\[
\lim_{\sin \phi \to 0} \frac{P_n^m(\cos \phi)}{\sin \phi} = \begin{cases} 
-\frac{\pi}{2} n(n+1)(\cos \phi)^{m+1} \bigg|_{\cos \phi = 1} & \text{if } m = 1 \\
0 & \text{if } m > 1 
\end{cases} \tag{47}
\]

4.2 Particular solution

Particular solutions are used to model local effects and improve thus the rate of convergence of the numerical model. A typical situation is the modelling of local singularities, which is highly demanding for the regular approximation basis (39).

This can be illustrated with the modelling of point radiators, associated with the following pressure and displacement fields, where \( q_0 \) is the source strength and \( a \) the radius:

\[
p_0 = j q_0 \frac{\rho \omega k}{4\pi} \frac{\exp(jka)}{1 + jka} \cdot \frac{\exp(-jkr)}{kr} \tag{48}
\]

\[
\mathbf{u}_0 = \begin{bmatrix} u_r \\ u_\theta \\ u_\phi \end{bmatrix} = -j q_0 \frac{\rho \omega}{4\pi K} \cdot \frac{\exp(jka)}{1 + jkr} \cdot \frac{\exp(-jkr)}{(kr)^2} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{49}
\]

The following assumption holds, as the radius is assumed to be small:

\[
\lim_{a \to 0} \frac{\exp(jka)}{1 + jka} = 1
\]

5. Finite element approximations

It is assumed that the domain of the fluid, \( V \), is discretized into elements. Letting \( V^e \) define the domain of a typical element, four regions can be distinguished, in general, on its boundary, \( \Gamma^e \),

\[
\Gamma^e = \Gamma_p^e \cup \Gamma_u^e \cup \Gamma_z^e \cup \Gamma_i^e
\]

namely, the boundaries whereon the pressure and the normal displacement (or velocity) are prescribed, \( \Gamma_p^e \) and \( \Gamma_u^e \), respectively, the absorbing boundary, \( \Gamma_z^e \), and the boundary of the element interior to the finite element mesh, \( \Gamma_i^e \).
5.1 Approximations in the domain

The homogeneous and the particular solutions defined above are used to approximate the displacement, volumetric strain and pressure fields in form,

\[ u = U X + u_0 \quad \text{in } V^e \]  
\[ \varepsilon = E X + \varepsilon_0 \quad \text{in } V^e \]  
\[ p = P X + p_0 \quad \text{in } V^e \]  

where matrix \( U \) and row vectors \( E \) and \( P \) collect the homogeneous solution modes, vector \( X \) defines their (unknown) amplitudes and the vector and scalar fields \( u_0, \varepsilon_0 \) and \( p_0 \) define the (prescribed) particular solution terms.

According to the results summarized in Section 4, the domain approximations (50) to (52) satisfy the following relations:

\[ E = \nabla^T U \]  
\[ P = -KE \]  
\[ -\nabla p + (\omega^2 \rho - j\omega \gamma) U = 0 \]  
\[ \varepsilon_0 = \nabla^T u_0 \]  
\[ p_0 = -K\varepsilon_0 \]  
\[ -\nabla p_0 + (\omega^2 \rho - j\omega \gamma) u_0 + b = 0 \]

This set of relations is called the Trefftz constraint in the context of the finite element method, as they typify the variant of this method based on the method proposed by E. Trefftz to solve boundary value problems [4].

5.2 Approximations on the boundary

As the amplitudes of the homogeneous solution modes, \( X \), do not represent physically displacements, strains or pressures at particular points (or nodes) of the element, they cannot be used to assemble directly one element to its adjacent elements. As it is shown below, this leads to the necessity of approximating independently either the pressure or the displacement field on the boundary of the element. Finite elements based on the independent approximation of domain and boundary fields, as the ones used here, are termed hybrid elements.

In displacement elements the pressure field is independently approximated on the boundaries where the pressure is not prescribed, to yield,

\[ p = B_\alpha p_\alpha \quad \text{on } \Gamma^e_\alpha \text{ with } \alpha = u, z, i \]  

(59)
where vector $p_\alpha$ defines the generalized pressure associated with each mode collected in row-vector $B_\alpha$. In the implementation considered here, it is defined as follows.

$$B_\alpha = \left[ \cdots \ T_m(\zeta_1) \ T_n(\zeta_2) \ \cdots \right] \quad (60)$$

When spherical bases (39) are used in the domain approximation, the functions, $T_m(\zeta)$, used in the boundary approximation are either Chebyshev polynomials of degree $m$,

$$T_m(\zeta) = \cos(m \cos^{-1}(\zeta)) \quad (61)$$
or the Legendre polynomials of the same degree,

$$T_m(\zeta) = \frac{1}{2^m m!} \frac{d^m}{d \zeta^m} (\zeta^2 - 1)^m \quad (62)$$
with $-1 \leq \zeta \leq +1$ defining the surface parametric coordinate system (see Section 7). Hence, the dimension of a complete basis of degree $N$ is, on a given surface:

$$\dim(B_\alpha) = \frac{1}{2}(N+1)(N+2) \quad (63)$$

If the half-range Fourier approximation (31) is used instead to set up the approximation in the domain of the element, a trigonometric approximation is used in the implementation of the boundary approximation basis:

$$T_m(\zeta) = \cos [m \pi (1+\zeta)] \quad (64)$$

Complementary, the normal displacement field is independently approximated on the boundaries of stress elements whereon it is not prescribed, to yield,

$$w = B_\alpha \ w_\alpha \ \text{on } \Gamma_\alpha \ \text{with } \alpha = p, z, i \quad (65)$$
where vector $w_\alpha$ defines the generalized normal displacement associated with each mode collected in row-vector $B_\alpha$, still defined by equation (60), meaning that dimension (63) still holds.

