Modelling Singularities and Discontinuities with Hybrid-Trefftz Stress Elements

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\textbf{Abstract} The paper reports on a preliminary study on the modelling of cohesive fracture using the stress model of the hybrid-Trefftz finite element formulation. The objective is to use relatively coarse meshes while allowing for damage and fracture to develop within and across the elements. This preliminary study addresses the modelling the effect of embedded cracks and point loads, typically present in fracture mechanics benchmarks. The solutions that model the local response of open and filled cracks (associated with weakly singular stress fields and discontinuous displacement fields) and point loads (associated with strongly singular stress fields and continuous, weakly singular displacement fields) are included in the approximation basis. As the finite element solving system is derived from the basic equations of the governing boundary value problem, the associated energy statements are recovered a posteriori using basic results of mathematical programming, which are also used to establish sufficient conditions for the existence, uniqueness and multiplicity of the finite element solutions. Numerical tests are presented to illustrate the application of the Trefftz concept in the modelling of singular and discontinuous fields.

\textbf{Keywords} Trefftz elements; stress singularity; displacement discontinuity.
1. Introduction

A project on the modelling of cohesive fracture using the alternative stress and displacement models of the hybrid-Trefftz finite element formulation has been started recently. The first stage of this study addresses the representation of high stress gradients caused by cracking and by concentrated loads, typically present in cohesive fracture applications. The central objective is to use finite element approximations containing the classical elastic solutions for cracks and point loads to support the representation of the stress field and avoid thus the use of highly refined meshes and/or generalized finite element formulations.

A particular application, the modelling of the effect of spray-on liners used to stabilize the rock walls of mining tunnels [3], motivated the implementation of hybrid-Trefftz elements with embedded unfilled (open) and filled (repaired) cracks [1]. The current extension of this study consists in replacing the elastic filler constitutive relation by a nonlinear cohesive fracture relation and couple the formulation with procedures to detect and implement the onset and propagation of discrete damage and fracture in structural elements.

This paper reports on the modelling of the strong singularities caused by the presence of point loads. The issue is the consistent implementation of the corresponding fundamental solutions in hybrid-Trefftz stress elements. However, the resulting finite formulation is used to explain also the implementation of the Trefftz method in a boundary element context.

It is known that the implementation of the Trefftz method using fundamental solutions has been hindered by the choice of the ‘optimal positioning’ of the stress poles, which should be ‘sufficiently close’ to the boundary to avoid ill-conditioning. The boundary element approach to the Trefftz method is recovered here simply by limiting the finite element approximation basis to the set of fundamental solutions. It is shown that the resulting governing system remains symmetric and the solution bounded when the stress poles are placed strictly on the boundary of the mesh. This is the ‘optimal positioning’ of the poles, in the sense that simplifies the implementation of semi-analytical procedures to integrate singular integrals and strengthens the conditioning of the governing system.

The finite element formulation is derived directly from the basic equations that define the boundary value problem. They are stated to establish the notation and the basic assumptions, and to justify the selection and the implementation of the finite element approximation criteria. To concentrate on essentials, the formulation is first derived for regular problems, that is, for problems that involve neither embedded displacement discontinuities nor strong singularities.
These modelling extensions are addressed next, both in terms of approximation criteria and numerical implementation. The paper closes with the presentation of a set of tests frequently used in fracture mechanics applications.

2. Finite element mesh

Flexibility in the description of shape and topology facilitates the modelling of evolving fracture processes. This led to the option of abandoning the master-element concept that supports the implementation of isoparametric mappings. Three entities are used instead to define the topography of the mesh: master nodes, boundary elements and domain elements, e.g. [6].

Master nodes are defined by their co-ordinates measured in the global Cartesian system of reference of the mesh, $x$. They are assigned to each boundary element of the mesh, that is the sides (or surfaces, in three-dimensional applications) of the mesh to support the definition of their geometry in parametric form,

$$ x = x(\eta) \quad on \quad \Gamma $$

with $-l \leq \eta \leq +l$, which is used to define the outward normal vector, $n$, the versor of the side coordinate $\xi$ in the illustration of Figure 1 for a typical domain element. The topography of a domain element is defined simply by direct assignment of its bounding elements.

Therefore, domain elements may not be convex, simply connected or bounded. Addition of a new side, namely, a developing crack segment, is implemented defining the co-ordinates of the new master node (or nodes) and the parametric description of its shape, in form (1), and assigning the new boundary element to that particular domain element. The eventual partition of a domain element is implemented simply by reassigning the sets of boundary elements involved.
3. Boundary value problem

The equilibrium, compatibility and elasticity conditions are written as follows for a typical element of the mesh with domain $V$ and boundary $\Gamma$,

$$\mathcal{D} \sigma + b = 0 \quad \text{in } V \quad (2)$$

$$\varepsilon = \mathcal{D}^* u \quad \text{in } V \quad (3)$$

$$\varepsilon = f (\sigma - \sigma_\theta) + \varepsilon_\theta \quad \text{in } V \quad (4)$$

where vectors $\sigma$ and $\varepsilon$ define the independent components of the stress and strain tensors and $b$ and $u$ are the body force and displacement vectors, respectively. The divergence and gradient matrices, $\mathcal{D}$ and $\mathcal{D}^*$, are linear and conjugate in geometrically linear applications. In the elasticity condition (4), the local flexibility matrix, $f$, is symmetric and positive definite, and vectors $\sigma_\theta$ and $\varepsilon_\theta$ define (alternative) residual states of stress and strain, respectively.

Three complementary regions are identified on the boundary of an element, namely the Neumann, Robin and Dirichlet boundaries,

$$t = \bar{t} \quad \text{on } \Gamma_n \quad (5)$$

$$t = k_i (u - \bar{u}) + \bar{t} \quad \text{on } \Gamma_r \quad (6)$$

$$u = \bar{u} \quad \text{on } \Gamma_d \quad (7)$$

where vectors $\bar{t}$ and $\bar{u}$ define prescribed forces and displacements, respectively, and $k_i$ represents the Robin boundary stiffness matrix. The surface forces that equilibrate the stress field, as implied by equations (5) and (6), are determined by the Cauchy condition,

$$t = N \sigma \quad \text{on } \Gamma \quad (8)$$

where matrix $N$ collects the relevant components of the unit outward normal vector, $n$.

It is assumed that the Neumann and Dirichlet conditions (5) and (7) account for mixed boundary conditions. Moreover, to lighten the derivation of the finite element equations, the interpretation of the Robin condition (6) is extended to include the Neumann condition (5) by setting $k_e = O$. It is also extended to include the interelement surface force continuity condition by interpreting the prescribed forces in equation (5), $\bar{t}$, as the reactions caused by connecting elements. The interelement boundary is denoted by $\Gamma_e$ in the illustration of Figure 1.

