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Mixed and hybrid stress elements for biphasic media

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1. Introduction

The application of hybrid and mixed formulations of the finite element method to the analysis of the response of mixtures has been motivated by the multi-field nature of the problem and frequently by difficulties intrinsic to the modelling of incompressibility.

Typical illustrations in the context of soft tissue modelling are the early contributions of Spilker and his co-workers [1,2]. The common feature of the formulations reported in the literature is the use of development concepts and implementation techniques that root on those of the conventional conform displacement model. The central objective of this paper is to illustrate the application of the alternative stress model of the hybrid-Trefftz finite element method to the analysis of the same family of problems.

The Trefftz concept [3] consists simply in basing the approximation on the formal solutions of the governing differential equation and to determine their weights by enforcing adequately the boundary conditions. In the present context the governing differential equation is the wave equation that models the response of incompressible and spectral and transient modelling of both bounded and unbounded domains. The performance of the element is illustrated in terms of sensitivity to distortion and to wavelength, full incompressibility and spectral and transient modelling of both bounded and unbounded domains.

The hybrid–mixed stress element for transient analysis of compressible and incompressible biphasic media is based on the approximation of the stress and pressure fields in the domain of the element and of the displacements in the solid and fluid phases independently in the domain and on the boundary. The hybrid and hybrid-Trefftz variants are derived constraining the domain approximation to satisfy locally the equilibrium condition and all domain conditions, respectively. As it is typical of stress elements, in these alternative formulations the inter-element and boundary continuity conditions are enforced on the surface forces acting on the solid and fluid phases of the medium. The hybrid-Trefftz stress element is applied to the transient analysis of two-dimensional and axisymmetric biphasic media. The performance of the element is illustrated in terms of sensitivity to distortion and to wavelength, full incompressibility and spectral and transient modelling of both bounded and unbounded domains.

The alternative hybrid–mixed, hybrid and hybrid-Trefftz formulations [5,6], as applied to the analysis of compressible and incompressible mixtures.

The formulation develops from the classical separation of variables in time and space. The time dimension is discretized first using a finite element approach designed to replace the parabolic/hyperbolic problem by an equivalent boundary value problem, which can be implemented either in the frequency domain or in the time domain [7]. The time domain applications reported here are implemented on a wavelet approximation basis.

The resulting elliptic problem is discretized next in the space dimension, using first a hybrid–mixed finite element formulation based on the direct approximation of the stress and pressure fields in the domain of the element and of the displacements in the solid and fluid phases both in the domain and on the boundary of the element. The hybrid formulation is established by constraining the domain approximation to satisfy locally the domain equilibrium condition independently in the solid and in the fluid phases of the mixture. The hybrid-Trefftz formulation is obtained next by constraining further the domain approximation basis to satisfy also the compatibility condition and the constitutive relations of the mixture. As the alternative formulations are derived from first principles, the associated energy statements are recalled and the information necessary to establish sufficient conditions for uniqueness of the finite element solutions is summarized.

The paper closes with the illustration of the application of the stress model of the hybrid-Trefftz finite element formulation to the spectral analysis of saturated soils and to the time domain analysis of incompressible soft tissues. The frequency domain applications are selected to illustrate the performance of the hybrid-Trefftz stress element in terms of sensitivity to mesh...
distortion and mesh sensitivity to wavelength, using both bounded and unbounded two-dimensional saturated soil tests. The second set of tests is selected to illustrate the quality of the results that are obtained when the hybrid-Trefftz stress element is used to model the response in time of two-dimensional and axisymmetric incompressible soft tissue specimens.

2. Discretization in time

Let \( x \) and \( t \) denote the time and space dimensions of the problem, respectively. In the time integration procedure used here, all variables are approximated independently in the time dimension using a hierarchical basis \( T_m(t) \), with dimension \( N \) and support \( 0 \leq t \leq \Delta t \).

In particular, the displacement and velocity fields are approximated in form,

\[
\mathbf{u}(x,t) = \sum_{n=1}^{N} T_m(t) \mathbf{u}_n(x) \quad (1)
\]

and the velocity definition is enforced on average in the time interval,

\[
\int_{0}^{\Delta t} \overline{\mathbf{m}} \mathbf{v} \, dt = \int_{0}^{\Delta t} \overline{\mathbf{m}} \mathbf{u} \, dt \quad (3)
\]

and integrated by parts, to enforce the initial condition \( \mathbf{u}(x,0) = \mathbf{u}(x,0) \). The following relation is obtained,

\[
\Delta t \sum_{n=1}^{N} H_m(x) \mathbf{v}_n = \sum_{n=1}^{N} G_{mn} \mathbf{u}_n - \overline{\mathbf{m}}(0) \mathbf{u}_0 \quad \text{with} \quad m = 1, \ldots, N \quad (4)
\]

\[
\Delta t H_m = \int_{0}^{\Delta t} \overline{\mathbf{m}} \mathbf{m} \, d\mathbf{t} \quad (5)
\]

\[
G_{mn} = \left( \overline{\mathbf{m}}(\mathbf{m}) \right)_{t=-\Delta t} - \int_{0}^{\Delta t} \overline{\mathbf{m}} \mathbf{m} \, d\mathbf{t} \quad (6)
\]

where \( \overline{\mathbf{m}} \) is the complex conjugate of \( \mathbf{m} \), \( \mathbf{m} \) its time derivative, and \( i \) is the imaginary unit.

Uncoupling of the velocity integration rule (4) is essential to ensure viability of the time integration procedure. This objective is attained by constructing the time approximation basis as to ensure that the following relation holds:

\[
G_{mn} = i\Delta t H_m \mathbf{a}_n \quad (7)
\]

It can be readily verified that, under this constraint, system (4) uncouples into form,

\[
\mathbf{v}_n(x) = \frac{i}{\omega_n} \mathbf{a}_x(x) - i\omega_n^2 \mathbf{a}_t(x) \quad (8)
\]

\[
i\Delta t \mathbf{a}_n = \sum_{n=1}^{N} H_m^\dagger \mathbf{v}_n(0) \quad (9)
\]

where \( H_n^\dagger \) denotes the coefficient of the inverse of matrix \( H \), defined by Eq. (5). In parabolic problems, a similar result is obtained for the approximation of the acceleration field, \( \mathbf{a}(x, t) = \mathbf{v}(x, t) \), under the initial condition \( \mathbf{v}_x(x) = \mathbf{v}(x,0) \):

\[
\mathbf{a}(x,t) = \sum_{n=1}^{N} T_m(t) \mathbf{a}_n(x) \quad (10)
\]

\[
\mathbf{a}_x(x) = \frac{i}{\omega_n} \mathbf{v}_n(x) - i\omega_n^2 \mathbf{v}_t(x) \quad (11)
\]

As it is shown in Ref. [7], constraint (7) does not limit the type of basis that may be used and requires the solution of a simple eigenvalue problem, with the dimension of the time basis, \( N \), Polynomial, trigonometric, wavelet and radial bases are used in Ref. [7] to characterize the performance of the method in terms of (unconditional) stability and accuracy.