As opposed to the strong set of conditions (53) to (58) enforced on approximations (50) to (52) in the domain of the element, completeness and linear independence are the only constraints set a priori on the basis (60) used to approximate either the pressure (59) or the displacement (65) fields on the boundary of the element.

### 5.3 Modelling of the solution in time and space

The solving systems for the alternative hybrid-Trefftz displacement and stress (pressure) finite elements are presented below. The solution of these systems yields the amplitudes $X$ of the domain approximations (50) to (52) for a given forcing frequency, $\omega_\alpha = \omega$, and the solution in
time, at any instant \( t \) of the periodic (or periodically extended) loading is recovered solving the
time approximation (12), extended to the volumetric and pressure fields:

\[
\varepsilon(x,t) = \sum_{n=-\infty}^{\infty} T_n(t) \varepsilon_n(x) \tag{66}
\]

\[
p(x,t) = \sum_{n=-\infty}^{\infty} T_n(t) p_n(x) \tag{67}
\]

Similar expressions are found for the pressure and normal displacement fields (59) and (65)
that are independently approximated in the implementation of displacement and stress elements,
respectively. The estimates for the velocity and acceleration fields are obtained using the
integration rules (16) and (17):

\[
v(x,t) = \sum_{n=-\infty}^{\infty} j \omega_n T_n(t) u_n(x) \tag{68}
\]

\[
a(x,t) = -\sum_{n=-\infty}^{\infty} \omega_n^2 T_n(t) u_n(x) \tag{69}
\]

6. Displacement element

Displacement elements are designed to produce kinematically admissible solutions, under strong
enforcement conditions. The kinematic admissibility conditions are the domain and boundary
compatibility conditions (19) and (22), respectively, with the latter being extended to include the
inter-element boundaries, \( \Gamma_i^e \). Therefore, term \( \bar{w} \) represents either the prescribed normal
displacement on boundary \( \Gamma_u^e \) of the typical element \( e \) or the normal displacement of the
connecting element that shares boundary \( \Gamma_i^e \).

6.1 Approximations

In order to attain the objective of obtaining kinematically admissible solutions, the displacement
approximation (50) is taken as the primary approximation and the volumetric strain
approximation (51) as the dependent approximation, as it is constrained to satisfy the domain
compatibility condition (19), as required by conditions (53) and (56). The pressure field is
approximated in form (59), that is, on all boundaries whereon it is not prescribed.

6.2 Dual variables

The dual transformations of the primary and dependent approximations (50) and (51), which
define (prescribed) generalized body forces and (free) generalized pressures,
\[ \bar{Q} = \int U^* b\, dV^e \]  
\[ s = \int E^* p\, dV^e \]  

are used to enforce on average, in the sense of Galerkin, the domain equilibrium condition (18) and the constitutive relation (20). In the notation used here, \( U^* = \hat{U}^T \) represents the complex conjugate of the transpose of array \( U \).

The dual transformation of the independent boundary approximation (59) defines generalized boundary displacements:

\[ \delta_a = \int B_a^* n^T u\, d\Gamma_a^e \text{ with } \alpha = u, z, i \]  

This definition is used to enforce the Dirichlet condition (22), in which case \( d_e \) is prescribed, and the impedance condition (23), which is written in its inverse form for the assumed boundary pressure (59). It also used to enforce on average the inter-element displacement continuity condition, in which case \( \bar{w} \) represents the normal displacement of the connecting element.

### 6.3 Equilibrium

The following expression is obtained after inserting the body-forces determined from the domain equilibrium condition (18) in definition (70) for the generalized body force vector:

\[ -\int U'\nabla p\, dV^e + \bar{Q} + \int U^*(\omega^2 \rho - j\omega \gamma)u\, dV^e = 0 \]  

The first term is integrated by parts to bring in the boundary term, and the primary approximation (50) is inserted in the last term of the equation:

\[ \int (\nabla^T U)^* p\, dV^e - \int (n^T U)^* p\, d\Gamma^e + \bar{Q} + \int U^*(\omega^2 \rho - j\omega \gamma)(U X + u_0)\, dV^e = 0 \]  

Result (53) is used to recover definition (71) for the generalized pressure and the definitions for the mass and damping matrices and for the generalized body forces associated with the particular solution are used to simplify the notation:

\[ s = \int (n^T U)^* p\, d\Gamma^e - \bar{Q} - (\omega^2 M - j\omega C) X - \bar{Q}_0 \]  

\[ M = \int U^* \rho U\, dV^e \]  

\[ C = \int U^* \gamma U\, dV^e \]  

\[ \bar{Q}_0 = \int U^*(\omega^2 \rho - j\omega \gamma)u_0\, dV^e \]  

As the fluid density and the volumetric drag are real, equations (76) and (77) show that the mass and damping matrices are Hermitian: \( M = M^*; \ C = C^* \).
The boundary term in equation (75) is uncoupled the Neumann condition (21) and approximation (59) whereon the pressure is not prescribed:

\[\int (n^T U)^* p \, d \Gamma^e = \int (n^T U)^* \overline{p} \, d \Gamma^e_p + \int (n^T U)^* B_s \, d \Gamma^e_z p_z + \int (n^T U)^* B_p \, d \Gamma^e_u p_u + \int (n^T U)^* B_i \, d \Gamma^e_i p_i \] (79)

