HYBRID-MIXED FINITE ELEMENT MODELLING OF THE EARLY THERMAL RESPONSE OF CONCRETE

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Abstract. The presentation addresses the modelling of the early age thermal response of concrete using a hybrid-mixed finite element formulation based on the direct approximation of the temperature and degree of hydration fields in the domain of the elements, and the heat flux on their boundary. The governing equations are stated first, to identify the assumptions made and to establish the notation. The incremental form (in time) of these equations is stated next and then the time dimension of the problem is discretized using a trapezoidal integration rule. The hybrid-mixed finite element formulation is derived next using a Galerkin weighed residual approach. The essential numerical implementation aspects are addressed next, namely the scaling of the governing system of differential equations, the definition of the master element mapping and of the independent functions used in the approximation of the temperature, hydration and heat flux fields, the computation of coefficients of the finite element solving system and the enforcement of the initial conditions of the problem. The presentation closes with the assessment of the numerical results obtained.
1. INTRODUCTION

The presentation addresses the modelling of the thermal response of young concrete [4, 5]. This study has been motivated by the research recently reported by Azenha [1], where the generation of heat during hydration of several cements used in Portugal and the development of mechanical properties were obtained through an extensive experimental campaign. The experimental results have been used to calibrate numerical models based on the conventional formulation of the finite element method.

The option reported here consists in approximating the temperature and degree of hydration fields in the domain of the elements and of the heat flux on their boundary. Having in mind the ultimate goal of developing a coupled thermo-mechanical model, the main objective here is to ensure a tighter control on the modelling of the variation of the temperature and hydration degree fields. The formulation is designed to be implemented using coarse meshes of macro-elements and naturally hierarchical bases to enhance adaptive refinement and parallelization.

The presentation is organized in four main parts. The mathematical model is defined first, to identify the assumptions and establish the notation. The criteria used to approximate the solution in time and in space are presented next. The hybrid-mixed finite element formulation is derived, which enhances the main conceptual differences regarding the conventional (conform) finite element formulation. After commenting briefly on the main numerical implementation aspects, the closing section addresses the assessment of the performance of the formulation using a relatively wide set of tests, ranging from sensitivity to distortion to accuracy, as well as on the enforcement of the domain and boundary conditions of the problem.

2. MATHEMATICAL MODEL

The system of equations governing the transient thermal response of a concrete specimen, with domain $V$, boundary $\Gamma$ and unit outward normal $n$, can be stated as follows [3], using a time frame $t$ and a Cartesian reference system $x = (x_1, x_2, x_3)$:

$$ -\nabla^T \sigma + \dot{Q} = \rho c \dot{T} \quad \text{in} V $$

$$ \dot{\varepsilon} = \nabla T \quad \text{in} V $$

$$ \sigma = -k \varepsilon \quad \text{in} V $$

$$ n^T \sigma = \bar{q} \quad \text{on} \Gamma_{q} $$

$$ T = \bar{T} \quad \text{on} \Gamma_{T} $$

$$ n^T \sigma = h_{cr} (T - T_a) \quad \text{on} \Gamma_{cr} $$

In the equations above, $T$ is the temperature field, and $\dot{T}$ and $\dot{\varepsilon}$ define its time and space gradients, respectively, with $\nabla = \{ \partial_{x_1}, \partial_{x_2}, \partial_{x_3} \}^T$. 
In the thermal equilibrium condition (1), vector $\sigma$ defines the velocity of heat flow, $\rho c$ is the volumetric specific heat and $\dot{Q}$ is the heat release rate due to hydration. It is defined by the Arrhenius law,

$$\dot{Q} = A_r f(\alpha) \exp[-\beta(T)]$$  \hspace{1cm} (7)

$$\beta(T) = \frac{E_a}{RT}$$  \hspace{1cm} (8)

where $E_a$ is the activation energy, $R$ is the universal gas constant, $A_r$ is the maximum value of the heat production rate and function $f(\alpha)$ defines the evolution of the normalised heat production rate as a function of the hydration degree, $\alpha$. The hydration degree at instant $t$ is defined as the ratio of heat released up to that instant and the total heat expected upon completion of cement hydration, $Q_c$:

$$\alpha(t) = \frac{Q(t)}{Q_c}$$  \hspace{1cm} (9)