In the applications reported below, a high-order wavelet basis is implemented on a single (large) time step, \( \Delta t \), and the solution at any instant of the time interval is recovered from approximations (11), (2) and (10) for the displacement, velocity and acceleration fields in the mixture. Similar approximations are implemented on the stress, pressure and strain fields present in the modelling of biphasic media.

When a periodic and orthonormal Fourier time basis is used, \( T_m(t) = \exp(\omega_n t \mathbf{a}_n) \), matrix \( \mathbf{H} \), defined by Eq. (5), is the identity matrix, and matrix \( \mathbf{G} \), defined by Eq. (6), is diagonal with coefficients \( G_{mn} = \omega_n \Delta t \). Moreover, and consequent upon periodicity, all terms associated with end conditions cancel. Hence the results above hold for spectral analysis problems, by identifying \( \alpha_n \) with the spectral frequency and setting \( \omega_n = 0 \) to eliminate the influence of the initial conditions of the problem.

3. Boundary value problem

After implementation of the time integration method, the system of equations governing the response of a biphasic element with domain \( V \) and boundary \( \Gamma \), subject to a typical forcing frequency \( \omega_n \), is stated as follows (subscript \( n \) on each variable is omitted from this point onwards, to lighten the notation):

\[
\mathbf{D} \mathbf{e} + \mathbf{b} = (i\omega_n \mathbf{C} - \omega_n^2 \rho) \mathbf{u} - \mathbf{b}_0 \quad \text{in} \quad V \quad (12)
\]

\[
\mathbf{e} = \mathbf{D}^{-1} \mathbf{u} \quad \text{in} \quad V \quad (13)
\]

\[
\mathbf{e} = \mathbf{f} \sigma \quad \text{in} \quad V \quad (14)
\]

\[
\mathbf{Na} = \mathbf{t} \quad \text{on} \quad \Gamma_N \quad (15)
\]

\[
\mathbf{u} = \overline{\mathbf{u}} \quad \text{on} \quad \Gamma_D \quad (16)
\]

The explicit form of system (12)–(16) is presented in the Appendix for two-dimensional problems. It can be verified that, in the domain equilibrium and compatibility conditions (12) and (13), vector \( \sigma \) collects the \( n \)th spectral decomposition of the dependent components of the (stress and pressure) tensor, vector \( \epsilon \) collects the corresponding strain components, vector \( \mathbf{u} \) defines the components necessary to characterize the displacement field in the solid and fluid phases, and vector \( \mathbf{b} \) defines the corresponding components of the mixture body forces. Matrices \( \mathbf{c} \) and \( \rho \) collect their damping and mass terms. They are assumed to be symmetric and the divergence matrix \( \mathbf{D} \) and its conjugate, \( \mathbf{D}^\dagger \), to be linear. The constitutive relations (14), where \( \mathbf{f} \) is the local (symmetric) flexibility matrix, can be extended to include creep and stress relaxation terms. Moreover, the domain conditions (12)–(14) hold for both compressible and incompressible mixtures, by adequate definition of the operators involved.

In the Neumann condition (15), vector \( \mathbf{t} \) combines the prescribed force and pressure, and their corresponding displacement components prescribed on the complementary part of the boundary are listed in vector \( \mathbf{n} \) in the Dirichlet condition (16). The equations above can be written to account for mixed boundary conditions on \( \Gamma = \Gamma_N \cup \Gamma_D \), with \( \Gamma_N \cap \Gamma_D = \emptyset \). The inter-element boundary conditions resulting from the decomposition of the domain are stated below.

In the domain equilibrium condition (12), the body-force term associated with the initial conditions is null for spectral analysis problems and defined as follows for the non-periodic time integration procedure summarized above:

\[
\mathbf{b}_0 = \frac{i}{\omega_n}(\alpha_n \mathbf{u}_n + \mathbf{c} \mathbf{u}_n + \rho \mathbf{v}_n) \quad (17)
\]

System (12)–(16) can be used to implement time integration procedures based on trapezoidal-type rules, e.g. Ref. [8]. The equivalent
(and duly adjusted) forcing frequency, in a spectrum with dimension \( N = 1 \), is \( \omega \rightarrow -i(\gamma_{\Delta t})^{-1} \) for parabolic problems, \( \rho = 0 \).

\[
\mathbf{u} = \mathbf{u}_d + (1 - \gamma)\Delta \mathbf{u}_a + \gamma \Delta \mathbf{v}
\]

and \( \omega \rightarrow -i(\beta\Delta t)^{-1} \), with \( \omega^2 \rightarrow -(\beta\Delta t)^2 \), for hyperbolic problems, \( \rho = 0 \):

\[
\mathbf{u} = \mathbf{u}_d + \Delta \mathbf{v}_0 + \Delta \Delta \mathbf{u}_a + \beta \Delta \mathbf{a}
\]

\[
\mathbf{v} = \mathbf{v}_s + (1 - \gamma)\Delta \mathbf{u}_a + \gamma \Delta \mathbf{a}
\]

The body-force vector associated with the initial conditions has to be adjusted to the relevant rule of time integration.

### 4. Hybrid–mixed stress element

In order to clarify the notation used below, and letting from now onwards \( V \) and \( \Gamma \) in system (12)–(16) denote the domain and the boundary of a typical element, which may not be convex, simply connected or bounded, it is convenient to recall that the Neumann boundary of a stress element is necessarily non-empty, as it combines its inter-element boundary, \( \Gamma_{1} \), with the Neumann boundary of the mesh it may share, \( \Gamma_N = \Gamma_{1} \cup \Gamma_{p} \), where \( \Gamma_{1} \) and \( \Gamma_{p} \) denote the portions of the boundary where forces are prescribed on the solid phase and the pressure is prescribed on the fluid phase. The exception is the single-element discretization of a Dirichlet problem.

Still in the context of a multi-element mesh, the complementary Dirichlet boundary, \( \Gamma_{p} \) of a stress element may be empty, as it is defined as the portion of the Dirichlet boundary of the mesh shared by the element. Hence \( \Gamma_{p} = \Gamma_{1} \cup \Gamma_{w} \), where \( \Gamma_{1} \) and \( \Gamma_{w} \) define the boundaries where displacements on the solid phase and the outward normal displacement of fluid phase are prescribed.