This leads to the following finite element description of the domain equilibrium condition (18):

\[s = [A_z \quad A_u \quad A_i] \begin{bmatrix} p_z \\ p_u \\ p_i \end{bmatrix} - (\omega^2 M - j \omega C) X - Q - \tilde{Q}_0 + \tilde{s} \] (80)

\[A_\alpha = \int (n^T U)^* B_\alpha \, d \Gamma^e \text{ with } \alpha = u, z, i \] (81)

\[\tilde{s} = \int (n^T U)^* \overline{p} \, d \Gamma^e_p \] (82)

6.4 Compatibility

As the domain compatibility condition (19) is locally satisfied by the primary and dependent approximations (50) and (51), see equations (53) and (56), the generalized boundary displacement definition is used to enforce, on average, the Dirichlet condition (22) on the boundary of the mesh and on inter-element boundaries for the assumed displacement field (50):

\[\delta_\alpha = \int B_\alpha^* n^T (U X + u_0) \, d \Gamma^e \text{ with } \alpha = u, z, i \] (83)

It is used, also, to enforce the (inverse form of the) impedance condition (23) for the assumed displacement and boundary pressure fields:

\[\int B_z^* n^T (U X + u_0) \, d \Gamma^e_z = -j \omega^{-1} \int B_z^* Z^{-1} B_z \, d \Gamma^e_z p_z \] (84)

The resulting finite element compatibility conditions are found,

\[\begin{bmatrix} A_z^* \\ A_u^* \\ A_i^* \end{bmatrix} X = \begin{bmatrix} -j \omega^{-1} F_z \, p_z - \delta_{z0} \\ \delta_u - \delta_{u0} \\ \delta_i - \delta_{i0} \end{bmatrix} \] (85)

where use is made of result (81), and:

\[F_z = \int B_z^* \overline{Z}^{-1} B_z \, d \Gamma^e_z \] (86)

\[\delta_{\alpha0} = \int B_\alpha^* n^T u_0 \, d \Gamma^e \text{ with } \alpha = u, z, i \] (87)

Definition (86) shows that the flexibility form of the impedance matrix is Hermitian.
6.5 Constitutive relations
The finite element constitutive relations are obtained enforcing the local condition (20) in
definition (71) for the generalized pressure and using next the dependent volumetric strain
approximation (51). In the resulting expression,

$$s = -K X - s_{k0}$$  \hspace{1cm} (88)

\(K\) is the stiffness matrix of the element and \(s_0\) the term that accounts for the contribution of the
particular solution:

$$K = \int E^* K E \, dV^e$$  \hspace{1cm} (89)

$$s_{k0} = \int E^* K \varepsilon_0 \, dV^e$$  \hspace{1cm} (90)

As the fluid bulk modulus is real, definition (89) shows that the element stiffness matrix is
Hermitian: \(K = K^*\).

6.6 Solving system
The finite element solving system is obtained eliminating the generalized pressure in the
equilibrium condition (80) using the constitutive relation (88) and adding the compatibility
condition (85) to the resulting equation:

$$\begin{bmatrix}
-D A_z^* A_u & A_j
\end{bmatrix}
\begin{bmatrix}
X

\end{bmatrix}
= \begin{bmatrix}
-\delta - s_0
\end{bmatrix}$$  \hspace{1cm} (91)

It is also shown in Section 8 that the element dynamic matrix,

$$D = -K - j\omega C + \omega^2 M$$  \hspace{1cm} (92)

and the particular solution dependent term,

$$s_0 = s_{k0} - \overline{Q} - Q_0$$  \hspace{1cm} (93)

can be defined using an alternative boundary integral expression.

7. Stress element
Stress (or pressure) and displacement elements are complementary, or dual, in concept. This
complementarity is stressed here by replicating next the dual of the derivation of the
displacement element.

Stress elements are designed to produce statically admissible solutions, under strong
enforcement conditions. The static admissibility conditions are the domain and boundary
equilibrium conditions (18) and (21), respectively, with the latter being extended to include the inter-element boundaries, $\Gamma^e_i$. Therefore, term $\bar{p}$ represents either the prescribed pressure on boundary $\Gamma^e_p$ of the typical element $e$ or the pressure in the connecting element that shares boundary $\Gamma^e_i$.

7.1 Approximations

In order to attain the objective of obtaining statically admissible solutions, the pressure approximation (52) is taken as the primary approximation and the displacement approximation (50) as the dependent approximation, as it is constrained to satisfy the domain equilibrium condition (18) through constraints (55) and (58). The normal displacement field is approximated in form (65), that is, on all boundaries whereon it is not prescribed.

7.2 Dual variables

The dual transformation of approximation (52) defines (free) generalized volumetric strains,

$$e = \int P^* \varepsilon \: dV^e$$

and are used to enforce on average, in the sense of Galerkin, the domain compatibility condition (19) and the (inverse form of the) constitutive relation (20).

The dual transformation of the independent boundary approximation (65) defines generalized boundary pressures:

$$\pi_\alpha = \int B^*_\alpha p \: d\Gamma^e_\alpha \text{ with } \alpha = p, z, i$$

This definition is used to enforce the Neumann condition (21), in which case $\pi_p$ is prescribed, and the impedance condition (23) for the assumed pressure and boundary displacement approximations (52) and (65), respectively. It also used to enforce on average the inter-element pressure continuity condition, in which case $\bar{p}$ represents the pressure of the connecting element.