In the last domain condition, the constitutive condition (3), $k$ is the thermal conductivity of concrete. It is defined as follows [8], where $k_c$ is the value of the thermal conductivity of concrete in its hardened state:

$$k = k_c (1.33 - 0.33 \alpha)$$  \hspace{1cm} (10)

In the Neumann condition (4), $n$ is the unit outward normal vector and $\bar{q}$ defines the prescribed heat flux. In the Dirichlet condition (5), $T$ is the prescribed temperature. The Robin-type boundary condition (6), where $T_a$ is the air temperature, models radiation and convection using the Newton’s cooling law and a single convection-radiation coefficient:

$$h_r = h_c + h_f$$  \hspace{1cm} (11)

The convection coefficient is given by,

$$h_c = \begin{cases} 5.6 + 3.95 v & v \leq 5 \text{ m/s} \\ 7.6 v^{0.78} & v > 5 \text{ m/s} \end{cases}$$

where $v$ is the wind speed at the surface of the concrete [6]. The radiation coefficient is defined as follows, where $\varepsilon$ is emissivity of concrete, which is usually in the range of 0.85-0.95 [2, 9]:

$$h_r = 4.8 \varepsilon \begin{cases} 1 + 0.015625(T_a - 278.15) & T_a \geq 278.15 \text{ K} \\ 1 & T_a < 278.15 \text{ K} \end{cases}$$

Boundary condition (6) can be easily generalized to model convective internal surfaces in layered bodies.

The initial conditions complete the definition of the thermochemical problem (1)-(6). They are stated either in terms of an initial temperature field or of its time rate:

$$T(x, t=0) = T_0(x)$$  \hspace{1cm} (12)

$$\dot{T}(x, t=0) = \dot{T}_0(x)$$  \hspace{1cm} (13)
3. INCREMENTAL FORMULATION

It is assumed that all variables are written in incremental form, say,

\[ v(x,t) = v_0(x) + \delta v(x) \tag{14} \]

for a variable \( v \) at instant \( t \), where \( v_0 \) defines its value at the beginning of the time increment, \( t_0 \). In the applications reported below, time integration is implemented with a trapezoidal rule,

\[ v = v_0 + \gamma_0 \delta t \dot{v}_0 + \gamma \delta t \ddot{v} \tag{15} \]

where \( \dot{v}_0 \) and \( \ddot{v} \) represent the values of the time derivative of the variable at instant \( t_0 \) (start of time step) and at instant \( t = t_0 + \delta t \) (end of time step), respectively, and \( \gamma \) and \( \gamma_0 \) are time integration factors (e.g., \( \gamma = 1 \) and \( \gamma_0 = 0 \) for the backward-Euler integration rule and \( \gamma = \gamma_0 = \frac{1}{2} \) for the Crank-Nicolson method).

Under condition (14) and approximation (15), the following incremental description is found for the incremental form of system (1)-(7), under definition (9):

\[ \rho \epsilon \delta T + \gamma \delta t \nabla \cdot \delta \sigma - Q_0 \delta \alpha = - (\gamma + \gamma_0) \delta t \nabla \cdot \sigma_0 \text{ \ in } V \tag{16} \]

\[ A_{\alpha_0} \delta \alpha = f_0 \delta T + R_{\alpha_0} + R_{\alpha} \text{ \ in } V \tag{17} \]

\[ \delta \epsilon = \nabla \delta T \text{ \ in } V \tag{18} \]

\[ \delta \sigma = - k_0 \delta \epsilon + R_k \text{ \ in } V \tag{19} \]

\[ n^T \delta \sigma = \delta \bar{q} \text{ \ on } \Gamma_q \tag{20} \]

\[ \delta T = \delta \bar{T} \text{ \ on } \Gamma_T \tag{21} \]

\[ n^T \delta \sigma = h_\nu \delta T + R_{\sigma_0} \text{ \ on } \Gamma_{cr} \tag{22} \]