Thus, the right–end side of the Neumann condition (15) defines either prescribed forces and pressures on the mesh Neumann boundary or the forces on the solid phase and the pressure on the boundary of a connecting element. Moreover, the displacement condition (16) holds for elements that share the Dirichlet boundary of the mesh.

#### 4.1. Approximation bases

The stress model develops from the direct approximation of the stress and pressure fields in the domain of the element, as stated by Eq. (21), where matrix \( S \) collects the approximation functions, weighted by the generalized stress vector, \( \mathbf{e}_s \), and \( \sigma^2 \) is a particular solution term:

\[
\sigma = S \mathbf{e}_s + \sigma^2 \quad \text{in} \quad V
\]

The dual transformation of this approximation, defined by Eq. (22) and where \( \mathsf{S} \) denotes the conjugate transpose of array \( S \), defines generalized strains used to enforce on (Galerkin) average the compatibility (13) and elasticity (14) conditions:

\[
\mathbf{e}_s = \int \mathsf{S} \mathbf{e} d\mathsf{V}
\]  \( \tag{22} \)

In the mixed formulation of the stress element, the displacements in the solid and fluid phases of the mixture are approximated independently in the domain of the element in a similar manner,

\[
\mathbf{u} = \mathbf{U}_a + \mathbf{u}^f \quad \text{in} \quad V
\]  \( \tag{23} \)

and its dual transformation defines generalized body forces that are used to enforce on average the domain equilibrium condition (12):

\[
\mathbf{b}_a = \int \mathsf{U}^t \mathbf{b} d\mathsf{V}
\]  \( \tag{24} \)

Thus, the generalized strains (22) and the generalized body forces (24) ensure the invariance of the inner product in the finite element mapping:

\[
\mathbf{x}_e \mathbf{e}_s = \int (\mathbf{e}^o \mathbf{e}^o)^t \mathbf{v} d\mathsf{V}
\]  \( \tag{25} \)

\[
\mathbf{x}_a \mathbf{b}_a = \int (\mathbf{u}^o \mathbf{u}^o)^t \mathbf{b} d\mathsf{V}
\]  \( \tag{26} \)

The domain approximations are complemented with the independent approximation of the solid displacements and of the normal displacement of the fluid on the Neumann boundary of the element, in form,

\[
\mathbf{u} = \mathsf{Z} \mathbf{y}_u \quad \text{on} \quad \Gamma_N
\]  \( \tag{27} \)

where matrix \( \mathsf{Z} \) collects the boundary approximation functions and the weighting vector \( \mathbf{y}_u \) defines generalized displacements. The dual transformation (28) defines generalized prescribed forces in the solid and fluid phases, which are used to enforce on average the Neumann condition (15) and ensure, also, the invariance of the inner product in the boundary finite element mapping:

\[
\mathbf{t}_a = \int \mathsf{Z} \mathbf{t}^o d\mathsf{V}_N
\]  \( \tag{28} \)

\[
\mathbf{y}_a \mathbf{t}_a = \int \mathbf{u}^o \mathbf{t}^o d\mathsf{V}_N
\]  \( \tag{29} \)

It is noted that no constraints are set a priori on the domain and boundary approximation bases (21), (23) and (27), except for linear independence and completeness. Thus, the major strength of the hybrid–mixed stress element that results from this approximation criterion is the easiness in setting up the approximation bases, typically built on naturally hierarchical orthogonal functions, which may range from polynomials to systems of wavelets.

Consequent upon this relaxation of the approximation bases, the generalized stress and displacement approximations (21) and (23) will not, in general, satisfy locally the domain equilibrium condition (12). In addition, their associated strain fields, defined by the local elasticity and compatibility conditions (14) and (13), respectively, will, in general, be disjoint.

Moreover, the forces and pressures that equilibrate the generalized stress approximation on the Neumann boundary of the element will violate, in general, the Neumann condition (15) on both the inter-element boundary of the element and on the Neumann boundary of the mesh it may share.

Similarly, the generalized displacement approximation (23) will not satisfy, in general, the continuity conditions, either on the inter-element boundary or on the Dirichlet boundary of the mesh the element may contain. In addition, the displacement estimates it produces on the Neumann boundary of the element will not be compatible, in general, with the independent boundary approximation (27).

The only conditions that are met a priori are the Dirichlet conditions on the boundary displacement approximation, as the prescribed values are assumed to hold locally on the Dirichlet boundary of the element and the same approximation (27) is enforced on boundaries shared by two connecting elements.

#### 4.2. Finite element elasticity condition

The finite element elasticity condition defines the average enforcement of the local constitutive relations. This is attained by inserting the local condition (14) in definition (22) for the generalised strain vector.

\[
\mathbf{e}_s = \int \mathsf{S} (f \mathbf{e}) d\mathsf{V}
\]  \( \tag{30} \)
to yield the following algebraic relation when approximation (21) is
used for the assumed stress and pressure fields:
\[ \mathbf{e}_s = \mathbf{F}_s \mathbf{x}_o + \mathbf{e}^d \] (31)

In this weak form of the constitutive relations, the (Hermitian) flexibility matrix and the “residual generalized strains” associated with the particular solution term are defined by:
\[ \mathbf{F} = \int \mathbf{S} \mathbf{S}^T dV \] (32)
\[ \mathbf{e}^d = \int \mathbf{S} \mathbf{f} dV \] (33)

4.3. Finite element compatibility condition

The same technique is used to enforce the local compatibility conditions (13) and (16). The strain–displacement relation is inserted in definition (22) for the generalized strain vector,
\[ \mathbf{e}_s = \int \mathbf{S} \mathbf{r}^T dV \] (34)
and integrated by parts to force the emergence of the boundary term needed to enforce subsequently the Dirichlet condition (16):
\[ \mathbf{e} = - \int (\mathbf{DS})^T \mathbf{u} dV + \int (\mathbf{NS})^T \mathbf{u} d\Gamma \] (35)

The displacement approximation (23) is inserted in the domain term of this weak form of the compatibility condition, and the boundary term is uncoupled into its Neumann and Dirichlet parts to enforce the boundary displacement approximation (27) and the local condition (16), respectively. In the resulting algebraic equation, which combines the kinematic admissibility conditions (13) and (16),
\[ \mathbf{e}_s = \left[ -\mathbf{A} \quad \mathbf{B} \right] \begin{bmatrix} \mathbf{x}_o \\ \mathbf{y}_o \end{bmatrix} + \mathbf{e}_d - \mathbf{e}_d^d \] (36)
the domain and boundary compatibility matrices are defined as follows,
\[ \mathbf{A} = \int (\mathbf{DS})^T \mathbf{u} dV \] (37)
\[ \mathbf{B} = \int (\mathbf{NS})^T \mathbf{u} d\Gamma \] (38)
and the residual generalized strains associated with the particular solution and the prescribed boundary displacements are defined by:
\[ \mathbf{e}_d^d = \int (\mathbf{DS})^T \mathbf{u}^d dV \] (39)
\[ \mathbf{e}_d = \int (\mathbf{NS})^T \mathbf{u} d\Gamma_D \] (40)