7.3 Compatibility

The following expression is obtained after enforcing the domain compatibility condition (19) in definition (94) for the generalized volumetric strain:

$$e = \int P^* \nabla^T u \: dV^e$$

This equation is integrated by parts to bring in the boundary term, which is uncoupled into each of the parts defined above:
The finite element compatibility condition is obtained inserting above the dependent approximation (50) for the displacements in the domain of the element, the boundary displacement approximations (65) and the Dirichlet condition (22):

\[
e = \left[ A_z \quad A_p \quad A_i \right] \begin{bmatrix} w_z \\ w_p \\ w_i \end{bmatrix} - (\dot{\omega}^2 M^* + j\omega C^*) X + \bar{e} - \bar{e}_0
\]  

(98)

Definitions (76) and (77) for the Hermitian mass and damping matrices are recovered in this equation using constraint (55). The boundary compatibility matrices and the terms associated with the prescribed displacement and the particular solution are defined as follows:

\[
A_\alpha = \int P^* B_\alpha \ d\Gamma_\alpha^e\text{ with } \alpha = p, z, i
\]  

(99)

\[
\bar{e} = \int P^* \bar{w} \ d\Gamma_\alpha^e
\]  

(100)

\[
\bar{e}_0 = \int (\nabla P)^* u_0 \ dV^e
\]  

(101)

7.4 Equilibrium

As the domain equilibrium condition (18) is locally satisfied by the primary and dependent approximations (52) and (50), see equations (55) and (58), the generalized boundary pressure definition (95) is used to enforce, on average, the Neumann condition (21) on the boundary of the mesh and on inter-element boundaries for the assumed pressure field (52):

\[
\pi_\alpha = \int B_\alpha^* (P X + p_0) \ d\Gamma_\alpha^e\text{ with } \alpha = p, i
\]  

(102)

It is used, also, to enforce the impedance condition (23) for the assumed pressure and boundary displacement fields:

\[
\int B_z^* (P X + p_0) \ d\Gamma_z^e = j\omega \int B_z^* \bar{Z} B_z \ d\Gamma_z^e \ w_z
\]  

(103)

The resulting finite element compatibility conditions are found,

\[
\begin{bmatrix} A_z^* \\ A_p^* \\ A_i^* \end{bmatrix} X = \begin{bmatrix} j\omega K_z \ w_z - \pi_{z0} \\ \pi_p - \pi_{p0} \\ \pi_i - \pi_{i0} \end{bmatrix}
\]  

(104)

where use is made of result (99), and:

\[
K_z = \int B_z^* \bar{Z} B_z \ d\Gamma_z^e
\]  

(105)

\[
\pi_{\alpha0} = \int B_\alpha^* p_0 \ d\Gamma_\alpha^e\text{ with } \alpha = p, z, i
\]  

(106)
Definition (105) shows that the stiffness form of the impedance matrix is Hermitian.

7.5 Constitutive relations

The finite element constitutive relations are obtained enforcing the inverse form of the local condition (20) in definition (94) for the generalized volumetric strain and using next the pressure approximation (52). In the resulting expression,

\[ e = -F \tilde{X} - e_{f0} \]  \hspace{1cm} (107)

\( F \) is the Hermitian stiffness matrix of the element and \( e_{f0} \) the term that accounts for the contribution of the particular solution:

\[ F = \int P^{*}K^{-1}P \, dV^e \]  \hspace{1cm} (108)

\[ e_{f0} = \int P^{*}K^{-1}p_0 \, dV^e \]  \hspace{1cm} (109)

7.6 Solving system

The finite element solving system is obtained eliminating the generalized volumetric strain in the compatibility condition (98) using the constitutive relation (107) and adding to the resulting equation the boundary equilibrium condition (102):

\[
\begin{bmatrix}
    D^* & -A_z & -A_p & -A_i \\
    -A_z & j\omega K_z & 0 & 0 \\
    -A_p & 0 & 0 & 0 \\
    -A_i & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
    X \\
    w_z \\
    w_p \\
    w_i
\end{bmatrix}
= 
\begin{bmatrix}
    \overline{e} - e_{f0} \\
    \pi_{z0} \\
    \pi_{p0} - \pi_p \\
    \pi_{i0} - \pi_i
\end{bmatrix}
\]  \hspace{1cm} (110)

Constraint (54) can be used to show that matrix,

\[ D^* = -F + j\omega C^* + \omega^2 M^* \]  \hspace{1cm} (111)

is the conjugate transpose of the element dynamic matrix (92). It is shown below that this matrix and the particular solution dependent term,

\[ e_{f0} = \overline{e}_0 - e_{f0} \]  \hspace{1cm} (112)

can be defined using alternative boundary integral expressions.

8. Trefftz constraint

According to the results presented in Section 4, the Trefftz constraint on the domain approximations (50) to (52) implies the compatibility conditions (53) and (56), the equilibrium conditions (55) and (58), and the constitutive relations (56) and (57). These constraints are used below to replace by boundary integral expression the domain integral definitions (92), (93), (111) and (112). The results summarized in Sections 6 and 7 show that all remaining terms
present in systems (91) and (110) have boundary integral expressions. This feature is typical of
the Trefftz variant of the finite element method.