The hydration coefficient and the residual terms present in the incremental form (17) of the Arrhenius law (7), are defined as follows,

\[ A_{\alpha_0} = \frac{T_0}{\beta_0} \left( \frac{Q_e \exp(\beta_0)}{A_T \gamma} \delta t - f_0' \right) \tag{23} \]

\[ R_{\alpha_0} = \frac{T_0}{\beta_0} f_0 \left( 1 + \frac{\gamma}{\gamma_0} \right) \tag{24} \]

\[ R_{\alpha} = f_0 T_0 R_A + f_0' \delta \alpha \left( \delta T + T_0 R_A \right) \tag{25} \]

where \( f_0 \) and \( f_0' \) represent the value of the hydration function and of its derivative at instant \( t_0 \). Parameter \( \beta_0 \) is determined from equation (8) at the same moment, and the Taylor series residual associated with the exponential term present in the Arrhenius law is:

\[ R_A = - \left( \frac{\delta T}{T_0} \right)^2 \left[ (1 - \frac{1}{2} \beta_0) - (1 - \beta_0 + \frac{1}{2} \beta_0^2) \frac{\delta T}{T_0} + \cdots \right] \]

Result (25) holds for piecewise linear descriptions of the hydration function. The definition
for (eventually, piecewise) nonlinear functions is:

\[ R_\alpha = -f_0 (\delta T + T_0 / \beta_0) - f'_0 \delta \alpha (T_0 / \beta_0) + (T_0 R_\delta + \delta T + T_0 / \beta_0) f(\alpha) \]

In equation (19), \( k_0 \) represents the thermal conductivity (10) at instant \( t_0 \) and the definition of the residual term is:

\[ R_k = 0.33 k_0 \delta \alpha (\varepsilon_0 + \delta \varepsilon) \]

In equation (22), \( h_{cr} \) represents the heat transfer coefficient (11) at the end of the time interval, \( t_0 + \delta t \), and residual term is defined by,

\[ R_{h0} = -h_{cr0} \delta T + \delta h_{cr} (T_0 - T_a) \]

where \( h_{cr0} \) and \( \delta h_{cr} \) define the same coefficient at the onset of the increment and its variation during the time increment, respectively.

4. HYDRATION FUNCTION

The model defined in the above sections, taken directly from [3], does not contain a water conservation equation, it being assumed that water is present at every point in quantities sufficient to ensure maturation. Besides this limitation, two other aspects of the assumed hydration law justify comments on formal consistency. The first is the local nature of condition (7) and the second the end conditions usually used in the definition of the hydration function, \( f(\alpha) \). These issues are discussed in detail in [5].

As the hydration law is not gradient dependent, the finite element formulation of the governing system of equations (1)-(7), or (16)-(22), cannot include the continuity condition on the hydration degree field. In the implementation of the local model implied by equations (7) and (17), the inter-element continuity condition on this field is enforced indirectly through that definition and condition (30).

In what regards the analytical definition of the experimentally determined hydration function, \( f(\alpha) \), different options are presented in the literature. For instance, a piecewise linear approximation is used in [1, 3],

\[ f(\alpha) = f_0 + f'_0 \delta \alpha \]

and a continuous fit is suggested, for instance, in [7]:

\[ f(\alpha) = a \alpha^b (1 - \alpha)^c \]

Both models meet the initial condition \( f(\alpha = 0) = 0 \). Although they differ in the definition of the initial gradient, \( f'(\alpha = 0) < +\infty \) and \( f'(\alpha = 0) = +\infty \), respectively, neither of the models can emulate the triggering of the onset of hydration, which implies a non-null instantaneous heat release, \( f(\alpha = 0) > 0 \), as required by the limit of definition (17) at the origin \( \alpha = 0 \). In what regards full hydration, both models satisfy the end condition \( f(\alpha = 1) = 0 \) but only the latter satisfies condition \( f'(\alpha = 1) = 0 \), as required by the limit of definition (17) close to full hydration, \( \alpha \approx 1 \). Both models have been tested using the hybrid-mixed formulation described below, and the results obtained confirm the comments made above.
Another issue is raised in the implementation of this variant of the finite element method, is the implementation of the hydration law in form (17), under definitions (23) to (25). As shown below, this definition may destroy the symmetry of the finite element solving system but it is numerically stable both at the onset of hydration and at full hydration. The alternative form (written for the simplest, piecewise linear law),

\[ A_{a_0} \delta \alpha = \delta T + R_{a_0} + R_{a} \text{ in } V \]  