4.4. Finite element equilibrium condition

The finite element equilibrium condition is defined by the dual transformation of Eq. (36),
\[ \begin{bmatrix} -\mathbf{A} \\ \mathbf{B} \end{bmatrix} \mathbf{x}_o = \begin{bmatrix} \alpha \mathbf{M} - i\omega \mathbf{C} \\ 0 \end{bmatrix} \mathbf{x}_o + \begin{bmatrix} \mathbf{e}_d \quad \mathbf{e}_d^d \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathbf{f}_s \quad \mathbf{t}_s \end{bmatrix} = \begin{bmatrix} \int \mathbf{U}^T \mathbf{p} \mathbf{U} dV \\ \int \mathbf{U}^T \mathbf{e} \mathbf{U} dV \end{bmatrix} \] (41)
where the (Hermitian) mass and damping matrices and the term associated with the initial conditions and the particular solution are:
\[ \mathbf{M} = \int \mathbf{U}^T \mathbf{p} \mathbf{U} dV \] (42)
\[ \mathbf{C} = \int \mathbf{U}^T \mathbf{e} \mathbf{U} dV \] (43)
\[ \mathbf{t}_s = \begin{bmatrix} \int \mathbf{U}^T \mathbf{\dot{\mathbf{\sigma}}} \mathbf{p} - i\omega \mathbf{c} \mathbf{\dot{\mathbf{u}}} + \mathbf{\dot{\mathbf{D}}} \mathbf{\sigma}^\prime + \mathbf{b}_s \mathbf{d} \mathbf{v} \\ \int \mathbf{Z} \mathbf{\dot{\mathbf{n}}} d\Gamma_N \end{bmatrix} \] (44)
\[ \mathbf{t}_d = \begin{bmatrix} \int \mathbf{Z} \mathbf{\dot{\mathbf{n}}} d\Gamma_N \end{bmatrix} \] (45)

The first equation in system (41) represents the average enforcement of the local domain equilibrium condition (12). This weak form is obtained inserting the local condition in definition (24) for the generalized body forces,
\[ \mathbf{t} = \int \mathbf{U}^T \mathbf{\dot{\mathbf{D}}} \mathbf{\sigma} - i\omega \mathbf{c} \mathbf{\dot{\mathbf{p}}} \mathbf{u} + \mathbf{b}_s \mathbf{d} \mathbf{v} \] (46)

enforcing the estimates for the generalized stress and displacement fields (21) and (23), and using results (37) and (42)–(44).

The second equation in system (41) defines the weak form of the boundary equilibrium condition (12) for the assumed stresses (21). It is obtained inserting this condition in definition (28) for the generalized boundary forces,
\[ \mathbf{t}_d = \int \mathbf{Z} \mathbf{\dot{\mathbf{n}}} d\Gamma_N \] (47)

for the stress field approximation and using results (38) and (45).

4.5. Indeterminacy numbers

The kinematic admissibility condition (36) shows that the element is kinematically indeterminate, as the number of unknowns exceeds the number of equations. According to the static admissibility condition (41), the (linearly independent) domain and boundary bases must be so balanced as to ensure a non-negative indeterminacy number,
\[ \alpha = N_x - N_y \geq 0 \] (48)
where \( N_x \) is the dimension of the stress and pressure approximation basis (21) and \( N_y \) is the dimension of the boundary displacement basis (27) in the limit situation of an isolated element with an empty Dirichlet boundary.

Eqs. (41) and (48) show that the smaller the indeterminacy number of the stress element the stronger the enforcement of the element boundary force continuity conditions.

4.6. Governing system

The governing system of the hybrid–mixed stress element is obtained equating the elasticity and compatibility equations (31) and (36), to eliminate the generalized strains as independent variables, and adding the equilibrium equation (41):
\[ \begin{bmatrix} \mathbf{F} & \mathbf{A} & -\mathbf{B} \\ \mathbf{A}' & \alpha \mathbf{M} - i\omega \mathbf{C} & 0 \\ -\mathbf{B}' & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_o \\ \mathbf{y}_o \\ \mathbf{y}_s \end{bmatrix} = \begin{bmatrix} \mathbf{e}_s - \mathbf{e}_d - \mathbf{e}_d^d \\ \mathbf{t}_s - \mathbf{t}_d \end{bmatrix} \] (49)

As the flexibility, mass and damping matrices are Hermitian, it can be readily seen that system (49) is Hermitian if the real part of the forcing frequency is null, \( \Re(\omega) = 0 \), as it is typically the case of trapezoidal rules. If the forcing frequency is real, \( \Im(\omega) = 0 \), as in spectral analysis problems, system (49) is Hermitian only for undamped hyperbolic problems, \( \mathbf{C} = \mathbf{0} \).

The solving system for the assembled mesh presents the same highly sparse structure of the elementary system (49), when this system is not condensed on the boundary variables, \( \mathbf{y}_s \), the option followed here. The other features of the assembled system (49) that are worth stressing are its suitability to adaptive refinement and to parallel processing.

The first characteristic results directly from the use of (non-nodal) naturally hierarchical approximation bases and from the fact that summation of structural coefficients is not involved in the
assemblage process, which involves only direct allocation operations. The flexibility, mass and damping matrices and the domain equilibrium matrix are block-diagonal, as they are assigned separately to each finite element, the boundary compatibility/equilibrium matrices are assigned to specific boundaries of the mesh. Consequently, the domain variables, \( \mathbf{x}_c \) and \( \mathbf{y}_c \), are strictly element-dependent, and the boundary variables, \( \mathbf{y}_b \), are shared by at most two connecting elements. Thus the suitability of system (49) to parallel processing.