8.1 Dynamic matrix
Definition (89) can be written as follows using constraints (53) and (54),
\[ K = \int (\nabla^T U)^* P \, dV^e \] (113)
to yield the following expression after integration by parts:
\[ K = - \int U^T \nabla P \, dV^e + \int (n^T U)^* P \, d\Gamma^e \] (114)

Enforcing constraint (55) in the equation above, the following relation is obtained recalling
definitions (76) and (77) for the mass and damping matrices, respectively:
\[ K = \omega^2 M - j\omega C + \int (n^T U)^* P \, d\Gamma^e \] (115)

Therefore, according to equation (92) the boundary integral definition of the dynamic matrix
is:
\[ D = \int (n^T U)^* P \, d\Gamma^e \] (116)

8.2 Particular solution terms
Definition (90) can be written as follows using constraints (53) and (57),
\[ s_{k_0} = \int (\nabla^T U)^* p_0 \, dV^e \] (117)
to yield the following expression after integration by parts:
\[ s_{k_0} = - \int U^T \nabla p_0 \, dV^e + \int (n^T U)^* p_0 \, d\Gamma^e \] (118)

Enforcing constraint (58) in the equation above, the following boundary integral expression is
found for definition (93) recalling expression (70) for the generalized body force and the
associated term (78) induced by the particular solution:
\[ s_0 = \int (n^T U)^* p_0 \, d\Gamma^e \] (119)

Similarly, definition (109) can be written as follows using constraints (56) and (57),
\[ e_{F_0} = \int P^* \nabla^T u_0 \, dV^e \] (120)
to yield the following expression after integration by parts:
\[ e_{F_0} = \int (\nabla P)^* u_0 \, dV^e - \int P^* n^T u_0 \, d\Gamma^e \] (121)

The following boundary integral expression results directly from definitions (101) and (112):
\[ e_0 = \int P^* n^T u_0 \, d\Gamma^e \] (122)
It is noted that the expression above is incomplete when the particular solution involves a strong singularity, as the one present in the modelling (49) of point radiators, taking now the following form,

\[
e_0 = \int P^* n^T u_0 \, d\Gamma^e + j q_0 \omega^{-1} \overline{P}^e
\]  

(123)

where \( \overline{P}^e \) is the pressure vector computed at the source of the radiator and \( q_0 \) the amplitude of the (Dirac) velocity dislocation at that point.

9. Conventional pressure element

The derivation presented in Section 7 holds, in general terms, for the conventional formulation of pressure elements. The main difference is that formal solutions (31) or (39) of the governing differential equation (28) are abandoned. The pressure approximation basis is still written in form (52) with two main differences. The particular solution corrective term, \( p_0 \), is discarded and the pressures modes are defined using the polynomial approximation functions defined for isoparametric elements,

\[
p = P p_N \quad \text{in} \, V^e
\]  

(124)

meaning that vector \( p_N \) defines now the value of the pressure field at the element nodes.

Due to the way the isoparametric approximation are defined, continuity of pressure at the element nodes ensures the continuity of the pressure on the interfaces of connecting elements, rendering it unnecessary to approximate the normal displacement component. That is, approximations (65) are not called upon.

In the weak form (97) of the compatibility condition, the terms associated with pressure and interelement boundaries are interpreted as the definition of (generalized) nodal displacements,

\[
e = -\int (\nabla P)^* u \, dV^e + \int P^* n^T u \, d\Gamma^e + e_p + e_i + \bar{e}_n
\]  

(125)

where definition (100) still holds and:

\[
e_\alpha = \int P^* n^T u \, d\Gamma^e + \quad \text{with} \quad \alpha = p, i
\]  

(126)

In order to satisfy the equilibrium condition (18) locally, this condition is used to define the displacement approximation (50), with \( u_0 = 0 \),

\[
u = U \, p_N \quad \text{in} \, V^e
\]  

(127)

consistent with the pressure approximation,

\[
U = (\omega^2 \rho - j \omega \gamma)^{-1} \nabla P \quad \text{in} \, V
\]  

(128)

to yield, according to definitions (76) and (77):

\[
e = -\left(\omega^2 M^* + j \omega C^*\right) p_N + \int P^* n^T u \, d\Gamma^e + e_p + e_i + \bar{e}
\]  

(129)
Finally, the impedance condition (23) is enforced locally for the assumed pressure approximation,
\[ e = -(\hat{\omega}^2 M^* + j \hat{\omega} C^* + j \hat{\omega}^{-1} F_z) p_N + e_p + e_i + \vec{e} \]  
(130)
to obtain the expression that replaces definition (86) for the admittance matrix:
\[ F_z = \int P^* \tilde{Z}^{-1} P d \Gamma_z \]  
(131)

As the weak enforcement (107) of the constitutive relation (20) remains valid, provided that the particular solution term is removed, the governing system that replaces definition (110) for the hybrid pressure element is:
\[ D^* p_N = e_N \]  
(132)
\[ D^* = -F + j \hat{\omega}^{-1} F_z + j \hat{\omega} C^* + \hat{\omega}^2 M^* \]  
(133)
\[ e_N = e_p + e_i + \vec{e} \]  
(134)

It is noted that the isoparametric approximation functions are real, meaning that the system is symmetric: \( D^* = D^T = D \). Moreover, it is now impossible to obtain boundary integral expressions for the system coefficients, where now vector \( e_N \) defines equivalent nodal displacements. System (132) is assembled using a technique in every way similar to the one used in the implementation of the conventional (conform) displacement formulation of the finite element method for solid mechanics problems.

10. Geometry of polytope elements

It is assumed, for simplicity, that domain \( V \) is discretized in polytopes, as illustrated in Figure 2. Each planar surface is assumed to be quadrangular, with unit external normal \( n \). Shown in the same figure is the master element used to implement the integral expressions presented above for the coefficients present in the solving systems of the hybrid-Trefftz displacement and stress elements.

Letting \( r^T = \{ x, y, z \} \) define the position vector and \(-l \leq \zeta \leq +l\), the parametric equation of the 4-node planar surface is defined in form:
\[ r = r_0 + A_1 \zeta_1 + A_2 \zeta_2 + A_{12} \zeta_1 \zeta_2 \]  
(135)
The definitions of the surface coefficients are given in Appendix 2, where general expressions are presented for the determination of the outward normal and of the surface area.
It is convenient, for numerical stability, to refer the domain approximation bases (50), (51) and (52) to the centre of gravity of the finite element. The expressions that can be used to determine its coordinates for a polytope are summarized in Appendix 3.