In this illustration $\Gamma_e$ represents the domain of (piecewise linear) cracks embedded in the element. According to the notation defined in Figure 2, continuity in the force field and the displacement discontinuity in the displacement field are defined as follows:

$$t = t^+ = t^- \quad \text{on } \Gamma_e \quad (9)$$
\[ u = u^+ + u^- \text{ on } \Gamma_c \]  (10)

Robin-type boundary conditions are used to model embedded cracks,

\[ t = k_c u \text{ on } \Gamma_c \]  (11)

with \( k_c \) representing the (symmetric) stiffness matrix of cracks with elastic fillers, and \( k_c = 0 \) for open, unfilled cracks. It is noted that the formulation used here can be readily extended to model embedded boundary force discontinuities and interelement force or displacement discontinuities.

4. Modelling problems

The elements developed here are designed to support three main modelling situations, namely: regular elements, to model stress and displacement fields that are neither singular nor discontinuous in its domain; elements with displacement discontinuities, to model embedded cracks; elements with strongly singular stress fields, to model the effect of point loads.

![Figure 3: Stress modes weakly singular on the boundary](image)

Weakly singular stress fields may develop on the boundary of regular elements. Typical situations are the modelling of stress concentrations associated with wedges and open cracks, as illustrated in Figure 3. The order of the singularity is classified as weak when the work dissipated in the vicinity of the source of singularity (see Figure 3) is bounded,

\[ \lim_{\epsilon \to 0} W_{r^e} = 0 \]  (12)

\[ \lim_{\epsilon \to 0} W_{e^e} = 0 \]  (13)

\[ W_{r^e} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sigma^e_i \varepsilon_j \, dV \epsilon = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sigma^e_i \varepsilon_j \, r \, dr \, d\theta \right] \]

\[ W_{r^e} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} T^T_i \varepsilon_j \, d\Gamma \epsilon = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[ T^T_i \varepsilon_j \right]_{r=\epsilon} \, d\theta \]
where vectors $\sigma_i$ and $t_i$ define equilibrated stress and boundary force modes and $\varepsilon_j$, and $u_j$ (eventually independent) compatible strain and boundary displacement fields.

The formulation of regular elements is extended to account for displacement discontinuities and strong stress singularities. Elements with embedded (filled or unfilled) cracks, as illustrated in Figure 4, can be used to model particular crack configurations or the process of crack propagation. The fundamental solutions associated with point loads, shown in Figure 5, can be used to set up an approximation basis, as in the Trefftz variants of the boundary element method, or to model the effect of applied loads in a finite element method context.

![Diagram of cracks](image)

**Figure 4: Modelling of embedded cracks**

![Diagram of point loads](image)

**Figure 5: Modelling of point loads ($0 < \omega \leq 2\pi$)**

The point loads can be placed either on the boundary or in the domain of the element. To ensure the boundedness of the strain energy associated with the strongly singular stress modes used in the modelling of point loads, as stated by equation (12), it is assumed that the support of the approximation functions does not contain the source of singularity, $r \geq \epsilon$ in Figure 5. The stress function is so defined as to ensure that equation (13) recovers the work dissipated by the point load:

$$
\lim_{\epsilon \to 0} \mathcal{W}_\epsilon = Fu
$$

(14)
5. Finite element formulation

This section is used to summarize the main concepts supporting the development of hybrid-
Trefftz stress elements and avoid thus undue repetitions in the presentation of the three main
modelling situations addressed here. To lighten the derivation of the finite element equations, the
element is assumed to be regular, in sense defined in the previous section.

5.1 Finite element approximations

The formulation of the hybrid stress element develops from the direct approximation of the stress
and boundary displacement fields in form,

$$\sigma = S_\alpha z_\alpha + S_\beta z_\beta + \sigma_0 \quad \text{in } V$$

$$u = Z_\gamma q_\gamma \quad \text{on } \Gamma_\gamma$$

where the columns of matrices $S$ and $Z$ stress and boundary displacement modes, respectively,
and vectors $z$ and $q$ list the corresponding amplitudes. They define generalized stresses and
displacements as the node concept is not used.

Two families of stress approximation modes are identified explicitly, $S_\alpha$ and $S_\beta$, to support
the alternative modelling situations addressed here. The optional particular solution, defined by
vector $\sigma_0$, is used to illustrate is typically used to equilibrate body-forces or to model local
effects that affect the rate of convergence of the finite element solution (see Section 6).

In equation (16), $\Gamma_\gamma$ defines the portions of the boundary of the element whereon the
displacements are unknown, $\Gamma_i$. This extended Robin boundary combines, the Neumann and
Robin boundaries of the mesh that the element may contain, as well as its interelement
boundaries:

$$\Gamma_i = \Gamma_n \cup \Gamma_r \cup \Gamma_e$$

(17)

Letting $\Gamma_u$ identify the portion of the Dirichlet boundary (7) of the mesh shared by the
element, the boundary of the element (which includes the boundary of the cells of multiply
connected elements) is defined by:

$$\Gamma = \Gamma_i \cup \Gamma_u$$

(18)

The stress and boundary displacement bases are assumed to be complete and linearly
independent. In addition the stress approximation is constrained to satisfy the domain
equilibrium condition (3) in strong form:

$$\mathcal{D}S_\delta = 0$$

(19)

$$\mathcal{D}\sigma_0 + b = 0$$

(20)
The forces that equilibrate the stress estimate (15), determined from the Cauchy condition (8),
\[
t = T_\alpha z_\alpha + T_\beta z_\beta + t_0
\]  
may not (and in general will not) satisfy the force continuity conditions on the boundary of the element. However, they are constrained to satisfy the continuity condition (9) in the domain of the element.

5.2 Dual variables

The dual transformations of approximations (15) and (16) define generalized strains (with \(\delta = \alpha, \beta\)) and generalized boundary forces (with \(\gamma = n, r, e\)),
\[
e_\delta = \int S_\delta^T \varepsilon \, dV
\]
\[
p_\gamma = \int Z_\gamma^T t \, d\Gamma_\gamma
\]
that ensure the invariance of the inner product in the finite element mappings:
\[
\sum_\delta z_\delta^T e_\delta = \int (\sigma - \sigma_0)^T \varepsilon \, dV
\]
\[
\sum_\gamma q_\gamma^T p_\gamma = \int u^T t \, d\Gamma_i
\]

5.3 Finite element equations

The generalized strains (23), which are bounded in consequence of the weak singularity condition (12), are used to enforce the domain compatibility and elasticity conditions (3) and (4), and the generalized boundary forces (24) are used to enforce the Neumann, Robin, interelement and embedded crack continuity conditions for the force fields (21) that equilibrate the stress estimate. The procedure is equivalent to the Galerkin version of the weighted residual method and leads to the finite element equations summarized in Table 1, with \(\delta = \alpha, \beta\) and \(\gamma = n, r, e\):

<table>
<thead>
<tr>
<th>Equilibrium</th>
<th>Compatibility</th>
<th>Elasticity</th>
</tr>
</thead>
<tbody>
<tr>
<td>[\sum_\delta A_{\delta\gamma}^T z_\delta = K_{\gamma\gamma} q_\gamma + \bar{p}<em>\gamma - p</em>{\gamma0}] Displ</td>
<td>[e_\delta = \sum_\gamma A_{\delta\gamma} q_\gamma + F_{\delta u} + e_{\delta0}] Displ</td>
<td>[e_\delta = F_{\delta u} z_\alpha + F_{\delta e} z_\beta + e_{\delta0}] Displ</td>
</tr>
</tbody>
</table>