5. Hybrid stress element

The hybrid formulation results directly from the hybrid-mixed formulation, by constraining the domain approximation to satisfy locally the domain equilibrium condition (12). To this effect, the stress and pressure approximation (21) remains the primary approximation and the mixture displacement approximation (23) is treated as a dependent approximation, under the following constraints:

\[
\mathbf{x}_c = \mathbf{x}_e \quad (50)
\]

\[
\mathbf{D} \mathbf{s} = (i \omega \mathbf{C} - \omega^2 \mathbf{P}) \mathbf{U} \quad (51)
\]

\[
\mathbf{D} \mathbf{q}^e + \mathbf{b} = (i \omega \mathbf{C} - \omega^2 \mathbf{P}) \mathbf{u}^e - \mathbf{b}_s \quad (52)
\]

This constraint has no effect in the elasticity condition (31), which remains valid for the hybrid formulation. However, and as it is shown below, the compatibility and equilibrium conditions (36) and (41) simplify to the following (still dual) relations:

\[
\mathbf{e}_c = B \mathbf{y}_a + (i \omega \mathbf{C} + \omega^2 \mathbf{M}) \mathbf{x}_e + \mathbf{e}_c - \mathbf{e}_c^o \quad (53)
\]

\[
\mathbf{B}^T \mathbf{x}_c = \mathbf{t}_e - \mathbf{t}_c^o \quad (54)
\]

Eq. (53), where, it is recalled, \( i \omega \) is the complex conjugate of \( \omega \), is obtained simply by enforcing constraints (50) and (51) in definition (37) for the domain compatibility matrix and enforcing expressions (42) and (43) for the mass and damping matrices, respectively. Moreover, as the domain equilibrium condition (12) is satisfied locally through constraints (50)–(52), the static admis-

6. Hybrid-Trefftz stress element

The hybrid-Trefftz formulation is the variant obtained by con-

straining the domain approximation to satisfy locally all domain conditions of the problem, that is, the compatibility and elasticity conditions (13) and (14), besides the equilibrium condition (12), as ensured by constraints (50)–(52).

The implication is that there exists now a uniquely defined strain field, coherent with the compatibility condition (13) and the constitutive relation (14):

\[
\mathbf{e} = E \mathbf{x}_c + \mathbf{e}_c^o \quad \text{in } V \quad (58)
\]

\[
\mathbf{E} = \mathbf{D}' \mathbf{U} = \mathbf{f}_s \quad (59)
\]

\[
\mathbf{e}^o = \mathbf{D}' \mathbf{u}^o = \mathbf{f}_s^o \quad (60)
\]

This so-called Trefftz constraint leads to naturally hierarchical bases combining two P-wave modes and one S-wave mode, for both compressible and incompressible media. Eqs. (31), (53), and (54) still represent the element conditions for elasticity, compatibility and equilibrium, and system (55), with the same features, still holds. However, the Trefftz constraints (51), (52), (59), and (60), and integration by parts, can be used to establish the follow-

7. Energy statements

As the finite element equations are derived here from first principles, it is convenient to recover the associated statements on the mechanical, potential and complementary potential energies of the system

\[
M = 2(E + C + T) - (W + W_c) = 0 \quad (63)
\]

\[
P = E + C + T - W \quad (64)
\]

\[
P_c = E + C + T - W_c \quad (65)
\]

as these concepts are often used in the derivation of finite element formulations.
Duality is called upon to establish the virtual work equation and standard mathematical programming concepts are used to recover the well-established energy concepts in the framework of the finite element formulation presented above. The technique used is straightforward: the mathematical program equivalent to a particular algebraic system of equations is derived; the objective function of the resulting mathematical program is interpreted as the structural energy form being minimized and the mathematical program constraints are identified with the relevant (equilibrium or compatibility) structural conditions. In addition, mathematical programming is also used to qualify the finite element solutions in terms of existence and uniqueness.

7.1. Virtual work

In a spectral decomposition framework, the strain, damping and kinetic energies, and the work associated with prescribed displacements and forces are defined by:

\[
E = \frac{1}{2} \int \sigma : \varepsilon \, dV
\]

(66)

\[
C = \frac{1}{2} \int \varepsilon : \varepsilon \, dV
\]

(67)

\[
T = \frac{1}{2} \int \beta : \varepsilon \, dV
\]

(68)

\[
W = \int \varepsilon \cdot \alpha \, d\Gamma_D
\]

(69)

\[
W_s = \int \varepsilon \cdot \beta \, dV + \int \varepsilon \cdot \alpha \, d\Gamma_N
\]

(70)

It can be confirmed that, consequent upon duality, the inner product of the compatibility and equilibrium conditions (36) and (41) recovers definition (63) for the mechanical energy:

\[
M = (\varepsilon - \varepsilon_e, \varepsilon - \varepsilon_e)_{\varepsilon_0} + \gamma \varepsilon \varepsilon_0 - \varepsilon_0 \varepsilon_0 - \varepsilon_0 \varepsilon_0
\]

(71)

As the equilibrium and compatibility conditions are independent of the constitutive relations, the equality above states the virtual work equation. The identification of definition (63) with the result stated by Eq. (71) is obtained substituting in this equation the finite element approximations and the definitions given in Section 4 for the finite element variables and arrays, while using definitions (66)–(70).

7.2. Potential energy

It can be verified that the Hermitian system (49) is associated with the following pair of dual quadratic programming problems:

\[
\text{Min} \quad z = \text{Re} \left[ \frac{1}{2} x^T F x + \frac{1}{2} x^T i \omega C x - \omega^2 M x \right] + \frac{1}{2} x^T \left( t_0 - t_0^e \right)
\]

(72)

subject to:

\[
F = x_e + A x_u - B y_u = e_e - e_e^0 - e_u^0
\]

Min \quad z_e = \text{Re} \left[ \frac{1}{2} x_e^T F x_e + \frac{1}{2} x_e^T i \omega C x_e - \omega^2 M x_e \right] + \frac{1}{2} x_e^T \left( t_e - t_e^0 \right)

(73)

subject to:

\[
A x_e + i \omega^2 M E x_e = (t_e - t_e^0);
\]

\[
B x_u = (t_u - t_u^e)
\]

The programs above recover the stationary conditions on the potential energy and on the complementary potential energy, as the following identifications hold:

\[
Z = \text{Re} (P) + \text{constant}
\]

(74)

\[
Z_e = \text{Re} (P_e) - \text{constant}
\]

(75)

and their feasible regions define the kinematic and static admissibility conditions (36) and (41), respectively, the former under the elasticity constraint (31).

If system (49) is non-Hermitian, the associated quadratic programming problem requires simply the minimization of the mechanical energy of the system:

\[
\text{Min} \quad w = \text{Re} (M) \quad \text{subject to system (49)}.
\]

(76)

7.3. Qualification of solutions

Although program (76) is computationally trivial, as the set of constraints involves both static and kinematic admissibility conditions, it is useful to support the following statements:

Optimal solutions (S1): If program (76) has optimal solutions, the solutions are (weak) statically and kinematically admissible and the Hermitian (conservative) part of the mechanical energy is null at optimality.