\[ A_{a_0} = \frac{T_0}{\beta_0} \left( \frac{Q_0 \exp(\beta_0)}{A_T \gamma \delta t} - f'_0 \right) \]  

\[ R_{a_0} = \frac{T_0}{\beta_0} (1 + \gamma / \gamma_0) \]  

\[ R_{a} = T_0 R_{a} + \frac{f'_0}{f_0} \delta \alpha \left( \delta T + T_0 R_{a} \right) \]  

ensures the symmetry of the finite element solving system. However, definitions (27) and (28) indicate that this form of implementing the hydration law may cause numerical instability when the hydration function tends to zero, that is, at the onset of hydration and in the vicinity of full hydration.

5. FINITE ELEMENT APPROXIMATION

Let \( V^e \) and \( \Gamma^e \) define the domain and the boundary of a typical element. Four parts are identified on the boundary, namely the interelement boundary, \( \Gamma_{i}^e \), and the parts of the Neumann, Dirichlet and convection-radiation boundaries of the mesh the element may contain,

\[ \Gamma^e = \Gamma_{i}^e \cup \Gamma_q^e \cup \Gamma_r^e \cup \Gamma_c^e \]  

on which conditions (20) to (22) hold. It is assumed that the interelement continuity condition is enforced on the temperature field,

\[ \delta T = \delta T_i \text{ on } \Gamma_{i}^e \]  

where \( \delta T_i \) represents the temperature field on the boundary of the connecting element.

The hybrid-mixed formulation reported here is based on the independent approximation of the temperature and hydration degree fields in the domain of the element,

\[ T(x) = \Psi_T \text{ in } V^e \]  

\[ \alpha(x) = \Psi_\alpha \text{ in } V^e \]  

and the heat flux on the boundaries of the element where it is not prescribed:

\[ q(x) = n^T \sigma = \Psi_q \text{ on } \Gamma_{\text{ne}} = \Gamma_{r}^e \cup \Gamma_{i}^e \]  

Besides the independent approximations (31) to (33), it is also assumed that the temperature gradient condition (18) is locally enforced:
\( \mathbf{e} = \mathbf{B}_T \mathbf{T} \text{ in } V^e \) \hspace{1cm} (34)

\( \mathbf{B}_T = \nabla \mathbf{\Psi}_T \) \hspace{1cm} (35)

In approximations above, the (row-) vectors \( \mathbf{\Psi}_T, \mathbf{\Psi}_\alpha \text{ and } \mathbf{\Psi}_q \) list the approximation functions and vectors \( \mathbf{T}, \mathbf{\alpha} \text{ and } \mathbf{t} \) define their associated weights. They represent non-nodal, generalized variables, as they define the amplitudes of the temperature, hydration degree and heat flux approximation modes. Polynomial bases are used in the numerical implementation of the formulation. Typically, Legendre polynomials to exploit their orthogonality and obtain highly sparse and well conditioned solving systems.

Approximations (31) to (33) are implemented on a master-element, with a natural coordinate system \( \xi \), where the approximation functions \( \mathbf{\Psi}_T(\xi), \mathbf{\Psi}_\alpha(\xi) \text{ and } \mathbf{\Psi}_q(\xi) \) are defined. The shape functions applied in the implementation of isoparametric elements, say \( N(\xi) \), are used here with the sole purpose of defining the finite element geometric mapping,

\[ \mathbf{x} = N(\xi) \mathbf{c}_e \]

where vector \( \mathbf{c}_e \) defines the nodal coordinates of the element in the global coordinate system.

6. FINITE ELEMENT EQUATIONS

The dual transformations of approximations (31) to (33) are used to enforce on average (in the sense of Galerkin) the thermal equilibrium condition (16), the heat of hydration law (17) and the Dirichlet conditions (21) and (30), respectively,

\[ \int \mathbf{\Psi}_T^T \left[ \rho c \frac{\partial \mathbf{T}}{\partial t} + \gamma \frac{\partial \mathbf{T}}{\partial \mathbf{T}} \right] dV^e = 0 \] \hspace{1cm} (36)