The finite element kinematic admissibility condition (26) is obtained enforcing the domain compatibility condition (3) in definition (23) for the generalized strains:
\[
e_\delta = \int S_\delta^T \varepsilon \, dV = \int S_\delta^T (D^* u) \, dV
\]
This equation is integrated by parts to mobilize the boundary term,

\[ e_\delta = -\int (\mathcal{D} S_\delta)^T u \, dV + \int (N S_\delta)^T u \, d\Gamma \]  \hspace{1cm} (29)

which is uncoupled in form (18) to implement the boundary approximation (16) and the Dirichlet condition (7). After enforcing condition (19) and definition (22), the following expressions are found for the boundary compatibility matrix and for the generalized strains associated with the prescribed displacements:

\[ A_{\delta \gamma} = \int T_\gamma^T Z_\gamma \, d\Gamma_\gamma \]  \hspace{1cm} (30)

\[ \bar{\epsilon}_{\delta u} = \int T_\delta^T \bar{u} \, d\Gamma_u \]  \hspace{1cm} (31)

As the domain equilibrium condition (2) is locally satisfied, the dual finite element static admissibility condition (25) reduces to weak enforcement of the static boundary conditions for the assumed stress field (15). Condition (25) is stated by equating the generalized boundary forces (24) associated with the forces (21) induced by the stress approximation with those developing on the (generalized) Robin boundary (6), for the assumed displacements (16):

\[ \int Z_\gamma^T (T_a z_a + T_\beta z_\beta + t_0) \, d\Gamma_\gamma = \int Z_\gamma^T [k_\gamma (Z_\gamma q_\gamma - \bar{u}) + \bar{T}] \, d\Gamma_\gamma \]  \hspace{1cm} (32)

Definition (30) is recovered and the following expressions are found for the boundary stiffness matrix and for the generalized forces associated with prescribed terms:

\[ K_{\gamma \gamma} = \int Z_\gamma^T k_\gamma Z_\gamma \, d\Gamma_\gamma \]  \hspace{1cm} (33)

\[ \bar{p}_\gamma = \int Z_\gamma^T (\bar{T} - k_\gamma \bar{u}) \, d\Gamma_\gamma \]  \hspace{1cm} (34)

\[ p_{\gamma 0} = \int Z_\gamma^T t_0 \, d\Gamma_\gamma \]  \hspace{1cm} (35)

Matrix \( k_\gamma \) is symmetric and positive definite on Robin boundaries, and \( k_\gamma = O \) on Neumann and interelement boundaries, \( \Gamma_n \) and \( \Gamma_e \) in definition (17).

The finite element elasticity condition (27) is derived enforcing the domain condition (4) in definition (23) for the generalized strains,

\[ e_\delta = \int S_\delta^T e \, dV = \int S_\delta^T [f(\sigma - \sigma_0) + \epsilon_0] \, dV \]  \hspace{1cm} (36)

and enforcing next the stress approximation (15). The following expression is found for the (symmetric) flexibility matrix and for the generalized strains associated with the particular solution and residual terms:

\[ F_{ab} = \int S_a^T f S_b \, dV \]  \hspace{1cm} (37)

\[ e_{a0} = \int S_a^T [f(\sigma_0 - \sigma_0) + \epsilon_0] \, dV \]  \hspace{1cm} (38)
5.4 Finite element solving system

The governing system is obtained combining the element compatibility and elasticity conditions (26) and (27) to eliminate the generalized strains as independent variables, and adding the equilibrium condition (25). In the resulting system,

\[
\begin{bmatrix}
F_{\alpha\alpha} & F_{\alpha\beta} & -A_{\alpha\alpha} & -A_{\alpha\beta} \\
F_{\beta\alpha} & F_{\beta\beta} & -A_{\beta\alpha} & -A_{\beta\beta} \\
-A_{\alpha\alpha}^T & -A_{\beta\alpha}^T & K_{rr} & \bullet \\
-A_{\alpha\beta}^T & -A_{\beta\beta}^T & \bullet & \bullet
\end{bmatrix}
\begin{bmatrix}
z_{\alpha} \\
z_{\beta} \\
q_r \\
q_i
\end{bmatrix}
= \begin{bmatrix}
\vec{e}_{\alpha\alpha} - e_{\alpha0} \\
\vec{e}_{\beta\alpha} - e_{\beta0} \\
p_{r0} - \vec{p}_r \\
p_{i0} - \vec{p}_i
\end{bmatrix}
\]

(39)

approximation (16) is uncoupled to distinguish the displacements associated with the Robin boundary displacements, \( q_r \), and with the Neumann and interelement boundaries, \( q_i = \{q_n, q_s\} \).

The first two equations in system (39) combine the weak forms of the kinematic admissibility and elasticity conditions, and the remaining equations define strictly the weak form of the element static admissibility conditions. Therefore, and assuming that the element flexibility matrix and the Robin boundary stiffness matrices are positive-definite, a condition that is easily fulfilled, the kinematic and static indeterminacy numbers of the element are,

\[
k = N_{qf} + N_{qe} \geq 0 \quad \text{(40)}
\]

\[
s = N_{z\alpha} + N_{z\beta} - N_{qf} \geq 0 \quad \text{(41)}
\]

where \( N_z \) and \( N_q \) define the dimensions of the generalized stress and displacement vectors, respectively. The sum of the static and kinematic numbers defines the number of deformable modes of the element.

5.5 Trefftz constraint

In the present application, the Trefftz constraint consists in limiting the domain approximation (15) to stress fields associated with strain and displacement fields,

\[
\begin{align*}
\epsilon &= E_{\alpha}\zeta_{\alpha} + E_{\beta}\zeta_{\beta} + \epsilon_0 \quad \text{in } V \\
u &= U_{\alpha}\zeta_{\alpha} + U_{\beta}\zeta_{\beta} + Rz + u_0 \quad \text{in } V
\end{align*}
\]

(42)

(43)

that satisfy in strong form the domain compatibility and elasticity conditions (3) and (4):

\[
\begin{align*}
E_{\delta} &= \mathcal{D}^*U_{\delta} = fS_{\delta} \\
\epsilon_0 &= \mathcal{D}^*u_0 = f(\sigma_0 - \sigma_0) + \epsilon_{\theta}
\end{align*}
\]

(44)

(45)

In the dependent displacement approximation (43), matrix \( R \) collects as columns the rigid-body modes and vector \( z \) the corresponding amplitudes: \( \mathcal{D}^*U = 0 \). The rigid-body modes remain undetermined after solving the finite element governing system (39). They can be
determined, in a non-unique way, by matching with the independent boundary displacement approximation (16).

Conditions (44) and (45) can be used to obtain boundary integral expressions to the only terms in system (39) defined by domain integral expressions, namely the element flexibility matrix (37) and the generalized deformations (38) associated with the particular solution in the stress approximation. The following expressions are found,

\[ T_{\alpha \beta} = \int_{\Gamma} F_{\alpha} U_{\beta} d\Gamma \]  \hspace{1cm} (46)

\[ e_{\delta 0} = \int_{\Gamma} T_{\delta} u_{0} d\Gamma \]  \hspace{1cm} (47)

enforcing conditions (44) and (45) in equations (37) and (38), respectively, integrating by parts and using the equilibrium constraints (19) and (20) and definition (22).