Multiple optimal solutions (S2): If \((x_e, x_u, y_u)\) is an optimal solution to program (76), the feasible solution \((x_e, x_u, y_u) + \varepsilon (Ax_e, A y_u, Ay_u)\) is also optimal if the variation is a solution to the homogeneous form of system (49), and:

\[
\text{Re} (\Delta x_e F \Delta x_u - \Delta x_e i \omega C + \omega^2 M \Delta x_u) = 0
\]

(77)

The statements above hold for Hermitian systems, in which case the following multiplicity condition must also be enforced:

\[
\Delta x_e = (t_e + t_e^0) - \Delta y_u (t_e - t_e^0) = 0
\]

(78)

Statement (S2) can be used to establish sufficient conditions for uniqueness of the finite element solutions. The results apply to the hybrid and hybrid-Trefftz formulations, under conditions (50)–(52).

8. Compressible biphasic media

The first set of tests is implemented on saturated soil specimens under low and high forcing frequencies. The mechanical data on the soil (Molsand) is taken from Ref. [9]: mass density of the mixture \(\rho = 2650 \text{ kg m}^{-3}\) and of the fluid \(\rho_f = 1000 \text{ kg m}^{-3}\); scalar tortuosity correction factor \(\alpha = 1.0\); volume fraction of the liquid \(n^v = 0.388\); dissipation \(\gamma = 1.48 \times 10^9 \text{ N m}^{-2}\); modulus of elasticity \(E = 2.98 \times 10^9 \text{ N m}^{-2}\) and Poisson’s ratio \(\nu = 0.333\); Biot’s first coefficient \(\alpha = 1.0\) and second coefficient \(M = 5.67 \times 10^9 \text{ N m}^{-2}\).

The compression box tests represented in Fig. 1 are used to illustrate the sensitivity of the hybrid-Trefftz stress element to mesh distortion and the level of accuracy it may attain in the modeling of the stress, pressure and displacement fields. The domains represented in Fig. 2 are used to illustrate the convergence and the
quality of the finite element solutions for unbounded media using an absorbing boundary condition, placed at the outer boundary. The results presented below are taken from Ref. [10].

8.1. Approximation bases

The domain approximation basis is extracted from the (non-singular) solution set of the homogeneous governing equation obtained by combining the domain conditions (12)–(14) of the problem,

$$DkD^T u = (i \omega c - \omega^2 \rho)u \quad \text{in } V$$ \hspace{1cm} (79)

where now $k = f^{-1}$ is the local stiffness matrix of the mixture.

Using rotational and irrotational displacement potentials, Eq. (79) reduces to the Helmholtz equation, which is solved to obtain three independent (and naturally hierarchical) displacement modes, namely two P- and one S-wave modes. These solutions are collected in format (23) and the local compatibility and elasticity equations (13) and (14) are used to define the associated strain and stress approximation bases, defined by Eqs. (21) and (58), respectively.

The results reported below are obtained using a polar description of the Helmholtz solutions, defined by the product of a trigonometric circular component with a Bessel radial variation, Ref. [11]. It is noted that the Helmholtz solutions satisfy locally the Sommerfeld condition for unbounded media, which is used to define the consistent (Robin-type) absorbing boundary condition used in the implementation of the unbounded saturated soil test.

Different bases can be used to implement the boundary approximation (27). In general, the displacement components of the solid phase and the component of the relative fluid phase displacement orthogonal to the boundary are approximated independently. In the testing problems shown in Fig. 2, the Chebyshev basis is used on the radial sides of the mesh and replaced by a complete trigonometric approximation on the circular sides.

Consequent upon these approximation criteria, a domain approximation basis of (Bessel-) order $d_V$ involves $3(2d_V + 1)$ independent approximation modes per element, and a boundary approximation basis of degree (or order) $d_C$ involves $d_C + 1$ ($2d_C + 1$ for trigonometric bases) independent modes for each displacement component approximated on each Neumann side of the mesh.

8.2. Mesh distortion

Sensitivity to mesh distortion is measured in energy, $E$, using as reference the energy, $\overline{E}$, determined for the undistorted mesh with a refined solution of order $d_V = 9$ in the domain and $d_C = 5$ on the boundary, according to approximations (21) and (27), respectively. They are implemented on the uniformly loaded drained saturated soil compression test represented in Fig. 1a.

The results shown in Fig. 3 confirm that the element is basically insensitive to gross mesh distortion, as measured by parameter $\eta$ identified in Fig. 1a. The residual sensitivity reported in Fig. 3b for the high forcing frequency test, $\omega = 87.5\text{ Hz}$, rapidly decreases with the refinement of the approximation, from $(d_V; d_C) = (3; 1)$ to $(d_V; d_C) = (5; 2)$. 

![Fig. 2. Saturated soil convergence tests under absorbing boundary conditions.](image1)

![Fig. 3. Sensitivity to mesh distortion tests.](image2)
The solution obtained with basis \((d_V; d_C) = (9; 5)\) for the triangular loading test, defined in Fig. 1b, is shown in Fig. 4, namely the stress and pressure fields and the vector representation of the displacements in the solid skeleton \((\mathbf{u}, \text{in red})\) and of the pore fluid seepage displacement \((\mathbf{w}, \text{in blue})\). The boundary conditions and the inter-element continuity conditions are adequately enforced with the four-element mesh used to model the response to a relatively high forcing frequency \((200 \text{ Hz})\). It is recalled that the Trefftz finite element solution is “exact” in the domain of each element.

The forcing frequencies are selected to ensure a relatively wide range in the variation of the ratio between the characteristic length of the element, \(L\), and the wavelength of the excitation, \(\lambda\):

\[
r = \frac{L}{\lambda} = \frac{\text{Re}(k)}{2\pi}
\]

where \(\text{Re}(k)\) is the real part of the wave number associated with the forcing frequency. For the Molsand specimen tests reported here, the variation covers the range \(0.1 \leq r \leq 1.0\), found for the \(P_1\) - and \(S\)-waves, respectively, which exceeds the limits that are usually recommended in the literature on the implementation of the finite element method in the solution of time-dependent problems.

8.3. Convergence

Convergence is assessed on the cylindrical domain \((a = 20 \text{ m}, b = 50 \text{ m})\) shown in Fig. 2a. Two formal solutions of the wave equation for compressible porous media are used to determine the exact energy, \(E\), and to define the (Neumann) boundary conditions.

Two distinct outgoing waves are modelled with Hankel functions of the second kind, for low and high forcing frequencies, namely a ninth-order \(P_2\)-wave and a fifth-order \(S\)-wave. Three meshes are used to illustrate convergence under \(h\)- and \(p\)-refinement strategies: a single-element mesh, a regular mesh of two radial elements and a regular mesh of four elements.

The results obtained are presented in Fig. 5, where \(N\) defines the number of degrees-of-freedom of the solving system in the sparse, non-condensed form \((55)\). The range covered by the tests is \(39 \leq N \leq 522\).