11. Integration on polytopes

Two main cases should be distinguished in the implementation of the integral expressions,

$$I = \int f \, d\Gamma$$

summarized in Sections 6 to 8, namely when the functions involved are regular and singular.

The function is regular in all terms involving the approximation functions (39), (43) and (61) or (62). The surface mapping (135) is used to express integral (136) in form,

$$I = \frac{1}{4} \Omega \int_{-1}^{1} \int_{-1}^{1} f(r) \, d\zeta_2 \, d\zeta_1$$

where \(\Omega\) represents the area of the polytope face. This integral can be computed using a Gaussian quadrature rule,

$$I = \frac{1}{4} \Omega \int_{-1}^{1} \int_{-1}^{1} f(r) \, d\zeta_2 \, d\zeta_1 = \frac{1}{4} \Omega \sum_{i=1}^{m} \sum_{j=1}^{m} w_i w_j \, f(\gamma_i, \gamma_j)$$

where \(\gamma_i \in ]-1, +1[\) and \(w_i\) are the Gauss points and weights, respectively. It is recalled that the \(m\)-point Gauss rule integrates exactly polynomials of degree \(2m-1\). For 3-node (triangular) faces (see Appendix 2), the expression above is replaced by the following,

$$I = 2 \Omega \int_{0}^{1} \int_{0}^{1} f(r) \, d\zeta_2 \, d\zeta_1 = 2 \Omega \sum_{i=1}^{3} w_i \, f(\gamma_i)$$
with different integration rules for triangular domains being proposed in the literature.

A clear advantage in the use of the Fourier basis for regular polytopes is the possibility of obtaining analytical expressions for the surface integrals (137), which can be reduced to form,

\[ I = \frac{1}{4} \Omega \int_{\Omega} \left[ \alpha (1 + \zeta_1) \right] dA \int_{\Omega} \left[ \beta (1 + \zeta_2) \right] dA \int_{\Omega} \left[ \gamma (1 + \zeta_3) \right] dA \]

where \( \exp [\gamma (1 + \zeta_3)] \) represents now the term that is constant at the particular surface on which the integration is performed. The logic that supports the implementation is rather more involved, as it demands the careful identification of the mapping of the functions on the polytope faces to define the adequate coefficients \( \alpha, \beta \) and \( \gamma \).

Expression (123) for the integration of strongly singular terms is justified in Appendix 4.

12. Closure

The bases used here are defined for interior problems. They can be extended to model exterior problems by taking the appropriate solution of the governing Helmholtz equation (25). To extend the implementation of the hybrid-Trefftz displacement and stress models to other that polytope elements it suffices to replace the planar description (135) of the bounding surfaces by appropriate parametric descriptions.

References


Appendix 1: Spherical coordinates

The following basic information on spherical coordinates is taken from Ref. [2], with the notation defined in Figure 1.

The spherical coordinates \((r, \theta, \phi)\) are related to the Cartesian coordinates \((x, y, z)\) by,

\[ r = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \tan^{-1}(y/x), \quad \phi = \cos^{-1}(z/r) \]
where \( r \in [0, \infty] \), \( \theta \in [-\pi, +\pi] \), and \( \phi \in [0, \pi] \), and the inverse tangent must be suitably defined to take the correct quadrant of \((x, y)\) into account. In terms of Cartesian coordinates:
\[
    x = r \cos \theta \sin \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \phi
\]

The line, area and volume elements are:
\[
    ds = dr \hat{r} + r \, d\phi \hat{\phi} + r \sin \phi \, d\theta \hat{\theta}, \quad da = r^2 \sin \phi \, d\phi \, d\theta \hat{r}, \quad dV = r^2 \sin \phi \, d\phi \, d\theta \, dr
\]

The Jacobian is:
\[
    \begin{vmatrix}
        \frac{\partial(x, y, z)}{\partial(r, \theta, \phi)}
    \end{vmatrix} = r^2 \sin \phi
\]

The position vector is,
\[
    \mathbf{r} = \begin{bmatrix}
        r \cos \theta \sin \phi \\
        r \sin \theta \sin \phi \\
        r \cos \phi
    \end{bmatrix}
\]

so the unit vectors are:
\[
    \hat{r} = \frac{dr}{dr} \begin{bmatrix}
        \cos \theta \sin \phi \\
        \sin \theta \sin \phi \\
        \cos \phi
    \end{bmatrix}, \quad \hat{\theta} = \frac{d\theta}{dr} \begin{bmatrix}
        -\sin \theta \\
        \cos \theta
    \end{bmatrix}, \quad \hat{\phi} = \frac{d\phi}{dr} \begin{bmatrix}
        \cos \theta \cos \phi \\
        \sin \theta \cos \phi \\
        -\sin \phi
    \end{bmatrix}
\]

Derivatives of the unit vectors are:
\[
    \frac{\partial \hat{r}}{\partial r} = 0, \quad \frac{\partial \hat{\theta}}{\partial r} = 0, \quad \frac{\partial \hat{\phi}}{\partial r} = 0
\]
\[
    \frac{\partial \hat{r}}{\partial \theta} = \sin \phi \hat{\theta}, \quad \frac{\partial \hat{\theta}}{\partial \theta} = -\cos \phi \hat{\phi} - \sin \phi \hat{r}, \quad \frac{\partial \hat{\phi}}{\partial \theta} = \cos \phi \hat{\theta}
\]
\[
    \frac{\partial \hat{r}}{\partial \phi} = \hat{\phi}, \quad \frac{\partial \hat{\theta}}{\partial \phi} = 0, \quad \frac{\partial \hat{\phi}}{\partial \phi} = -\hat{r}
\]