This procedure is equivalent to the implementation of result (29) identifying the displacement field \( u \) with the dependent (Trefftz) approximation (43). As the stress field is self-equilibrated, under condition (19), the term associated with the rigid-body movement vanishes, as it represents the force and moment resultants of the boundary force distribution:

\[ \int R^T T_\delta \, d\Gamma = 0 \]

The results summarized above 5 are specialized next to the modelling of weakly singular stress fields associated with wedges and cracks and strongly singular stress fields caused by the presence of point loads. The presentation is designed to introduce separately each modelling problem.

It is assumed first that the discretization of the body ensures that the faces of the existing wedges and/or open cracks coincide with the boundaries of the element, as assumed in [18,19], for instance.

6. Modelling of wedges and surface cracks

This section is used to define the basis adopted in the implementation of the domain and boundary approximations (15) and (16), respectively, and to clarify particular aspects concerning the numerical implementation of the finite element solving system (39).

6.1 Domain and boundary approximation bases

The regular stress modes collected in approximation (15) combine two independent bases, each of which satisfy the regularity and Trefftz constraints defined in Sections 4 and 5.

In simply (multiply) connected elements, the body of the approximation, \( S_{\alpha} \), is defined by the polynomial (rational) stress fields presented in Appendix A (a multiply connected element may combine both types of solutions):
\[ S_\alpha = [S_p S_r] \]

In elements affected by weak stress singular fields, caused by the presence of wedges or (boundary) cracks, the functions that model these local effects, presented in Appendices B and C, are used to construct the enrichment term, \( S_\beta \), in approximation (15):

\[ S_\beta = [S_w S_s] \]

The dimension of the bases, \( N_{z\alpha} \) and \( N_{z\beta} \) in equation (41), can be determined from the information given in Appendices A, B and C.

The boundary approximation (16) is polynomial to match the boundary mapping of the main body of the domain approximation. They are defined by (intrinsically scaled) Chebyshev polynomials,

\[ Z_n(\eta) = \cos\left(n \cos^{-1}\eta\right) \quad (48) \]

where \( \eta \) is the variable used in the parametric description (1) of the boundary, the support of the approximation. When polynomials of (uniform) degree \( d_\gamma \) are used on Robin \( (\gamma = r) \) and on Neumann and interelement \( (\gamma = n) \) boundaries, the dimension of the bases (16) is, at element level, \( N_{z\gamma} = c(d_\gamma + 1) \) where \( c \) is the number of components being approximated on each type of boundary. These dimensions are used to determine the kinematic indeterminacy number (40) of the element.

The displacement components are independently approximated on each boundary in the numerical applications reported here, leading to a discontinuous approximation at the vertices of the element, where the outward normal is not defined. Boundary displacement continuity can be enforced either by using continuous frame-functions [8,9] or by enforcing the null relative displacement conditions. Neither of these techniques is used in the implementation of the numerical tests presented here.

6.2 Numerical implementation

Assemblage of the elementary systems (39) to obtain the solving system of the finite element mesh consists in listing the generalized stress and Robin boundary displacements vectors, \( z_\alpha \) and \( q_\gamma \), for all domain and boundary elements (the flexibility and stiffness matrices \( F \) and \( K \) are block-diagonal), and assigning the same Neumann boundary displacement vector, \( q_n \), to the pairs of elements that connect on that particular boundary. The assembled system preserves the same general structure and is stored and solved exploiting its high sparsity.
The usual practice in the implementation of hybrid-stress elements, designed to emulate conventional displacement elements, is to condense the elementary systems (39) on the boundary displacement degrees-of-freedom by exploiting the positive-definiteness of the element flexibility matrix, \( F \).

This option is not followed in the approach adopted here to enhance the implementation of adaptive refinement techniques and to exploit the suitability of the structure of the solving system to parallel processing. The first feature results directly from the naturally hierarchical bases used in both domain and boundary approximations, and the second results from the fact that all domain variables present in the assembled system (39) are strictly element-dependent and the boundary variables are shared by at most two connecting elements.

As these aspects have been discussed in detail elsewhere, e.g. [2,4,5,7], the comments below address only the computation of the coefficients present in system (39) to clarify the solution the singular terms there involved. Consequent upon the Trefftz constraint, all coefficients present in system (39) are defined by boundary integral expressions of the form,

\[
\mathcal{Z}_y = \int_{\eta} t_i(\eta) \cdot u_j(\eta) |J| \, d\eta
\]

using the parametric description (1), where \( t_i \) and \( u_j \) represent boundary force and displacement functions, respectively (see Sections 5.3 and 5.5), and \( |J| \) is the Jacobian of the mapping.

### 6.3 Wedges and cracks

A Gauss-Legendre quadrature rule is used as both functions are polynomial for simply connected elements, assuming that the prescribed terms in boundary conditions (5) to (7) are also defined in polynomial form. When rational stress modes are used in the implementation of the domain approximation basis (15), the associate force and displacement fields remain regular on the boundary of multiply connected elements.

It is noted that the surface forces are null, \( t_i(\eta) = 0 \), on the edges that contain the (weak) singularity of wedge and crack stress modes, meaning that integral (49) is not computed on the free sides of wedges and open cracks. However, meshing may lead to internal boundaries that contain the source of the singularity, at \( \eta = \pm 1 \), and whereon \( t_i(\eta) \neq 0 \). As the displacement term, \( u_j(\eta) \), is necessarily regular when function \( t_i(\eta) \) is singular at position \( \eta_0 \), the integral is written in the following form to extract the singularity:

\[
\mathcal{Z}_y = \int_{-1}^{1} t_i(\eta) \cdot [u_j(\eta) - u_j(\eta_0)] |J| \, d\eta + u_j(\eta_0) \int_{-1}^{1} t_i(\eta) |J| \, d\eta
\]

The first (regularized) term is integrated numerically and the second is solved analytically.
This is typically the case of wedge and crack functions, for which the order of singularity is
\( r^{-\lambda} \) with \( \lambda \leq 0.5 \) and \( r = \frac{\ell}{2} (l \pm \eta) \), with \( \ell \) representing the side length. In the integration of
diagonal terms of the flexibility matrix (46), integral (49) is solved analytically using the stress
and displacement definitions presented in Appendices B and C.

6.4 Prescribed point loads

The regularity and Trefftz constraints defined in Sections 4 and 5 are not violated when
prescribed point loads are included in the loading conditions and modelled using the particular
term, \( \sigma_0 \), in the stress approximation (15). The Dirac, \( \delta(r - r_0) \), point solutions for infinite and
semi-infinite domains are recalled in Appendix D.