The effect of decreasing the element indeterminacy number, stated by Eq. \((48)\), to strengthen the enforcement of the Neumann and inter-element force continuity conditions, is illustrated for the single-element mesh in Fig. 5 (black triangular marks): the order of the domain approximation is kept constant, with \(d_V = 7\) and \(d_C = 5\) in the low and high frequency tests, respectively, while the degree in the boundary approximations is increased from constant, \(d_f = 0\), to cubic and quadratic in each test. The remaining test results shown in Fig. 5 are implemented increasing the degree in the boundary approximation and using the lowest order in the domain approximation that ensures the enforcement of condition \((48)\) at element level.

The symmetry simplification of the half-space problem shown in Fig. 2b is solved for a forcing frequency of \(100 \text{ Hz}\), using a regular five-element mesh enclosed by an absorbing boundary, placed at \(10 \text{ m}\) of the origin. The stress and pressure field solutions shown in Fig. 6 confirm an adequate modelling of the Neumann and inter-element stress continuity conditions for the relatively high forcing frequency being tested. The vector fields represented in Fig. 7 (as in Fig. 4) are the displacement field in the solid phase and the relative displacement in the mixture. It is noted that they satisfy the Dirichlet symmetry condition, although this condition is not explicitly enforced in the finite element solving system \((55)\).

9. Incompressible biphasic media

The compression of cartilage specimens under confined and unconfined conditions, represented in Fig. 8, is frequently used in the literature on soft tissue modelling. These tests are used to illustrate the application of hybrid-Trefftz stress elements to time domain analysis of incompressible biphasic media, under both plane strain and axisymmetric conditions \([12]\).
The data and the domains of discretization used are taken from Vermilyea and Spilker [2] and hold for all tests reported in this section. As for the drained saturated soil compression tests, the confining chamber is modelled as rigid and perfectly lubricated, now with dimensions $w = 6.35$ mm and $h = 1.78$ mm. The mechanical properties of the soft tissue specimen are taken from the same reference: modulus of elasticity $E = 0.675$ MPa; Poisson’s ratio $\nu = 0.125$; fluid fraction $u_f = 0.83$; permeability $k = 7.6 \times 10^{-15}$ m$^4$ N$^{-1}$ s$^{-1}$.

The procedure used to establish the approximation bases implemented in the tests reported below, presented in Ref. [13], is similar to that described in Section 8.1 for compressible biphasic media.

The distinguishing aspect is that the Navier-type governing differential equation (79) is now replaced by a mixed Navier–Bernoilli system of differential equations. Due to the incompressibility condition, these equations are now written in terms of the relative displacement of the solid and fluid phases and of the pressure field in the fluid phase. An additional consequence of this condition is the presence of constant pressure and rigid-body modes in the domain approximation basis. Moreover, and consequent upon the geometry of the problems being tested, a polynomial Chebyshev approximation is used to set up the boundary approximation bases.

The solutions shown below (no smoothing and non-uniform, scaled colour scales, as in the previous section) are frames extracted from the animations that can be accessed using the address http://www.civilist.ist.utl.pt/HybridTrefltz under link ‘Animation of the response of hydrated soft tissues’.

### 9.1. Plane strain problems

The test consists in modelling the response of the cartilage specimen subject to a prescribed displacement increasing in time according to the ramp loading programme shown in Fig. 8b. The displacement of the loading platen is increased to reach the prescribed deformation of 5% at instant $t_0 = 500$ s, for a prescribed displacement $\bar{u} = 0.0890$ mm in the confined compression tests, and $\bar{u} = 0.0445$ mm in the unconfined compression test. The results obtained with a single-element mesh and with a regular mesh of $2 \times 2$ elements, in a single time step, $\Delta \tau = t_{\text{max}} = 1000$ s, are shown in Figs. 9 and 10. It is noted that the relatively coarse discretization used in space and in time is sufficient to capture the sharp variation in velocity at the end of the loading phase. It is noted, also, that these results recover the values reported in Ref. [2], obtained with hybrid elements and a trapezoidal rule implemented with a time step $\Delta \tau = 5$ s. The meshes used there are a biased mesh with $5 \times 2$ pairs of elements for the confined compression test and a regular mesh of $12 \times 6$ pairs of elements for the unconfined compression test.

Non-uniform colour scales are used Figs. 11 and 12 to enhance the representation of the stress, pressure and displacement fields (defined separately for the solid and fluid phases). They are defined by the bounds found for each field, at each instant. The units used are meter for displacements and Pascal for stresses and pressure. The instants chosen are close to mid-loading phase and to the end of the time frame shown in Fig. 8b.

The results presented in Figs. 11 and 12 capture well the one-dimensional and two-dimensional nature of the response of the
Fig. 9. Evolution in time of the confined compression test solution.

Fig. 10. Evolution in time of the unconfined compression test solution.

Fig. 11. Confined test solutions at instants $t_A = 230 \text{ s}$ and $t_B = 850 \text{ s}$ ($2 \times 2$ element mesh, single time step).
specimen under the two alternative loading conditions. They confirm, also, the correct enforcement of the prescribed and inter-element continuity conditions, in both forces and displacements. In addition, the results presented in Fig. 11 model adequately the stress relaxation mechanism, under a vanishing pressure field. The tests are implemented using bases of order $dV = 11$ and $dC = 5$ in the domain and on the boundary of the elements.

9.2. Axisymmetric problems

The unconfined compression test defined in Fig. 8c is solved under axisymmetric conditions ($r/x$ and $z/y$), assuming that the loading platens are impermeable and adhesive. A regular $5 \times 1$-element mesh is used to model the full height of the specimen ($-h/2 \leq y \leq +h/2$, with $0 \leq x \leq w/2$), involving a total of 625 degrees-of-freedom.

The variation in time of the results shown in Fig. 13 recovers the values reported in Vermilyea and Spilker [14], using a relatively highly refined finite element approximation implemented in 200 time steps ($\Delta t = 5 \text{s}$). As for the plane strain test reported above, the results summarized in Fig. 13 confirm that the discretization in five Trefftz elements and single time step model adequately the response of the specimen.