\[ \int \mathbf{\Psi}_\alpha^T Q_\alpha \left( A_{\alpha_0} \frac{\partial \mathbf{\alpha}}{\partial \mathbf{\alpha}} - \mathbf{f}_0 \frac{\partial \mathbf{T}}{\partial \mathbf{T}} - \mathbf{R}_{\alpha_0} - \mathbf{R}_\alpha \right) dV^e = 0 \] \hspace{1cm} (37)

\[ \int \mathbf{\Psi}_q^T \left( \frac{\partial \mathbf{T}}{\partial \mathbf{T}} - \mathbf{T}^c \right) d\Gamma^e = 0 \] \hspace{1cm} (38)

where now \( \frac{\partial \mathbf{T}}{\partial \mathbf{T}} \) represents either the prescribed variation of temperature on the Dirichlet boundary of the element or the variation of the temperature on a connecting element.

The weak forms of the hydration law and of the Dirichlet condition are obtained simply by enforcing approximations (31) to (33) in equations (37) and (38), to yield,

\[ \mathbf{S}_{\alpha T} \frac{\partial \mathbf{\alpha}}{\partial \mathbf{\alpha}} = \mathbf{S}_{\alpha T} \mathbf{T} = \mathbf{R}_\alpha \] \hspace{1cm} (39)

\[ \mathbf{S}_{\alpha T} \frac{\partial \mathbf{T}}{\partial \mathbf{T}} = \mathbf{R}_T \] \hspace{1cm} (40)

where the following definitions hold:

\[ \mathbf{S}_{\alpha T} = \int \mathbf{\Psi}_\alpha^T Q_\alpha A_{\alpha_0} \mathbf{\Psi}_\alpha dV^e \]

\[ \mathbf{S}_{\alpha T} = \int \mathbf{\Psi}_\alpha^T Q_\alpha \mathbf{f}_0 \mathbf{\Psi}_\alpha dV^e \]

\[ \mathbf{S}_{\alpha T} = \int \mathbf{\Psi}_\alpha^T \mathbf{\Psi}_T d\Gamma^e \]

\[ \mathbf{R}_\alpha = \mathbf{R}_{\alpha_0} + \mathbf{R}_\alpha \] \hspace{1cm} (41)
The weak form (36) of the thermal equilibriu m condition is first integrated by parts to enforce the boundary conditions of the element, as follows, where result (35) is used:

\[
\int \Psi_T^T \left[ \rho c \delta T - Q_x \delta \alpha + (\gamma + \gamma_0) \delta t \nabla^T \sigma \right] dV^e - \gamma \delta t \int B_T^\gamma \delta \sigma dV^e + \gamma \delta t \int \Psi_T^T n^T \delta \alpha d\Gamma^e = 0
\]

After enforcing the local domain conditions (18) and (19) and uncoupling the boundary term, in form (29), to enforce the local conditions (20) and (22), as well as approximation (33), the following result is found for the assumed approximations (31) and (32) for the temperature and hydration degree fields of the element,

\[
S_{TT} \delta T - S_{Tq} \delta \alpha + S_{qt}^T \delta q = \gamma \delta t \left( \bar{R}_T - \bar{R}_a \right)
\]

where the following definitions hold for the intervening matrices,

\[
S_{TT} = H + \gamma \delta t \left( K + C \right)
\]

\[
H = \int \Psi_T^T \rho c \Psi_T dV^e
\]

\[
K = \int B_T^\gamma k_0 B_T^\gamma dV^e
\]

\[
C = \int \Psi_T^T h_0 \Psi_T d\Gamma^e
\]

\[
S_{Tq} = \int \Psi_T^T Q_x \Psi_T dV^e
\]

and stipulation vectors:

\[
\bar{R}_{a0} = (1 + \gamma_0 / \gamma) \bar{R}_Q + \bar{R}_q + \bar{R}_h
\]

\[
\bar{R}_Q = \int \Psi_T^T \nabla^T \sigma_q dV^e
\]

\[
\bar{R}_q = \int \Psi_T^T \delta \tilde{q} d\Gamma_q^e
\]

\[
\bar{R}_h = \int \Psi_T^T R_{h0} d\Gamma_h^e
\]

\[
\bar{R}_h = \int B_T^\gamma R_h dV^e
\]

To avoid the computation of second derivatives implied in definition (46), the residual term is integrated by parts and result (45) is used, to yield:

\[
\bar{R}_Q = (K - P) T_0
\]

\[
P = \int \Psi_T^T k_0 n^T B_T d\Gamma^e
\]
7. FINITE ELEMENT SOLVING SYSTEM

The system defined by the finite element equations (39), (40) and (43) is not symmetric because condition $S^T_{\alpha T} \neq S_{\alpha T}$ holds. This loss of symmetry is caused by the option of implementing the hydration law in form (17) instead of the alternative description (26), to safeguard numerical stability.