They affect differently definitions (35) and (47), the only terms in system (39) associated with
prescribed point loads. Definition (35), present in the boundary equilibrium condition (25),
extends into the following form (\( \gamma = \ell \)),

\[
\mathbf{p}_{\gamma=0} = \int Z^T_\gamma t_0 d\Gamma_\gamma + Z^T_\gamma (r_0) \mathbf{L} \mathbf{F}_0
\]

where \( \mathbf{F}_0 \) is the intensity vector of the Dirac force field, see Figure 5, and \( \mathbf{L} \) is the rotation
matrix that defines the force components associated with the boundary displacement
components.

Equation (47) results from the generalized strain definition (23), which can be written in
form,

\[
e_{\varepsilon_0} = \int T^T_\sigma \mathbf{u}_0 d\Gamma + \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{+\varepsilon} S^T_\sigma \varepsilon_0 r dr d\theta + \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{+\varepsilon} T^T_\sigma \mathbf{u}_0 r dr d\theta
\]

after isolating the singularity and integrating by parts the regular term. As terms \( \varepsilon_0 r \) and \( \mathbf{u}_0 r \)
are regular, see Appendix D, the limits are null and definition (47) remains unchanged.

7. Embedded cracks

Two types of embedded cracks are modelled, namely open cracks and filled (or repaired) cracks
using two alternative solutions, the Mitchell solution for unbounded cracks and the Griffith
solution for bounded cracks, as illustrated in Figure 4. Their associated stress and displacement
fields are defined in Appendix C, where explicit forms of the constitutive relation (11) of filled
cracks are also presented [3,1].

These results show that the order of the singularity of the stress field decreases for filled
cracks, as expected. Consequently, the procedure presented above to handle singular terms
remains valid. Although the filled crack constitutive relation (11) involves a singularity of order
\( r^{-1} \), which requires particular attention, the main modelling aspect addressed here is the
representation of the displacement discontinuity embedded in the element caused by the presence of either open or filled cracks.

7.1 Finite element approximations

The domain approximation (15) still holds, under the provision that term \( S_a \) combines now the bases defined above for regular elements, namely polynomial and/or rational stress modes and wedge and boundary crack stress solutions, and matrix \( S_\beta \) lists, as columns, the stress modes that model the local effect of embedded (open or filled) cracks:

\[
[S_a | S_\beta] = [S_p, S_w, S_{bc}|S_{wc}]
\] (51)

In addition, the boundary approximation (16) is extended to include the independent approximation of the displacement discontinuity (10) on crack boundaries (see Figure 2). Definition (17) of the boundary of the element where on the displacements are unknown is also extended to include the embedded cracks:

\[
\Gamma_i = \Gamma_n \cup \Gamma_r \cup \Gamma_c \cup \Gamma_e
\] (52)

It is noted that the stress approximation (15), besides satisfying the Trefftz constraints and the linear independence and completeness conditions, satisfies also the force continuity condition (9) in the domain of the element. In order to preserve the invariance of the inner product in the finite element mapping, the generalized strains (23) and the generalized boundary forces (24) are now complemented with the terms associated with the new degrees-of-freedom in the domain and on the boundary of the element.

7.2 Finite element equations

The finite element equations summarized in Table 1 and the definitions given above for the structural arrays remain valid, provided that the arrays involved are redefined to account for the discontinuity of the displacement field in embedded cracks.

Regarding the derivation of the kinematic admissibility conditions, the preliminary result (29), where condition (19) still holds for all stress approximation modes, is written as follows to account for the displacement discontinuity:

\[
e_\delta = \int T_{\delta} u^d \Gamma + \int T_{\delta}^r (u^+ + u^-) d\Gamma_c
\] (53)

Equation (26) is recovered, under definitions (30) and (31), enforcing above decomposition (52) and implementing the Dirichlet condition (7) and boundary approximations condition (16).

The dual, static admissibility condition (25), where definitions (30) and (31) still hold, extends the weak enforcement of the Robin, Neumann and interelement boundaries to include
the effect of the additional embedded crack stress modes. Thus, the equation obtained setting \( \gamma = c \), and \( \bar{\gamma}_c = \theta \), states the weak enforcement of the embedded crack constitutive relation (11) for the assumed displacement discontinuity, with \( K_{\gamma\gamma} = O \) on open embedded cracks.

Consequent upon the Trefftz constraint on the domain approximation basis (15), the extension of the weak form of the elasticity condition can be determined directly from result (53) identifying the displacements \( u \) with the dependent approximation (43) (see closing note in Section 5.5). Definition (46) holds for regular displacement fields, namely for \( F_{\alpha\alpha} \) and \( F_{\beta\alpha} \) under notation (51), and generalizes into form (with \( \delta = \alpha, \beta \)),

\[
F_{\delta\delta} = \int T_\delta^T U_\beta d\Gamma + \int T_\delta^T (U_\beta^+ + U_\beta^-) d\Gamma_c
\]

for the terms associated with the embedded crack discontinuity. It is noted that \( T_\delta = O \) on \( \Gamma_c \) for open embedded cracks. Definition (47) holds for both regular and embedded crack stress modes, even when the particular solution is used to model prescribed point loads.

System (39) is used to model elements with embedded cracks, under identification (51) and assuming that arrays \( q_i \) and \( q_r \) combine now the degrees-of-freedom associated with Neumann and interelement boundaries, \( q_i = \{ q_n, q_e \} \), and with the Robin and embedded crack displacements, \( q_r = \{ q_r, q_c \} \).

7.3 Stress intensity factors

The amplitudes of the fields inserted in the basis to model the different modes of fracture define directly the stress intensity factors. Moreover, system (39) can be used to assess the effect of prescribed stress intensity factors. The column associated with a particular boundary or embedded crack mode is multiplied by the corresponding stress intensity factor and moved into the stipulation vector, while the corresponding row is eliminated from the system.

7.4 Numerical implementation

The general comments made in Section 6.2 on the assemblage of the solving system for the finite element mesh remain valid, as well as the techniques summarized in Sections 6.3 and 6.4 on the computation of the system coefficients. The only distinguishing aspect is the singularity present in the definition of the stiffness matrix (33) for embedded filled cracks, \( \gamma = c \) and \( k_c \neq O \).

According to the results presented in Appendix C, the order of this singularity is \( r^{-1} \), with one pole for Mitchell cracks and two poles for Griffith cracks. Boundedness of the deformation energy dissipated in filled cracks is modelled by ensuring that displacement discontinuity is null.
at the stress poles. Thus, in the implementation of the boundary discontinuity approximation (16) for filled cracks, the polynomial approximation (48) is replaced by the boundary mapping of the crack displacement discontinuity field in forms,

\[ Z_n(\eta) = (1 - \eta)^{\lambda_n} \]
\[ Z_n(\eta) = (1 - \eta^{\frac{1}{2}})^{\lambda_n} \]

for Mitchell and Griffith cracks, respectively, where \( \lambda_n \geq 0.5 \) represents the filled crack eigenvalues (see Appendix C). Under this provision, the integral present in definition (33) for the stiffness matrix of embedded filled cracks can be easily solved analytically.

8. Prescribed point displacements

It may be necessary, or convenient, to prescribe either the displacement at particular points or the relative displacement of pairs of points on the Neumann boundary of the element.