The stress and pressure solutions found for the fields directly associated with the problem boundary conditions are presented in Fig. 14. The four time frames extracted from the animation of the response of the axisymmetric specimen show that both the boundary conditions and the inter-element flux continuity

---

Fig. 12. Unconfined test solutions at instants $t_a = 230 \text{s}$ and $t_b = 850 \text{s}$ (single-element mesh, single time step).
conditions are adequately modelled using a relatively coarse mesh of 5 \( /C_2 \) hybrid-Trefftz stress elements. A non-uniform colour scale is used in the presentation of the results shown in Fig. 14. They are scaled to the stress parameter \( \sigma_{zz} = E \frac{u}{w} = \frac{w}{E} \), where \( E \) is the modulus of elasticity, \( u \) is the prescribed displacement (see Fig. 1), and \( w \) is the width of the specimen. The peak stress found is \( \sigma_{zz} = 20 \) kPa at point \((x, y) = (w/2, 0)\), when the forcing displacement reaches the prescribed value, \( t_0 = 500 \) s. The ensuing stress relaxation process is well captured by the solutions presented in Fig. 14.

### 10. Closure

The alternative hybrid–mixed, hybrid and hybrid-Trefftz formulations of the finite element stress model are presented. They are derived as to establish a clear distinction on the set of constraints that are increasingly enforced on the approximation bases, using an approximation criterion that preserves the role of the static and kinematic admissibility equations of the resulting finite element model.

The computational advantages offered by the option of implementing directly the resulting highly-sparse, naturally hierarchical and weakly structured solving systems are briefly recalled. The associated energy statements are recovered and the use of mathematical programming to establish uniqueness conditions on the finite element solutions is suggested.

Illustration of numerical performance is based on the least-used variant, the hybrid-Trefftz stress element. It is shown that, for a wide range of forcing frequencies and for different excitation waves, the element performs well in terms of insensitivity to mesh distortion, rate of convergence and accuracy in the enforcement of displacement and flux continuity conditions. Time domain analysis is based on a time integration procedure that exploits the use of a wavelet system to solve each problem in a single time step. Coarse meshes of high-order elements are used to show that the hybrid-Trefftz element models equally well the response of fully incompressible mixtures.

Moreover, all tests reported here model linear behaviour, which is the best suited to illustrate the high performances offered by the hybrid-Trefftz variant of the finite element method. However, they
are essential to support the extension of this study to physically and geometrically nonlinear modelling, namely the elastoplastic analysis of saturated soils and the analysis of incompressible soft tissues under large deformations, as they show clearly the advantages that can be secured by using (eventually linearized) approximation bases that embody the physics of the problem being modelled.

A parallel study has been developed on the alternative displacement model of the hybrid-Trefftz finite element formulation for saturated porous media, under both compressibility and incompressibility conditions, Refs. [10,12]. The main distinction is that the displacement approximation is taken as the primary domain approximation, constrained to be associated with a compatible strain field (subject to the incompressibility condition, for incompressible mixture models). In addition, the boundary forces (the forces on the solid phase and the fluid pressure) are now approximated directly instead of the displacements of the solid phase and the outward normal component of the fluid displacement) and the dual transformation is used to enforce the Dirichlet and inter-element displacement continuity conditions (instead of the Neumann and inter-element force and pressure continuity conditions).

This extension offers no particular difficulty, either in development or in implementation. The domain approximation basis is the same used for the stress model (the so-called Trefftz basis), the boundary basis is qualitatively the same, and mesh assembly procedure is adapted to enforce explicitly the same force approximation on boundaries shared by connecting elements (instead of the boundary displacement approximation). In terms of performance, the tests basically reproduce the experience gained in modelling of single-phase media: as should be expected, and as compared to the displacement model, the stress model of the hybrid-Trefftz finite element formulation produces (marginally) better estimates for the stress and pressure fields and (marginally) weaker estimates for the displacement field in the solid and fluid phases, in either case in terms of enforcement of the continuity conditions.

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Appendix. Basic equations

Omitting, for simplicity, the initial solution term, the explicit form of the equilibrium equation (12) can be written as follows in Cartesian co-ordinates, e.g. Ref. [9]:

$$
\begin{align*}
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} &\left[ \begin{array}{ccc}
\rho & 0 & 0 \\
0 & \rho & 0 \\
0 & 0 & \rho
\end{array} \right] + \left[ \begin{array}{ccc}
\rho f_1 & 0 & 0 \\
0 & \rho f_2 & 0 \\
0 & 0 & \rho f_3
\end{array} \right] \\
&+ \alpha^2 \left[ \begin{array}{ccc}
\rho & \rho w & 0 \\
0 & \rho & \rho w \\
\rho w & 0 & \rho w
\end{array} \right] \left[ \begin{array}{c}
u_1 \\
\nu_2 \\
\nu_3
\end{array} \right] = 0
\end{align*}
$$

where $\sigma_{ij}$ are the components of the total stress tensor, $p$ is the pore fluid stress, $f_i$ is the component of the body force per unit mass density, $u_i$ is the (small) displacement of the solid skeleton and $w_i$ is the pore fluid seepage displacement.

The mass densities of the mixture and of the fluid are $\rho$ and $\rho_w$, respectively, and in the following definition $a$ is the scalar tortuosity correction factor, $n^w$ is the volume fraction of the liquid and $\zeta$ is the dissipation:

$$
\rho w = \frac{\rho^a a}{n^w} \frac{\dot{\zeta}}{\rho w}
$$

Hence, the mass and damping density matrices present in Eq. (12) are:

$$
\rho = \begin{bmatrix}
\rho & 0 & 0 \\
0 & \rho & 0 \\
0 & 0 & \rho
\end{bmatrix},
\begin{bmatrix}
\rho w^a & 0 & 0 \\
0 & \rho w^a & 0 \\
0 & 0 & \rho w^a
\end{bmatrix}
$$

$$
c = \frac{\zeta}{n^w} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
$$

In the compatibility condition (13), $e_{ij}$ are the components of the (small) strain tensor and $\zeta$ is the fluid content:

$$
e_{11} = \begin{bmatrix} d_1 & 0 & 0 \\
0 & d_2 & 0 \\
2d_{12} & 0 & 0
\end{bmatrix},
\begin{bmatrix}
u_1 \\
\nu_2 \\
\nu_{12}
\end{bmatrix}
$$

$$
\zeta = \begin{bmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
$$

The Neumann and Dirichlet conditions (15) and (16) are written as follows, where $n_1$ and $n_2$ are the components of the unit outward normal vector, $n$:

$$
\begin{bmatrix} n_1 & 0 & n_2 \\
0 & n_1 & 0 \\
0 & 0 & n_2
\end{bmatrix}
\begin{bmatrix}
u_1 \\
\nu_2 \\
\nu_{12}
\end{bmatrix} = \begin{bmatrix}
u_1 \\
\nu_2 \\
\nu_{12}
\end{bmatrix}
$$

The equations for incompressible mixtures are obtained by direct adaptation of the compatibility and constitutive equations, e.g. Ref. [2].

References

[8] Tamma KK, Zhou X, Sha D. The time dimension: a theory towards the evolution, classification, characterization and design of computational