To preserve symmetry, independently of the definition used for the hydration function, the weak thermal equilibrium condition (43) is implemented in form:

$$S_{\alpha T} \delta T - S^T_{\alpha T} \delta \alpha + S^T_{qT} \delta q = \overline{R}_e$$  \hspace{1cm} (48)

$$\overline{R}_e = R_{\alpha a} + \gamma \delta t \left( \overline{R}_e - \overline{R}_0 \right)$$

$$\overline{R}_{\alpha a} = \int \Psi_T Q_\alpha (I-f_0) \delta \alpha \, dV$$  \hspace{1cm} (49)

The additional residual term, $\overline{R}_{\alpha a}$, has a marginal effect on the rate of convergence of the solution of the nonlinear system equations, defined by equations (39), (40) and (48):

$$\begin{bmatrix} S_{TT} & -S^T_{\alpha T} & S^T_{qT} \\ -S^T_{qT} & S_{aa} & 0 \\ S_{qT} & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta T \\ \delta \alpha \\ \delta q \end{bmatrix} = \begin{bmatrix} \overline{R}_e \\ \overline{R}_a \\ \overline{R}_T \end{bmatrix}$$  \hspace{1cm} (50)

Besides being symmetric, this system is highly sparse, particularly when the mesh contains a significant number of geometrically regular (undistorted) elements, a feature of most applications.

These features are preserved in the solving system for the assembled finite element mesh. Moreover, the assembled system (50) is well suited to parallel processing because the temperature and hydration degrees-of-freedom, $\delta T$ and $\delta \alpha$, are strictly element-dependent, and the only shared degrees-of-freedom, the heat flux amplitudes $\delta q$, are shared at most by a pair of connecting elements.

8. NUMERICAL IMPLEMENTATION

To improve numerical stability and enhance speed of convergence, the thermochemical problem is implemented and solved in non-dimensional form, using a scaling criterion that ensures the same order of magnitude for all relevant variables involved in the calculations. In addition, the finite element solving system is also scaled prior to factorization.

Besides the length and temperature scales, $L_s$ and $T_s$, the basic scaling factors are the total heat and the thermal conductivity at full hydration, $Q_s = Q_s$ and $k_s = k_s$. The resulting scaling factors are: $\varepsilon_s = T_s / L_s$ for the temperature gradient; $\sigma_s = k_s \varepsilon_s$ for the velocity of heat flow; $t_s = Q_s L_s / \sigma_s$ for time; $(\rho c)_s = Q_s / T_s$ for the volumetric specific heat; $h_s = k_s / L_s$ for the heat transfer coefficient. To stabilize the exponential term in the Arrhenius law, the scaling used for the heat production rate is $A_s = \exp (\beta_s) Q_s / t_s$, with $\beta_s = E_s / (RT_s)$. 
The first step in numerical implementation is the solution of the initial condition of the problem, in the alternative forms (12) or (13). Three configurations are modelled, namely fully hydrated elements, $\alpha_0 = 1$, partially hydrated elements, with $\alpha_0$ computed from approximation (32), as determined in a previous analysis, and non-hydrated elements, $\alpha_0 = 0$.

After computing the time-independent terms (41) and (44), the incremental analysis is implemented using a standard procedure to solve the non-linear system of equations (50), with the usual provision of computing the increment-dependent terms at the onset of the time increment – the remaining system matrices and the residual terms subscripted $R_0$ in the equations presented in Section 6 –, and updating the solution-dependent vectors in each iteration, the residual terms (42), (47) and (49). All arrays are stored in sparse form.