Assume that these conditions are enforced, by collocation, on the boundary displacement approximation condition (16) with \( \gamma = n \). The additional condition,

\[ \vec{u}_d = A_{dy} \vec{q}_y \]

where vector \( \vec{d}_y \) defines the prescribed displacements, is enforced and added to the element kinematic admissibility equations (26). The dual transformation that replaces the element equilibrium condition (25),

\[ \sum_\delta A_{\delta y}^T \delta_y + A_{dy}^T p_d = K_{\gamma\gamma} q_{\gamma} + \bar{p}_y - p_{\gamma0} \]

accounts for the effect of forces, \( p_d \), induced by the displacement constraint. The extended form of the finite element solving system is the following:

\[
\begin{bmatrix}
F_{ax} & F_{a\beta}
\end{bmatrix}
\begin{bmatrix}
-A_{ax} & -A_{at}
\end{bmatrix}
\begin{bmatrix}
z_{ax}
\end{bmatrix}
+ \begin{bmatrix}
\bar{e}_{au} - e_{a0}
\end{bmatrix}
= \begin{bmatrix}
\bar{e}_{au} - e_{a0}
\end{bmatrix}
\]

A typical application is the removal of rigid-body movements of the finite element mesh, to yield \( p_d = 0 \) upon solution. This extension can be applied also to enforce continuity of approximation (16) at the element vertices, when frame functions are not used, in which case \( p_d \) tends to zero with the convergence of the finite element solution.

This technique has been applied also to model strain-inducing fixed support conditions [5], in which case the high stress gradients caused by the support reaction has to be captured by the
polynomial term of the stress field approximation, term $S_p$ in description (51). In general, and when h-refinement is not called upon, this implies the use of very high degrees in the approximation, which tends to deteriorate the condition number of the solving system.

This can be avoided by modelling directly the effect of the support reactions using the Boussinesq point load solution. As it is shown below, this extension also covers the modelling of prescribed displacements in points of the domain of the element using the Kelvin solution for point loads.

9. Point loads

The information summarized in this section is useful in two different contexts. The first is the use of fundamental solutions to construct the approximation basis in the implementation of the Trefftz method. The second is the modelling of the effect of non-prescribed point loads in a finite element framework, typically the effect of reaction forces in rigid supports.

In what regards the interpretation of the Trefftz method from a boundary element standpoint, this section addresses two issues discussed in the literature: where to place the source point and how to ensure symmetry in the resulting solving system. In what regards the use of fundamental solutions in finite element modelling, the issue is not to model loads that are unrealistic from a practical point of view. It is, instead, to exploit the limit situation that point loads represent to capture with simplicity and accuracy the high stress gradients associated with highly concentrated loading conditions.

9.1 Neumann and Dirichlet point loads

Two types of loads are relevant in a finite element context, namely the prescribed or Neumann point loads addressed in Section 6.4 and the Dirichlet point loads, that is point loads associated with prescribed displacements, typically the reaction forces in point supports. In this case, addressed in this section, equation (7) is written in form,

$$u = \bar{u}(r_{\beta}) \quad \text{on } \Gamma_d$$

(56)

with $r_{\beta}$ defining the support position. The support reaction identifies with the weight of the point load stress mode, or, in the notation used below:

$$p(r_{\beta}) = z_{\beta}$$

(57)

9.2 Finite element approximations

As the Robin and Neumann-type boundary conditions defined above still hold, the definition of the boundary displacement approximation (16) remains unchanged. However, the stress approximation (15) is reinterpreted to include in term $S_{z\alpha}$ the stress modes analyzed previously,
\[
S_\alpha = \left[ S_p \ S_r \ S_w \ S_{bc} \ S_{vc} \right]
\]
and identifying matrix \(S_\beta\) with the list of stress modes that model the local effect of unit point loads,
\[
S_\beta = S_{\beta l}
\]
as defined in Appendix D, to yield,
\[
\lim_{\epsilon \to 0} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[ T_\beta^T L r \right]_{r=\epsilon} d\theta = I
\]
with \(L\) representing the rotation matrix (the first two columns of the rigid-body displacement matrix, \(R\), defined in Appendix A). Their origins are placed at the point support positions, and they are so oriented as to ensure that their weights represent the (unknown) support reactions.

The generalized strains associated with the point load solution weights are still defined by equation (23), under the constraint that the energy measures they represent are bounded,
\[
e_\beta = \int_{\Omega} S_\beta^T \epsilon dV
\] (58)
where a finite part integration is assumed. As the Trefftz constraint holds, the generalized strains are defined in form (53), after isolating the point load stress poles:
\[
e_\beta = e_\beta^i + e_\beta^r + \int T_\beta^T u d\Gamma + \int T_\beta^T (u^+ + u^-) d\Gamma_c
\]
\[
e_\beta^i = \lim_{\epsilon \to 0} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^\epsilon S_\beta^T \epsilon r dr d\theta
\]
\[
e_\beta^r = \lim_{\epsilon \to 0} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[ T_\beta^T u r \right]_{r=\epsilon} d\theta
\]
The terms above are bounded and defined by,
\[
e_\beta^i = 0
\] (59)
\[
e_\beta^r = u(r_p)
\]
when the strain and displacement fields are regular at the stress pole. When these fields are singular, the definitions above still hold under the finite part integration argument.

Different arguments can be use to justify the fact that the formulation remains free of unbounded terms. One is that these terms must cancel, upon convergence, when the generalized strains computed from the kinematic admissibility condition (28) are equated to the generalized computed from the implementation of the elasticity condition (36), as implied by the second set of equations in systems (39) and (55). A simpler argument to justify condition (59) in all instances is to remove the pole from the definition of the support of the point load functions,
$r \geq \epsilon$, as illustrated in Figure 5. However, under these conditions the boundary term remains unbounded for singular displacement fields, as shown in Appendix D.

Under these conditions, the finite element equations summarized in Table 1 and the finite element solving system, in either of the formats (39) or (55), remain valid. However, the solving system coefficients directly related with point load solutions, typically all arrays affected by vector $z_{\beta}$, must be reassessed to account for the effect of the stress poles.

9.3 Effect on the equilibrium and compatibility conditions

Condition (25) and the supporting definitions (30) and (33) to (35) remain unchanged if the point loads are not applied to boundary $\Gamma_\gamma$. The same conclusion applies to the case of Dirichlet points loads applied on the boundary, when the equation (32) takes the following form (see Section 6.4) under condition (57):

$$
\left[ Z_f^T (T_{\alpha} z_{\alpha} + T_{\beta} z_{\beta} + t_0) d\Gamma_\gamma + Z_f^T (r_{\beta}) L z_{\beta} = \int Z_f^T [k_i (Z_f q_f - \bar{u}) + \bar{r}] d\Gamma_\gamma + Z_f^T (r_{\beta}) L p(r_{\beta}) \right] (60)
$$

It can be readily verified that condition (26) and definition for the boundary compatibility matrix (30) remain unchanged when the procedure summarized in Section 5.3 is applied to definition (58), while equation (31) extends into the following form to account for the additional Dirichlet condition (56):

$$
\bar{u}_{\beta} = \int T_{\beta}^T \bar{u} d\Gamma_{\alpha} + \bar{u}(r_{\beta})
$$

9.4 Effect on the elasticity condition

It is convenient to define the displacement term associated with the point solution modes in the dependent approximation (43) in form,

$$
U_{\beta} = U_{\beta}^l + L \delta(r, r_{\beta}) \bar{z}_{\beta}
$$

where $L$ still represents a rigid-body displacement matrix, $\delta(r, r_{\beta})$ is the Dirac function matrix and vector $\bar{z}_{\beta}$ the (finite part) of the of the displacement vector at the stress pole.