The gradient is,
\[
    \nabla = \hat{r} \frac{\partial}{\partial r} + \frac{1}{r} \hat{\phi} \frac{\partial}{\partial \phi} + \frac{1}{r \sin \phi} \hat{\theta} \frac{\partial}{\partial \theta}
\]

and its components are:
\[
    \nabla \cdot \hat{r} = 0, \quad \nabla \cdot \hat{\theta} = -\frac{\cot \phi}{r} \hat{r} - \frac{1}{r} \hat{\phi}, \quad \nabla \cdot \hat{\phi} = 0
\]
\[ \nabla_r \hat{\phi} = 0, \nabla_0 \hat{\phi} = \frac{\cot \phi}{r}, \nabla_\phi \hat{\phi} = -\frac{1}{r} \]

The divergence is, in vector notation:

\[ \nabla \cdot \mathbf{F} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 F_r) + \frac{1}{r \sin \phi} \frac{\partial}{\partial \phi} (\sin \phi F_\phi) + \frac{1}{r \sin \phi} \frac{\partial F_\theta}{\partial \theta} \]

The curl is:

\[ \nabla \times \mathbf{F} = \frac{1}{r \sin \phi} \left[ \frac{\partial}{\partial \phi} (\sin \phi F_\phi) - \frac{\partial F_\phi}{\partial \theta} \right] \hat{r} + \frac{1}{r} \left[ \frac{1}{r} \frac{\partial F_r}{\partial \theta} - \frac{\partial (r F_\phi)}{\partial r} \right] \hat{\phi} + \frac{1}{r} \left[ \frac{\partial (r F_\theta)}{\partial \phi} - \frac{\partial F_r}{\partial \phi} \right] \hat{\theta} \]

The Laplacian is:

\[ \nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \phi} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2 \sin \phi} \frac{\partial}{\partial \phi} \left( \sin \phi \frac{\partial}{\partial \phi} \right) \]

To express partial derivatives with respect to Cartesian axes in terms of partial derivatives of the spherical coordinates:

\[
\begin{bmatrix}
\frac{dx}{dr} & \frac{dy}{dr} & \frac{dz}{dr} \\
\frac{dx}{d\theta} & \frac{dy}{d\theta} & \frac{dz}{d\theta} \\
\frac{dx}{d\phi} & \frac{dy}{d\phi} & \frac{dz}{d\phi}
\end{bmatrix}
\]

Upon inversion, the result is:

\[
\begin{bmatrix}
\frac{dr}{dx} & \frac{dr}{dy} & \frac{dr}{dz} \\
\frac{d\theta}{dx} & \frac{d\theta}{dy} & \frac{d\theta}{dz} \\
\frac{d\phi}{dx} & \frac{d\phi}{dy} & \frac{d\phi}{dz}
\end{bmatrix}
\]

The Cartesian partial derivatives in spherical coordinates are therefore:

\[
\frac{\partial}{\partial x} = \cos \theta \sin \phi \frac{\partial}{\partial r} - \frac{\sin \theta}{r \sin \phi} \frac{\partial}{\partial \theta} + \cos \theta \cos \phi \frac{\partial}{\partial \phi}
\]

\[
\frac{\partial}{\partial y} = \sin \theta \sin \phi \frac{\partial}{\partial r} + \frac{\cos \theta}{r \sin \phi} \frac{\partial}{\partial \theta} + \sin \theta \cos \phi \frac{\partial}{\partial \phi}
\]

\[
\frac{\partial}{\partial z} = \cos \phi \frac{\partial}{\partial r} - \frac{\sin \phi}{r} \frac{\partial}{\partial \phi}
\]

**Appendix 2: Mappings of the faces of polytopes**

In equation (135), \( r_0 \) is the position of the centre point of the master element,

\[
r_0 = r (\zeta_1 = 0, \zeta_2 = 0) = \frac{1}{4} (r_1 + r_2 + r_3 + r_4)
\]

and the remaining coefficients define the gradients of the surface at the same point:
Thus, the definition of the outward normal vector is,
\[ N_1 \times N_2 = 0 \]
and can be used to confirm that the four vertices belong to the plane,
\[ \Omega = 2 \left( |N_1| + |N_3| \right) \]
and its unit outward normal, e.g.:
\[ n = \frac{N_1}{|N_1|} \]

The mapping (135) can be inverted for a particular position vector, \( r(\zeta_1, \zeta_2) = \mathbf{r} \), to yield,
\[
\begin{align*}
\begin{bmatrix}
\zeta_1 \\
\zeta_2 \\
\zeta_3
\end{bmatrix}
&= \begin{bmatrix}
A_1^T A_1 & A_1^T A_2 & A_1^T A_4 & A_1^T A_2 \\
A_2^T A_1 & A_2^T A_2 & A_2^T A_4 & A_2^T A_2 \\
A_4^T A_1 & A_4^T A_2 & A_4^T A_4 & A_4^T A_2
\end{bmatrix}^{-1}
\begin{bmatrix}
A_1^T (\mathbf{r} - \mathbf{r}_0) \\
A_2^T (\mathbf{r} - \mathbf{r}_0) \\
A_4^T (\mathbf{r} - \mathbf{r}_0)
\end{bmatrix}
\end{align*}
\]

with \( \zeta_3 = \zeta_1 \zeta_2 \) for consistency, or, for faces with \( A_{12} = 0 \), (e.g. rectangular faces):
\[
\begin{align*}
\begin{bmatrix}
\zeta_1 \\
\zeta_2 \\
\zeta_3
\end{bmatrix}
&= \begin{bmatrix}
A_1^T A_1 & A_1^T A_2 & A_1^T A_4 \\
A_2^T A_1 & A_2^T A_2 & A_2^T A_4 \\
A_4^T A_1 & A_4^T A_2 & A_4^T A_4
\end{bmatrix}^{-1}
\begin{bmatrix}
A_1^T (\mathbf{r} - \mathbf{r}_0) \\
A_2^T (\mathbf{r} - \mathbf{r}_0) \\
A_4^T (\mathbf{r} - \mathbf{r}_0)
\end{bmatrix}
\end{align*}
\]