The solution of system (50) is coupled with the control non-holonomic nature of hydration, $\delta \alpha \geq 0$, and with the control of the full-hydration condition, $\alpha_0 + \delta \alpha \leq 1$. These controls are implemented by collocation, at Gauss points placed on the boundary and in the domain of the hydrating elements. The full hydration condition is also checked on average,\

$$\int (\alpha_0 + \delta \alpha) \, dV^e \leq (1 - \varepsilon) \, V^e$$

for a given small tolerance, $\varepsilon$, and where $V^e$ defines the volume of the element.

9. NUMERICAL TESTS

The testing problem defined in Figure 1 is taken from [1]. The data on the cement (I42.5R) is: $\rho_c = 400 \, \text{kg/m}^3$; $k = 2.674 \, \text{W/mK}$; $\rho c = 2.4 \cdot 10^6 \, \text{J/m}^2 \, \text{K}$; $\varepsilon = 0.88$; $A_r = 2.15 \cdot 10^5 \, \text{J/m}\text{s}$; $E_a = 4.383 \cdot 10^4 \, \text{J/mol}$; $Q_v = 3.552 \cdot 10^5 \, \text{J/Kg}$. Regarding the boundary conditions identified in the same figure, the flux is assumed to be null and the convection condition is implemented assuming that the air temperature and the wind speed are constant: $T_0 = 20^\circ C$; $v = 0.3 \, \text{m/s}$.

![Figure 1. One-dimensional test (single element and 2x2 element meshes).](image)

The problem is one-dimensional under these boundary conditions, a feature that is useful to assess the implementation of complete, two-dimensional finite element bases, particularly in what concerns sensitivity to mesh distortion.

The results presented in Figures 2 and 3 illustrate the estimate of the temperature and of the hydration degree at the centre point, at different instants, using different degrees of approximation: the sequence $(d_T - d_h - d_q)$ defines the degrees used in the approximation of temperature, hydration degree and the heat flux, the latter being relevant only for the internal boundaries of the 2$\times$2 element mesh. The expected convergence patterns are recovered.
The test on sensitivity to mesh distortion, with the same boundary conditions, is implemented on a flattened domain \((1.0\times0.5)\) and measured at the most sensitive point, the centre point of the \(2\times1\) element mesh defined in Figure 4a). The results presented in the same figure show that, unexpectedly, the estimate of the degree of hydration is insensitive to gross mesh distortion. However, the results obtained for the temperature estimate recover the patterns obtained in the elastostatic tests reported in the literature: they are sensitive only to gross mesh distortion (most applications can be implemented with moderate ranges of distortion).

The variation in time of the temperature and hydration degree fields obtained for the one-dimensional test defined in Figure 1 are presented in Figures 5 and 6. The \(2\times2\) element mesh solution is presented to illustrate the enforcement of the continuity condition in both fields.
The last test is used to model, a two-phase concreting operation, still under the same set of boundary conditions. As illustrated in Figure 7, the second layer is placed immediately after the first layer is taken into full hydration. A convective boundary condition is implemented between the layers, each modelled with 2 elements and approximations of degree 5 for all fields, namely the temperature, degree of hydration and flux fields. The results obtained in the analysis of the response of the two-layer problem are presented in Figures 8 and 9.
A (null) temperature rate condition is assumed in the second stage, meaning that both layers have the same initial temperature (20°C). The results show that the heat of hydration generated in the second layer is partially transmitted to the fully hydrated layer. Its temperature raises and this variation is now two-dimensional in both layers. The discontinuity in the temperature field is caused by the inter-layer convective boundary condition. However, and throughout the process, the temperature and degree of hydration fields are continuous across the pairs of elements used to model each layer.
Figure 7. Two-layer testing problem.

Figure 8. Variation of the temperature field (two-layer test problem).
10. CLOSURE

A hybrid-mixed finite element formulation is used to model of the early age thermal response of concrete. The formulation is mixed because it is based on the direct approximation of two fields in the domain of the element, namely temperature and the degree of hydration fields. Its hybrid nature results from the option of approximating independently the heat flux on the boundary of the elements.
These options result from three main objectives, namely to directly control the modelling of the heat of hydration field, to obtain a system well-suited to present-day computation facilities, particularly in what regards manipulation of highly sparse systems and solution in parallel processing environments, and, lastly, to set up thermal model that could be easily coupled with alternative finite element models for the mechanical response of concrete structures.

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