Definitions (46), (47) and (54) remain valid when subscripts $\alpha$ and $\delta$ identify regular stress modes. When subscript $\beta$ identifies a point load mode, the integration is performed semi-analytically on the sides containing the stress pole, replacing equation (50) by the following:

$$
\mathcal{Z}_{\beta} = \int_{-1}^{+1} \left[ t_i(\eta) - t_i(\eta_0) \right] \cdot u_j(\eta) \cdot d\eta + t_i(\eta_0) \int_{-1}^{+1} u_j(\eta) \cdot d\eta
$$

When subscripts $\alpha$ and $\delta$ identify point load stress modes, definitions (46) and (47) extend into the following forms:

$$
F_{\alpha\beta} = \int T_{\alpha}^T U_{\beta} d\Gamma + U_{\beta}(r_{\beta})
$$
\[ e_{\beta 0} = \int T_\beta^T u_0 \, d\Gamma + u_0(r_\beta) \]

Moreover, the rigid-body term in definition (61) is so chosen as to ensure that the block diagonal term is the identity matrix:

\[ F_{\beta\beta} = \int T_{\beta\beta}^T U_{\beta\beta} \, d\Gamma + \bar{z}_\beta = I \]

### 9.5 Fundamental solutions and the Trefftz method

A typically boundary element formulation of the Trefftz method consists in limiting the domain approximation basis to a set of point load solutions applied to the Dirichlet boundary of the mesh. System (39) simplifies into form,

\[
\begin{bmatrix}
F_{\beta\beta} & -A_{\beta\gamma} & -A_{\beta\beta} \\
-A_{\beta\beta} & K_{\gamma\gamma} & \cdot \\
-A_{\beta\gamma} & \cdot & \cdot \\
\end{bmatrix}
\begin{bmatrix}
z_{\beta} \\
q_{\gamma} \\
q_{\beta} \\
\end{bmatrix}
=
\begin{bmatrix}
\bar{e}_{\beta 0} - e_{\beta 0} \\
p_{\gamma 0} - \bar{p}_{\gamma} \\
p_{\beta 0} - \bar{p}_{\beta} \\
\end{bmatrix}
\]

which remains symmetric but less sparse, as all matrices are now highly populated. The position of the stress poles is strictly on the Dirichlet boundary.

Another alternative is to interpret set \( S_\alpha \) in approximation (15) as a basis of Neumann point loads (point loads with poles at points where the displacement are not known, in the terminology used here) and set \( S_\beta \) as a basis of Dirichlet point loads (point loads applied on the Dirichlet boundary), as in the previous option. It can be verified that equations summarized in Table 1 and the finite element solving system (39) remain valid provided that definition (30) for the compatibility matrix is replaced by:

\[
A_{\alpha\gamma} = \int T_\alpha^T Z_\gamma \, d\Gamma \gamma + \bar{Z}_\gamma Z_\gamma (r_\alpha) \]

Equation (60) shows that duality is preserved, as it now reads:

\[
\int Z_\gamma^T \left( T_\alpha z_{\alpha} + T_\beta z_{\beta} + t_\alpha \right) d\Gamma \gamma + \int Z_\gamma^T \left( r_\alpha \right) L z_{\alpha} = \int Z_\gamma^T \left[ k_\gamma (Z_\gamma q_{\gamma} - \bar{u}) + \bar{\Gamma} \right] d\Gamma \gamma
\]

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References


**Appendix A: Polynomial and rational solutions**

The polynomial basis is determined [13,15] from the real and imaginary parts of the source fields, defined in a polar co-ordinate system \((r, \theta)\),

\[
\begin{align*}
\sigma_{rr} &= r^m \frac{m - n^2 + 2}{(m + l)(m + 2)} \exp(\pm in\theta) \\
\sigma_{r\theta} &= r^m \frac{m - n^2 + 2}{2G\kappa \pm in(m + l)} \exp(\pm in\theta)
\end{align*}
\]

where \(m\) is non-negative integer, \(n = m\) and \(n = m + 2\) and \(i\) is the imaginary unit. The dimension of a polynomial basis with degree \(d_p = m_{\text{max}}\) is \(3 + 4d_p\), as there are only three independent constant stress modes \((m = 0)\). In the associate displacement field,

\[
\begin{align*}
u_r &= \frac{r^{m+1}}{2G\kappa} \frac{m(\alpha - m - l)}{\pm in(\alpha + m + l)} \exp(\pm in\theta) \\
\end{align*}
\]

\(G\) is the shear modulus, \(\alpha = \kappa\) for \(n = m\) and \(\alpha = -l\) for \(n = m + 2\); \(\kappa = (3 - \nu)/(1 + \nu)\) for plane stress problems and \(\kappa = 3 - 4\nu\) for plane strain problems, with \(\nu\) representing the Poisson’s ratio. To improve the conditioning of the finite element solving system (39), the origin of the basis is placed at the baricentre of the element, with the orientation of its principal directions.

The rigid-body modes are obtained with \(m = 0\) and \(m = -l\):

\[
\begin{bmatrix}
\nu_r \\
\nu_\theta
\end{bmatrix} = \begin{bmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & r
\end{bmatrix}
\]

The bases used in the implementation of multiply connected elements are (the independent solutions) obtained by defining \(m\) as a negative integer. The dimension of a basis with order \(d_r = m_{\text{max}}\) is \(3 + 4d_r\). The origin of the system of reference is placed at the baricentre of the
element cells, to ensure that the source of the singularity is not contained in the domain of the element: \( r \geq \epsilon \) with \( \epsilon \) small.

**Appendix B: Wedge solutions**

Linear combinations \([16]\) of the solution presented above lead to the mode I (first column) and mode II (second column) wedge solutions,

\[
\begin{align*}
\begin{bmatrix}
\sigma_{rr} \\
\sigma_{\theta \theta} \\
\sigma_{r \theta}
\end{bmatrix}
&= r^{2-j} \begin{bmatrix}
(3-\lambda)c^- + \alpha c^+ \\
(1+\lambda)c^- - \alpha c^+ \\
(1-\lambda)s^- - \alpha s^+
\end{bmatrix} \\
\begin{bmatrix}
u_r \\
v_\theta
\end{bmatrix}
&= \frac{r^j}{2\lambda G} \begin{bmatrix}
(\kappa-\lambda)c^- + \alpha c^+ \\
-(\kappa+\lambda)s^- - \alpha s^+
\end{bmatrix}
\end{align*}
\]

where \( c^\pm = \cos((1\pm\lambda)\theta) \) and \( s^\pm = \sin((1\pm\lambda)\theta) \). The origin of the polar co-ordinate system is placed at the tip of the wedge, with \( \theta = \pm \omega / 2 \) defining the wedge faces, as shown in Figure 2.