The results above can be easily specialized for a triangular face, letting in equation (135),
\[
\begin{align*}
\mathbf{r}_0 &= \mathbf{r}_1 \\
A_1 &= \mathbf{r}_2 - \mathbf{r}_1 \\
A_2 &= \mathbf{r}_3 - \mathbf{r}_1 \\
A_{12} &= 0
\end{align*}
\]
where now \( 0 \leq \zeta_1 \leq 1 \) and \( 0 \leq \zeta_2 \leq 1 - \zeta_1 \) are the natural coordinates of the 3-node master element. Consequently, the definition of the area of the surface is:
\[ \Omega = \frac{1}{2} |N| \]

**Appendix 3: Centre of gravity of polytopes**

The coordinates \( \overline{r}^T = \{ \overline{x}, \overline{y}, \overline{z} \} = \{ \overline{x}_1, \overline{x}_2, \overline{x}_3 \} \) of the centre of gravity of a solid with volume,

\[ V = \int dV \]

are defined by:

\[ \overline{x}_i = V^{-1} \int x_i dV \]

The expressions above can be written in form in the following form, where \( \nabla \) is the gradient vector, and \( r^T = \{ x, y, z \} = \{ x_1, x_2, x_3 \} \) is the position vector (\( \nabla^T r = 3 \)):

\[ V = \frac{1}{3} \int \nabla^T r \ dV \]

\[ \overline{x}_i = \frac{1}{3} V^{-1} \int x_i \nabla^T r \ dV \]

After integration by parts, the following boundary integral expressions are found,

\[ V = \frac{1}{3} \sum_{j=1}^{N} \int n^T r \ d\Gamma_j \]

\[ \overline{x}_i = \frac{1}{3} V^{-1} \sum_{j=1}^{N} \int n^T r \ x_i \ d\Gamma_j \]

where \( N \) is the total number of planar surfaces of the polytope. The expressions above are computed using the general expressions presented in Section 8 for quadrangular planar surfaces.

**Appendix 4: Integration of point radiator solutions**

The point radiator solution (48) is defined for \( r \geq a \), with \( a \) representing the radius of the source. As this definition does not include the origin of the source, \( r = 0 \), where the displacement (and velocity field) are strongly singular (a Dirac distribution), it is possible to integrate by parts definition (109), to yield,

\[ e_{f_0} = \int P^* \mathbf{K}^{-1} \rho_0 dV^c = -\int (\nabla \mathbf{P})^T \mathbf{u}_0 dV^c + \int P^* n^T \mathbf{u}_0 d\Gamma^c \]

or, according to definition (101),

\[ e_{f_0} + \bar{e}_0 = \int P^* n^T \mathbf{u}_0 d\Gamma^c + \int P^* n^T \mathbf{u}_0 d\Gamma_{\text{int}}^c + \int P^* n^T \mathbf{u}_0 d\Gamma_{\text{a}}^c \]

where \( \Gamma^c \) is the surface of the element and \( \Gamma_{\text{int}}^c \) is the internal double surface that cuts through the sphere \( \Gamma_{\text{a}}^c \) centred on the point radiator, as shown for a plane cut in Figure 3.
The second term in the equation above is null because the pressure and displacement fields, \( P \) and \( \mathbf{u}_0 \), are continuous through the cut, where on the normal, \( \mathbf{n} \), as opposite signs:

\[
\int P^* \mathbf{n}^\top \mathbf{u}_0 \, d\Gamma^{e}\text{int} = 0
\]

The third term is written as follows,

\[
\int P^* \mathbf{n}^\top \mathbf{u}_0 \, d\Gamma^{e}\text{int} = \int (P - \overline{P})^* \mathbf{n}^\top \mathbf{u}_0 \, d\Gamma^{e}\text{a} + \overline{P}^* \int \mathbf{n}^\top \mathbf{u}_0 \, d\Gamma^{e}\text{a}
\]

where \( \overline{P} \) defines the pressure field at the origin of the source. The first term tends to zero as the radius of the radiator decreases and the second can be integrated analytically on the spherical surface using definition (49), to yield the following result when the wave number definition (27) is used for null damping:

\[
\int \mathbf{n}^\top \mathbf{u}_0 \, d\Gamma^{e}\text{a} = \int u_{r0} \, d\Gamma^{e}\text{a} = \left[ r^2 u_{r0} \right]_{r=a} \int_0^\pi sin\phi \, d\phi \int_{-\pi}^{\pi} d\theta = -j \frac{q_0}{\omega}
\]

This result recovers the identification of \( q_0 \) as the magnitude of the velocity dislocation:

\[
\int \mathbf{n}^\top \mathbf{v}_0 \, d\Gamma^{e}\text{a} = j \omega \int \mathbf{n}^\top \mathbf{u}_0 \, d\Gamma^{e}\text{a} = q_0
\]