The wedge eigenvalues are defined as follows,

**Mode I:**

\[
\begin{align*}
\lambda \sin \omega + \sin \lambda \omega &= 0 \\
\alpha &= \lambda \cos \omega + \cos \lambda \omega
\end{align*}
\]

**Mode II:**

\[
\begin{align*}
\lambda \sin \omega - \sin \lambda \omega &= 0 \\
\alpha &= \lambda \cos \omega - \cos \lambda \omega
\end{align*}
\]

and the dimension of the basis is \( 2d_\lambda \), with \( d_\lambda \) representing the number of eigenvalues taken in each mode.

**Appendix C: Crack solutions**

The wedge solution holds for the Mitchell crack shown in Figure 2, with \( \theta = 0 \) defining the crack alignment and \( \theta = \pm \pi \) the crack sides. The mode I and mode II eigenvalues for open cracks are defined as follows, where \( m \) is a non-negative integer:

\[
\begin{align*}
\lambda_i &= \lambda_{II} = 0.5 + m
\end{align*}
\]

The eigenvalues for a Mitchell crack with elastic filler are \([3]\),

\[
4(\kappa - I)\lambda_i \pi \cos(\lambda_i \pi) + \gamma (\kappa + I)(\kappa + I) \sin(\lambda_i \pi) = 0
\]

\[
4 \lambda_{II} \pi \cos(\lambda_{II} \pi) + \gamma (\kappa + I) \sin(\lambda_{II} \pi) = 0
\]

where \( \kappa \) is the parameter given in Appendix A, now defined for the filling material, and \( \gamma \) is a measure of the relative shear modulus,

\[
\gamma = \frac{\bar{G}}{\varepsilon \rho G}
\]
where $\varepsilon$ is a small positive number ($\varepsilon^2 \approx 0$), and $\rho = +1$ for $G \ll G$ ($\rho = -1$ for $G \gg G$).

According to the notation of Figure 1, the constitutive relation (11) of the filled crack is [1]:

$$
\begin{bmatrix}
\{t_z\} \\
\{t_\eta\}
\end{bmatrix} = \frac{\gamma G}{2\pi r}
\begin{bmatrix}
(\bar{\kappa} + 1)/(\bar{\kappa} - 1) & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\{\delta u_z\} \\
\{\delta u_\eta\}
\end{bmatrix}
$$

The Griffith crack solution [12,14] is defined as follows, now in Cartesian co-ordinates (see Figure 3),

$$
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{bmatrix} = \begin{bmatrix}
Re Z' - y Im Z'' & 2 Im Z' + y Re Z'' \\
Re Z' + y Im Z'' & - y Re Z'' \\
-y Re Z'' & Re Z' - y Im Z''
\end{bmatrix}
$$

$$
\begin{bmatrix}
u_x \\
u_y
\end{bmatrix} = \frac{I}{4G}
\begin{bmatrix}
(\kappa - l) Re Z - 2 y Im Z' & (\kappa + l) Im Z + 2y Re Z' \\
(\kappa + l) Im Z - 2 y Re Z' & -(\kappa - l) Re Z - 2 y Im Z'
\end{bmatrix}
$$

where $Z = (z^2 - a^2)^{i}$ and $z = x + iy$. The eigenvalue solutions presented for the Mitchell crack still apply for both open and filled cracks. The definition of the filled crack constitutive relation is:

$$
\begin{bmatrix}
\{t_z\} \\
\{t_\eta\}
\end{bmatrix} = \frac{\gamma Gr}{\pi(\alpha^2 - r^2)}
\begin{bmatrix}
(\bar{\kappa} + 1)/(\bar{\kappa} - 1) & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\{\delta u_z\} \\
\{\delta u_\eta\}
\end{bmatrix}
$$

Appendix D: Point load solutions

The Boussinesq solution [11] for point loads applied to semi-infinite domains is defined as follows, where $\beta = \alpha + 2/(1 + \kappa)$:

$$
\begin{bmatrix}
\sigma_{rr} \\
\sigma_{\theta \theta} \\
\sigma_{r \theta}
\end{bmatrix} = \begin{bmatrix}
2 \exp(i\theta) \\
\frac{\gamma r}{\gamma r} \\
0
\end{bmatrix}
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
$$

$$
\begin{bmatrix}
u_r \\
u_\theta
\end{bmatrix} = \frac{\exp(i\theta)}{4\gamma G}
\begin{bmatrix}
(\kappa + l)(\ln r + \alpha) - i(\kappa - l)\theta \\
i(\kappa + l)(\ln r + \beta) + (\kappa - l)\theta
\end{bmatrix}
$$

The real (imaginary) part recovers the unit point load solution $F_x = 1$ ($F_y = 1$) in Figure 5 by setting $\gamma = \omega + \sin \omega$ ($\gamma = \omega - \sin \omega$).

The real (imaginary) part of the Kelvin solution [11] models the effect of a unit point load $F_x = 1$ ($F_y = 1$) applied on an infinite domain letting $\gamma = 2\pi(\kappa + l)$:

$$
\begin{bmatrix}
\sigma_{rr} \\
\sigma_{\theta \theta} \\
\sigma_{r \theta}
\end{bmatrix} = \begin{bmatrix}
\exp(i\theta) \\
\frac{3 + \kappa}{\gamma r} \\
\frac{i(\kappa - l)}{1 - \kappa}
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
i(\kappa - l)
\end{bmatrix}
$$
\[
\begin{align*}
\begin{cases}
    u_r = \exp(i\theta) \left\{ \frac{\kappa \ln r - \bar{\alpha}}{\gamma G} \right\} \\
    u_\theta = \frac{i(\kappa \ln r - \bar{\alpha} + 1)}{\gamma G}
\end{cases}
\end{align*}
\]

The amplitudes of the rigid-body displacement modes present in the definitions above are so chosen as to eliminate the bounded part of the external work dissipated in the vicinity of the load:

\[
\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left[ (t_r' u_r' + t_\theta' u_\theta') r \right]_{r=\varepsilon} \, d\theta = -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left[ (\sigma_r' u_r' + \sigma_\theta' u_\theta') r \right]_{r=\varepsilon} \, d\theta = -\frac{\ln \varepsilon}{4\gamma G} \delta \delta_{ij}
\]

In this equation, the superscript identifies one type of load, \( F_x \) or \( F_y \), \( \delta_{ij} \) is the Kronecker symbol, and \( \delta = \kappa + 1 \) (\( \delta = 4\kappa \)) for Boussinesq (Kelvin) forces. The amplitude of the rigid-body displacement modes thus found for the unit Kelvin solution is \( \bar{\alpha} = \frac{1}{2}(\kappa - 1)/(\kappa + 1) \). The values found for the unit Boussinesq solution are \( \alpha = \bar{\alpha} \bar{\gamma}/\gamma \) and \( \alpha = -\bar{\alpha} \bar{\gamma}/\gamma \) for the \( F_x \) and \( F_y \) components, respectively, with \( \bar{\gamma} = \omega \cos \omega - \sin \omega \).